# Evaluation of Decoherence and Induced Interactions for Quantum Control and Computing

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Universal Decoherence in Solids

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(Received 3 December 2003; published 26 March 2004)

Symmetry implications for the decoherence of quantum oscillations of a two-state system in a solid are studied. When the oscillation frequency is small compared to the Debye frequency, the universal lower bound on the decoherence due to the atomic environment is derived in terms of the macroscopic parameters of the solid, with no unknown interaction constants.

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Field Dependence of the Electron Spin Relaxation in Quantum Dots

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The interaction of the electron spin with local elastic twists due to transverse phonons is studied. The universal dependence of the spin-relaxation rate on the strength and direction of the magnetic field is obtained in terms of the electron gyromagnetic tensor and macroscopic elastic constants of the solid. The theory contains no unknown parameters and it can be easily tested in experiment. At high magnetic field it provides a parameter-free lower bound on the electron spin relaxation in quantum dots.

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Quantum dynamics for short time steps, followed by Markovian approximation, etc.:

\[
\frac{1}{T_2} = \frac{1}{2T_1} + \text{pure decoherence}
\]

Valid only for \( t > O(\hbar / kT) \)
Deviation from Pure/Ideal Quantum States: Can we escape the $T_1$-$T_2$ paradigm?


We have developed new short-time approximation schemes for evaluation of decoherence. At low temperatures, the approximation is argued to apply at intermediate times as well. It then provides a tractable approach complementary to Markovian approximations, and is appropriate for evaluation of deviations from pure states in quantum computing models.
Single qubit:
- Internal dynamics
- Decoherence, relaxation
- Dressing by the bath modes
- Hybridization with the bath modes
- Reduction: how do we “trace” over the bath?
- Initial state?

Two qubits:
- Internal dynamics: Induced Interaction and its Coherence
- Decoherence, Loss of Entanglement, relaxation

Many qubits:
- Additivity

Interacting bath modes, realistic environment:
- The origin of thermalization
- “Foundational” thermodynamic issues
- Quantum Measurement
A simple model: Adiabatic Decoherence
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\[ H = H_S + H_B + H_I \]

\[ [H_S, H_I] = 0 \]

\[ H_I = \Lambda_S P_B \]

\[ [\Lambda_S, H_S] = 0 \]

Bath of bosons (harmonic oscillators) labeled by \( k \):

\[ H = H_S + \sum_k \omega_k a_k^{\dagger} a_k + \Lambda_S \sum_k \left( g_k^{\ast} a_k + g_k a_k^{\dagger} \right) \]
\[ |\rho_{mn}(t)| = |\rho_{mn}(0)| \exp \left[ -\frac{1}{4} (\lambda_m - \lambda_n)^2 \Gamma(t) \right] \]

\[ \Gamma(t) = 8 \sum_k \omega_k^{-2} |g_k|^2 \sin^2 \frac{\omega_k t}{2} \coth \frac{\beta \omega_k}{2} \]

\(\Gamma(t)\) is a sum of positive terms. For true decoherence, i.e., in order for this sum to diverge for large times, one needs a \textit{continuum} of frequencies and interactions with the bath modes that are strong enough at \textit{low frequencies}. We introduce the density of states \(G(\omega)\) and sum over frequencies rather than modes, with \(\frac{dk}{d\omega}\) calculated from the dispersion relation of the bosonic modes:

\[ \Gamma(t) \propto \int d\omega \frac{dk}{d\omega} G(\omega) |g(\omega)|^2 \omega^{-2} \sin^2 \frac{\omega t}{2} \coth \frac{\beta \omega}{2} \]

Most model studies use

\[ \frac{dk}{d\omega} G(\omega) |g(\omega)|^2 \propto \omega^n e^{-\frac{\omega}{\omega_c}} \]
For time-independent interactions, we have used relations of the type:

\[ e^{i(H_S + H_B + H_I)t + O(t^3)} = e^{iH_S t/2} e^{i(H_B + H_I)t} e^{iH_S t/2} \]

For time-dependent, “gate control” external interactions, a more complicated approach is needed, based on relations similar to the Magnus expansion in quantum chemistry (that avoids time-ordering when approximating the evolution operator):

\[
\sqrt{\pi} \rho(t) = \int_{-\infty}^{\infty} dy e^{-y^2} e^{-iH_2t/2} e^{i [yB(t)\Lambda_s - C(t)\Lambda_s^2]} e^{-iH_2t/2} \rho(0) e^{iH_2t/2} e^{-i[yB(t)\Lambda_s - C(t)\Lambda_s^2]} e^{iH_2t/2}
\]

\[
B^2(t) = 8 \sum_K \frac{|g_K|}{\omega_K^2} \sin^2 \frac{\omega_K t}{2} \coth \frac{\beta \omega_K}{2}
\]

\[
C(t) = \sum_K \frac{|g_K|^2}{\omega_K^2} (\omega_K t - \sin \omega_K t)
\]

For \(\rho(0) = |\uparrow\rangle\langle\uparrow|\) or \(|\downarrow\rangle\langle\downarrow|\), we get

\[
\text{Tr}_s [\rho^2(t)] = \frac{1}{2} [1 + e^{-2B^2(t)}]
\]

For \(\rho(0) = |\psi_0\rangle\langle\psi_0|\), the deviation from a pure state is apparent: \(\rho(t > 0)\) is obviously a \textit{mixture} (integral over \(y\)) of pure-state projectors \(|\psi(y, t)\rangle\langle\psi(y, t)|\), where

\[
|\psi(y, t)\rangle = e^{-iH_2t/2} e^{i [yB(t)\Lambda_s - C(t)\Lambda_s^2]} e^{-iH_2t/2} |\psi_0\rangle
\]
• **Quantum entropy** $S(t) = -\text{Tr}(\rho \ln \rho)$, and **idempotency defect**, also termed the first order entropy, $s(t) = 1 - \text{Tr}(\rho^2)$. Both expressions are basis independent, have a minimum, 0, at pure states and measure the degree of the state’s “purity.”

• **Fidelity** $F(t) = \text{Tr}\left[\rho_\text{ideal}(t)\rho(t)\right]$, where $\rho_\text{ideal}(t)$ represents the pure-state evolution of the system without the environment. The fidelity attains its maximal value, 1, provided $\rho(t) = \rho_\text{ideal}(t)$. 
• **Norm of deviation**: a new measure of decoherence, introduced in our works, e.g., L. Fedichkin, A. Fedorov & V. Privman, Phys. Lett. A 328, 87-93 (2004)

• Measure deviation from the ideal evolution by a norm of the deviation operator \( \sigma(t) \equiv \rho(t) - \rho_{\text{ideal}}(t) \). We can use the **eigenvalue norm**, \( \|\sigma\|_{\lambda} = \max_i \lambda_i \), etc. It has its minimal value, 0, for \( \rho(t) = \rho_{\text{ideal}}(t) \), even if the latter is not a projection operator.
The Maximal Norm and Its Properties

To characterize decoherence for an arbitrary initial state, pure or mixed, we define the maximal norm, $D$, which is determined as norm-of-deviation maximized over all initial density matrices:

$$D(t) = \sup_{\rho(0)}(\|\sigma(t, \rho(0))\|_\lambda).$$

$D$ is approximately additive for weakly interacting qubits, as long as it is small (close to 0) for each, namely for short times. This is similar to the approximate additivity of relaxation rates for weakly interacting qubits at large times.
The Maximal Norm and Its Properties

Technical point: why are norms like these candidate for “additivity”? The answer: their definition resembles that of length.

We used \( \| \sigma \|_{\lambda} = \max_i |\lambda_i| \), with \( \sigma \) given by the difference of two density matrices.

A more rigorous definition: \( \| A \| = \sup_{\phi \neq 0} \left[ \frac{\langle \phi | A^\dagger A | \phi \rangle}{\langle \phi | \phi \rangle} \right]^{1/2} \).
The Maximal Norm and Its Properties

Averaging over the initial density matrices removes time-dependence at the frequencies of the system, leaving only the relaxation temporal dynamics.

Spin-1/2 interacting with an Ohmic bath of bosonic modes, in the short-time approximation.
Models with Interactions of the form \( \Lambda S \frac{P^\dagger}{B} + \Lambda^\dagger \frac{P}{S} \)

D. Tolkunov & V. Privman,

Comparison between the \( O(t^2) \) expansion, (i), and the short-time approximation, (ii), for a Jaynes-Cummings type model, for the idempotency defect measure.
Decoherence and Loss of Entanglement
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Effect of uncorrelated quantum noise on the entanglement of two two-state systems (two qubits):

Suppression of entanglement is \textit{at least as fast} as the \textit{product} of the suppression factors describing decoherence of each of the two subsystems:

\textbf{D. Tolkunov, V. Privman & P. K. Aravind},
Exchange Interaction, Entanglement and Quantum Noise Due to a Thermal Bosonic Field

D. Solenov, D. Tolkunov and V. Privman,

\[ H_{\text{int}} = -\frac{2\alpha_n \Gamma(n) \omega_c^n}{(1 + \omega_c^2 d^2 / c_s^2)^{n/2}} \cos \left[ n \arctan \left( \frac{\omega_c |d|}{c_s} \right) \right] S_1 S_2 \]

Induced interaction between two localized spins in the bath of thermalized bosonic modes.

The concurrence calculated for Ohmic dissipation. The inset gives the time dependence for different temperatures (the top curve corresponds to the lowest temperature).
Exchange interaction, entanglement, and quantum noise due to a thermal bosonic field

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1. INTRODUCTION

The idea that exchange of fermionic or bosonic excitations can lead to physical interactions in solid state is not new.1 Recently, such induced interactions have received attention due to the possibility to experimentally observe quantum dynamics in nanoscale devices.2–9 In this work we explore the dynamics of two qubits (two two-state quantum systems), e.g., electron spins 1/2, placed a distance d apart, as they are entangled by common thermalized bosonic environment without direct spatial electron wave function overlap. At the same time quantum noise originating from the same bosonic field (e.g., phonons) ultimately erases the generated entanglement for large enough times. We demonstrate that the indirect exchange interaction induced by the bosonic thermal field10 can, in some cases, be comparable with other qubit-qubit couplings.

Extensive studies have been reported11–14 of the decay of quantum correlations between qubits subject to individual (local) or common environmental noise. In the presence of quantum noise, entanglement was shown to decay very fast and, in some cases, vanish at finite times.12,13 On the other hand, the idea that common bosonic as well as fermionic environment is able to entangle the qubits has also been advanced.10,11,15–23 For fermionic environment, this effect has been attributed to Rudermann-Kittel-Kasuya-Yosida (RKKY)-type interactions15,17–20,22,23 and it has recently been experimentally demonstrated for two-qubit semiconductor nanostructures.3,5

In this work, we investigate the two competing physical effects of a thermalized bosonic environment (bath) in which two qubits are immersed. Specifically, the induced interaction, which is effectively a zero-temperature effect, and the quantum noise, originating from the same bath modes, are derived within a uniform treatment. We study the dependence of the induced coherent vs noise (decoherence) effects on the parameters of the bath modes, the qubit system, and their coupling, as well as on the geometry. Specific applications are given for spins interacting with phonons in semiconductors.

We assume that the spins are identically coupled with the modes of a thermalized bosonic bath, described by $H_B$ and, in some cases, vanish at finite times.12,13 On the other hand, the idea that common bosonic as well as fermionic systems are able to entangle the qubits has also been explored.4,5

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the initially unentangled spins can develop entanglement. We find that the degree of the entanglement and the time scale of its ultimate erasure due to noise can be controlled by varying several parameters, as further discussed for various bath types in Sec. V. Estimates for the induced coherent interaction in Si-Ge type semiconductors are presented in Sec. VI. The coherent interaction induced by phonons is, expectedly, quite weak. However, we find that in strong magnetic fields it can become comparable with the dipole-dipole coupling.

II. COHERENT INTERACTION AND QUANTUM NOISE INDUCED BY THERMALIZED BOSONIC FIELD

In this section we present the expressions for the induced interaction and also for the noise effects due to the bosonic environment, calculated perturbatively to the second order in the spin-boson interaction, and with the assumption that the environment is constantly reset to thermal. Specific applications and examples are considered towards the end of this section, as well as in Secs. V and VI.

The dynamics of the system can be described by the equation for the density matrix

$$i\dot{\rho}(t) = [H,\rho(t)].$$

(2.1)

In order to trace over the bath variables, we carry out the second-order perturbative expansion. This dynamical description is supplemented by the Markovian assumption of resetting the bath to thermal equilibrium, at temperature $T$, after each infinitesimal time step, as well as at time $t=0$, thereby decoupling the qubit system from the environment. This is a physical assumption appropriate for all but the shortest time scales of the system dynamics. It can also be viewed as a means to phenomenologically account in part for the randomness of the bath modes due to their interactions with each other (anharmonicity) in real systems. This leads to the master equation for the reduced density matrix of the qubits $\rho_S(t) = Tr_B \rho(t)$,

$$i\dot{\rho}_S(t) = [H_S,\rho_S(t)] - i\int_0^\infty dt' Tr_B[H_{SB}[H_{SB}(t'-t),\rho_B(t')]],$$

(2.2)

where $H_{SB}(\tau) = e^{i(H_{SB}+H_T)\tau} H_{SB} e^{-i(H_{SB}+H_T)\tau}$, $\rho_B = \rho_B(1)-e^{-\omega_B^2 t} \rho_B T$, and $Z = 1/\{1+(1-e^{-\omega_B^2 t})\}$ is the partition function. Analyzing the structure of the integrand in Eq. (2.2), one can obtain the equation with explicitly separated coherent and noise contributions, see Appendix A,

$$i\dot{\rho}_S(t) = [H_{eff},\rho_S(t)] + i\tilde{M}\rho_S(t).$$

(2.3)

Here the effective coherent Hamiltonian $H_{eff}$ is

$$H_{eff} = H_S + \sum_{m=x,y,z} 2\chi^m_{\delta}(d)|\sigma_m^1|^2 \sigma_m^2 - \chi^i(d)(|\sigma_i^2|^2 + |\sigma_i^3|^2) + \chi^i(d)(|\sigma_i^2|^2 + |\sigma_i^3|^2) - [\eta^m_0(0) + \eta^m_0(0)](\sigma_i^1 + \sigma_i^2).$$

(2.4)

The expressions for the amplitudes $\chi^m_{\delta}(d)$, $\chi^i(d)$, $\eta^m_0(d)$, and $\eta^i_0(d)$ will be given shortly. The first three terms following $H_T$ constitute the interaction between the two spins. We will argue below that the leading induced exchange interaction is given by the first added term, proportional to $\chi^m_{\delta}(d)$. The last term corrects the energy gap for each qubit, representing their Lamb shifts.

The explicit expression for the noise term is very cumbersome. It can be represented concisely by introducing the noise superoperator $\tilde{M}$, which involves single-qubit contributions, which are usually dominant, as well as two-qubits terms

$$\tilde{M} = \sum_{m,j} [\tilde{M}_{m,j}^{0}(0) + \sum_{j\neq i} \tilde{M}_{m,j}^{ij}(d)],$$

(2.5)

where the summations are over the components, $m=x,y,z$, and the qubits, $i,j=1,2$. The quantities entering Eq. (2.5) can be written in terms of the amplitudes $\chi^m_{\delta}(d)$, $\eta^m_0(d)$, and $\eta^i_0(d)$, in a compact form, by introducing the superoperators $L_{\Delta}(O_1)O_2 = \{O_1, O_2\}$, $\Delta(O_1)O_2 = O_1 O_2$ and $\hat{L}_{\Delta}(O_1, O_2) = \hat{L}(O_1, O_2) = \hat{L}(O_2, O_1)$.

$$\tilde{M}_{m,j}^{0}(d) = \eta^m_0(d)[2\hat{L}(\sigma_m^1,\sigma_m^0) - \hat{L}_{\Delta}(\sigma_m^0,\sigma_m^1)] + \eta^m_0(d)\hat{M}_{\Delta}(\sigma_m^0,\sigma_m^1) - \hat{L}_{\Delta}(\sigma_m^0,\sigma_m^1),$$

(2.6)

where we defined $s_m^j = \frac{1}{2}(\sigma_m^j, \sigma_m^0)$ and

$$\tilde{M}_{m,j}^{ij}(d) = \eta^m_0(d)[2\hat{L}(\sigma_m^1,\sigma_m^0) - \hat{L}_{\Delta}(\sigma_m^0,\sigma_m^1)] + \eta^m_0(d)\hat{M}_{\Delta}(\sigma_m^0,\sigma_m^1) - i\chi^m_{\delta}(d)\hat{M}_{\Delta}(\sigma_m^0,\sigma_m^1).$$

(2.7)

The amplitudes in Eqs. (2.4), (2.6), and (2.7) calculated for the interaction defined in Eqs. (1.2) and (1.3), are

$$\chi^m_{\delta}(d) = -\sum_{\xi} \frac{\pi}{4} \int_{-\infty}^{\infty} \frac{Vdk}{(2\pi)^3} e^{2m \xi k^2} \frac{\omega_{k,\xi} \cos(k \cdot d)}{\omega_{k,\xi} - \Delta^2 (1 - \delta_{m,\xi})},$$

(2.8)

$$\eta^m_0(d) = \frac{\pi}{2} \sum_{\xi} \frac{Vdk}{(2\pi)^3} e^{2m \xi k^2} \frac{\omega_{k,\xi} \cos(k \cdot d)}{\omega_{k,\xi} - \Delta^2 (1 - \delta_{m,\xi})},$$

(2.9)

and

$$\chi^i(d) = -\sum_{\xi} \frac{\pi \xi}{4} \frac{Vdk}{(2\pi)^3} e^{2m \xi k^2} \frac{\omega_{k,\xi} \cos(k \cdot d)}{\omega_{k,\xi} - \Delta^2 (1 - \delta_{m,\xi})},$$

(2.10)

$$\eta^i_0(d) = \frac{\pi}{2} \sum_{\xi} \frac{Vdk}{(2\pi)^3} e^{2m \xi k^2} \frac{\omega_{k,\xi} \Delta \cos(k \cdot d)}{\omega_{k,\xi} - \Delta^2}.$$
Note that $\chi^n_c(d)$ appears only in the induced interaction Hamiltonian in Eq. (2.4), whereas $\eta^n_c(d)$, $\chi^n_s(d)$, and $\eta^n_s(d)$ enter both the interaction and noise terms. Therefore, in order to establish that the induced interaction can be significant for some time scales, we have to demonstrate that $\chi^n_c(d)$ can have a much larger magnitude than the maximum of the magnitudes of $\eta^n_c(d)$, $\chi^n_s(d)$, and $\eta^n_s(d)$. The third and fourth terms in expression for the interaction (2.4) are comparable to the noise and therefore have no significant contribution to the coherent dynamics.

Because of the complexity of the expressions for the noise terms within the present Markovian treatment, in this section we will only compare the magnitudes of the coherent vs noise effects. In the next section, we will discuss a different model for the noise which will allow a more explicit investigation of the time dependence.

For the rest of this section, we will consider an illustrative one-dimensional (1D) example favored by recent experiments,42 leaving the derivations for higher dimensions to Secs. V and VI. We comment that the 1D geometry is also natural for certain ion-trap quantum-computing schemes, in which ions in a chain are subject to Coulomb interaction, developing a variety of phonon-mode lattice vibrations.21,34,35

In 1D geometry we allow the phonons to propagate in a single direction, along $d$, so that $k \cdot d = k |d|$. Here, for definiteness, we also assume the linear dispersion, $\omega_c = c_s k$, since the details of the dispersion relation for larger frequencies usually have little effect on the decoherence properties. Furthermore, we ignore the polarization, $g^m_k - \delta^m_k$. Another reason to focus on the low-frequency modes is that an additional cutoff $\omega$ resulting from the localization of the electron wave functions, typically much smaller than the Debye frequency, will be present due to the factors $g^m(\omega)$. The induced interaction and noise terms depend on the amplitudes $\chi^n_c(d)$, $\eta^n_c(d)$, $\chi^n_s(d)$, and $\eta^n_s(d)$, two of which can be evaluated explicitly for the 1D case, because of the $\delta$ functions in Eqs. (2.9) and (2.10). However, to derive an explicit expression for $\chi^n_c(d)$ and $\eta^n_c(d)$, one needs to specify the $\omega$ dependence in $|g^m(\omega)|^2$. For the sake of simplicity, in this section we approximate $|g^m(\omega)|^2$ by a linear function with superimposed exponential cutoff. For a constant 1D density of states $Y(\omega) = V/2 \pi c_s$, this is the Ohmic-dissipation condition,27 i.e.,

$$|g^m(\omega)|^2 Y(\omega) = \alpha^m_{cs} \omega^3 \exp(-\omega/\omega_c),$$

with $n=1$ (the case when $n>1$ is considered in Sec. V).

In most practical applications, we expect that $\Delta \ll \omega_c$. With this assumption, we obtain

$$\chi^n_c(d) = \frac{\alpha^n_c \omega_c}{1 + (\omega_c |d|/c_s)^2},$$

$$\chi^n_s(d) = \alpha^n_s \omega_c \frac{\pi}{2} (1 - \delta_{mc,z}) \frac{\Delta}{\omega_c} \cos \frac{\Delta |d|}{c_s},$$

(2.14)

and

$$\eta^n_c(d) = \frac{\alpha^n_c \omega_c}{2} (1 - \delta_{mc,z}) \frac{\Delta}{\omega_c} \coth \frac{\Delta}{2k_BT} \cos \frac{\Delta |d|}{c_s}.$$  

(2.15)

The expression for $\eta_c(d)$ could not be obtained in closed form. However, numerical estimates suggest that $\eta_c(d)$ is comparable to $\eta^n_c(d)$. At short time separations $d$, $\eta_c(d)$ is approximately bounded by $-\alpha^n_c \Delta \ln \omega_c \exp(-|d|/c_s)$, while at larger distances it may be approximated by $\alpha^n_c \Delta^2 \frac{\sin \frac{\Delta d}{c_s} \coth \frac{\Delta}{2k_BT}}{2k_BT}$. The level of noise may be estimated by considering the quantity

$$\mathcal{M} = \max_{|d|} |\eta^n_c(d), \chi^n_s(d), \eta^n_s(d)|.$$  

(2.16)

The interaction Hamiltonian takes the form

$$H_{int} = - \frac{2}{1 + (\omega_c |d|/c_s)^2} \sum_{m=x,y,z} \alpha^n_{cm} \omega_c \sigma^m_m \sigma^s_m.$$  

(2.17)

This induced interaction is temperature independent. It is long-range and decays as a power law for large $d$. For the super-Ohmic case, $n>1$, one obtains similar behavior, except that the interaction decays as a higher negative power of $d$, as will be shown in Sec. V.

If the noise term were not present, the spin system would be governed by the Hamiltonian $H_S + H_{int}$. To be specific, let us analyze the spectrum, for instance, for $\alpha^n_c = \alpha^n_s = 0$. The two-qubit states consist of the singlet $\left| \uparrow \downarrow \right> - \left| \downarrow \uparrow \right> / \sqrt{2}$ and the split triplet $\left| \uparrow \uparrow \right>, \left| \uparrow \downarrow \right>, \left| \downarrow \uparrow \right>/\sqrt{2}$, and $|11\rangle$, with energies $E_2 = -4 \chi_c$, $E_0 = -\Delta$, $E_1 = 4 \chi_c$, $E_3 = \Delta$, respectively. The energy gap $|E_1 - E_2|$ between the two entangled states is defined by $4 \chi_c (\approx 4 \chi_c)$. In the presence of noise, the oscillatory, approximately coherent evolution of the spins can be observed over several oscillation cycles provided that $2 \alpha^n_{c} \omega_c / [1 + (\omega_c |d|/c_s)^2] > \mathcal{M}$. The energy levels will acquire effective width due to quantum noise, of order $\eta^n_c(d)$. This interplay between the interaction and noise effects is further explored within an exactly solvable model in the next section.

III. EXACTLY SOLVABLE MODEL

In this section, we consider a model appropriate for short times,31–33 which does not invoke the Markovian assumption of the rethermalization of the bath modes. This model is particularly suitable for investigating the onset of the system’s dynamics. While the noise effects are quantitatively different in this model, the qualitative interplay of the coherent and noise effects in the dynamics is the same as in the Markovian approach. Furthermore, we will show that the induced interaction is consistent with the one obtained within Markovian approach in the previous section.

We point out that, due to the instantaneous rethermalization assumption (resetting the density matrix of the bath to thermal), in the Markovian formulation it was quite natural to assume that the bath density matrix is also thermal at time $t=0$; the total density matrix retained an uncorrelated-product form at all times. In the context of studying the
short-time dynamics, in this section the choice of the initial condition must be addressed more carefully. In quantum computing applications, the initially factorized initial condition has been widely used for the qubit-bath system.\(^{21-33}\)

\[ \rho(0) = \rho_{3}(0)\rho_{6}(0). \]  

(3.1)

This choice allows comparison with the Markovian results, and is usually needed in order to make the short-time approximation schemes tractable.\(^{33}\) Specifically, it is necessary for exact solvability of the model considered in this and the next sections.

A somewhat more “physical” excuse for factorized initial conditions has been the following. Quantum computation is carried out over a sequence of time intervals during which various operations are performed on individual qubits and on pairs of qubits. These operations include control gates, measurements, and error correction. It is usually assumed that these “control” functions, involving rather strong interactions with external objects, as compared to interactions with sources of quantum noise, erase the fragile entanglement with the bath modes that qubits can develop before those time intervals when they are “left alone” to evolve under their internal (and bath induced) interactions. Thus, for evaluating relative importance of the quantum noise effects on the internal (and bath induced) qubit dynamics, which is our goal here, we can assume that the state of qubit-bath system is “reset” to uncorrelated at \( t = 0 \).

It turns out that the resulting model is exactly solvable for the Zeeman splitting \( \Delta = 0 \), and provided that only a single system operator enters the expression (1.2) for the interaction. Here we take \( \alpha_{n}^{g} = \alpha_{n}^{d} = 0 \), while \( \alpha_{n}^{f} \neq 0 \). We derive the exact solution and demonstrate the emergence of the interaction (2.17).

With the above assumptions, one can utilize the bosonic operator techniques\(^{29}\) to obtain the reduced density matrix for the system (1.2) and (1.3),

\[ \rho_{3}(t) = \sum_{\lambda, \lambda'} P_{\lambda} \rho_{3}(0) P_{\lambda'} e^{i \hbar \mathcal{L}_{\lambda, \lambda'}(t)}, \]  

(3.2)

where the projection operator is defined as \( P_{\lambda} = | \lambda \rangle \langle \lambda | \), and \( | \lambda \rangle \) are the eigenvectors of \( \sigma_{n}^{f} \). The exponent in Eq. (3.2) consists of the real part, which leads to decay of off-diagonal density-matrix elements resulting in decoherence,

\[ \text{Re} \mathcal{L}_{\lambda, \lambda'}(t) = - \sum_{k} G_{k}(t, T) \left[ (\lambda_{l}^{f} - \lambda_{l}^{d})^{2} + (\lambda_{l}^{g} - \lambda_{l}^{d})^{2} + 2 \cos \left( \frac{\omega_{k}}{c_{s}} | \mathbf{d} | \right)(\lambda_{l}^{f} - \lambda_{l}^{d})(\lambda_{l}^{g} - \lambda_{l}^{d}) \right] \]  

(3.3)

and the imaginary part, which describes the coherent evolution,

\[ \text{Im} \mathcal{L}_{\lambda, \lambda'}(t) = \sum_{k} C_{k}(t) \cos \left( \frac{\omega_{k}}{c_{s}} | \mathbf{d} | \right)(\lambda_{l}^{f} - \lambda_{l}^{d})(\lambda_{l}^{g} - \lambda_{l}^{d}). \]  

(3.4)

Here we defined the standard spectral functions\(^{27,31}\)

![Image](353x629 to 521x734)

FIG. 1. Short-time correction to the induced exchange interaction for the Ohmic case. The arrow shows the order of the curves for increasing \( \omega_{c} | \mathbf{d} | / c_{s} = 0.0, 0.1, 0.2 \).

\[ G_{k}(t, T) = \frac{2 | g_{k} |^{2}}{\omega_{k}^{2}} \sin^{2} \left( \frac{\omega_{c} t}{2} \right) \operatorname{coth} \left( \frac{\omega_{c} t}{2 k_{B} T} \right) \]  

(3.5)

and

\[ C_{k}(t) = \frac{2 | g_{k} |^{2}}{\omega_{k}^{2}} (\omega_{c} t - \sin \omega_{c} t). \]  

(3.6)

Calculating the sums by converting them to integrals over the bath-mode frequencies \( \omega \) in Eqs. (3.3) and (3.4), assuming the Ohmic bath \( n = 1 \) for \( T > 0 \) one obtains a linear in time \( t \), large-time behavior for both the temperature-dependent real part and for the imaginary part. The coefficient for the former is \( \sim k_{B} T \), whereas for the latter it is \( \sim \omega_{c} \).

For super-Ohmic models, \( n > 1 \), the real part grows slower, as was also noted in the literature.\(^{28,31,36}\)

Let us first analyze the effect that the imaginary part of \( \mathcal{L}_{\lambda, \lambda'}(t) \) has on the evolution of the reduced density matrix, since this contribution leads to the induced interaction. If the real part were not present, i.e., omitting Eq. (3.3) from Eqs. (3.2), (3.4), and (3.6), we would obtain the evolution operator in the form \( \exp[-iH_{\text{int}}t - iF(t) t] \). The interaction \( H_{\text{int}} \) comes from the first term in Eq. (3.6),

\[ H_{\text{int}} = - \frac{2 a_{n}^{g} \Gamma(n) c_{s}^{2} \omega_{c}^{n} }{\left( c_{s}^{2} + \omega_{c}^{2} | \mathbf{d} |^{2} \right)^{n/2}} \cos \left[ n \arctan \left( \frac{\omega_{c} | \mathbf{d} |}{c_{s}} \right) \right] \sigma_{n}^{f} \sigma_{n}^{d}. \]  

(3.7)

This expression is the same as the results obtained within the Markovian scheme, cf., Eqs. (2.17) and (2.8), and Sec. V. The operator \( F(t) \) is given by

\[ F(t) = 2 \sigma_{n}^{2} \int_{0}^{\infty} \frac{D(\omega) | g(\omega) |^{2} \sin \omega t}{\omega} \cos \left( \frac{\omega | \mathbf{d} |}{c_{s}} \right). \]  

(3.8)

It commutes with \( H_{\text{int}} \) and therefore could be viewed as the initial time-dependent correction to the interaction. In fact, it describes the onset of the induced coherent interaction; note that \( F(0) = -H_{\text{int}} \), but for large times \( F(t) = a_{n}^{g} a_{n}^{d} \). In Fig. 1, we plot \( F(t) \), defined via \( F(t) = F(t) \sigma_{n}^{f} \sigma_{n}^{d} \) for the Ohmic case as a function of time for various spin-spin separations.
the one presented in Fig. 2. In the next section, additional quantities are considered, namely, the density matrix elements $P_{\uparrow\uparrow} = \langle \uparrow \downarrow | \rho_0 | \uparrow \downarrow \rangle$, etc., on the same time scale.

Let us now explore the role of the decoherence term (3.3). In the exact solution of the short-time model, the bath is thermalized only initially, while in the perturbative Markovian approximation, one assumes that the bath is reset to thermal after each infinitesimal time step. Nevertheless, the effect of the noise is expected to be qualitatively similar. Since the short-time model offers an exact solution, we will use it to compare the coherent vs noise effects in the two-spin dynamics. We evaluate the concurrence $C(\rho)$ which measures the entanglement of the spin system and is monotonically related to the entanglement of formation. For a mixed state of two qubits, $\rho_0$, we first define the spin-flipped state, $\tilde{\rho}_0 = \sigma_z \rho_0 \sigma_z$, and then the Hermitian operator $R = \sqrt{\rho_0} \tilde{\rho}_0 \sqrt{\rho_0}$, with eigenvalues $\lambda_i = 1, 2, 3, 4$. The concurrence is then given by

$$C(\rho) = \max \left\{ 0, 2 \max_i \lambda_i - \sum_{j=1}^4 \lambda_j \right\} \tag{3.9}$$

In Fig. 2, we plot the concurrence as a function of time and the spin-spin separation, for the (initially unentangled) state $| \uparrow \uparrow \rangle$, and $n = 1$. One observes decaying periodic oscillations of entanglement. We should point out that the measure of entanglement we use here—the concurrence—provides the estimate of how much entanglement can be constructed provided the worst possible scenario for decomposing the density matrix is realized; see Refs. 37 and 38 for details and definitions. Therefore, one expects the entanglement that one can make use of in quantum computing to be no smaller than the one presented in Fig. 2. In the next section, additional quantities are considered, namely, the density matrix elements, which characterize the degree of coherence in the system’s dynamics.

IV. ONSET OF THE INTERACTION AND DYNAMICS OF THE DENSITY MATRIX

Let us now investigate in greater detail the onset of the induced interaction, the time-dependence of which is given by $F(t)$. In Fig. 1 we have shown the magnitude of $F(t)$, as a function of time for various qubit-qubit separations and $n = 1$. The correction is initially nonmonotonic, but decreases for larger times as mentioned above. The behavior for other non-Ohmic regimes is initially more complicated, however the large time behavior is similar.

It may be instructive to consider the time dependent correction $H_F(t)$ to the interaction Hamiltonian during the initial evolution, corresponding to $F(t)$. Since $F(t)$ commutes with itself at different times, as well as with $H_{\text{int}}$, it generates unitary evolution according to $\exp[-i \int_0^t dt' H_F(t')]$, with $H_F(t) = d(t F(t))/dt$, therefore

$$H_F(t) = \sigma_z^1 \sigma_2^a \alpha_n \Gamma(n) [u(\omega_c |d|/c_t - \omega_c t) + u(\omega_d |d|/c_r + \omega_c t)], \tag{4.1}$$

where $u(\xi) = \cos[n \arctan(\xi)]/[1 + \xi^2]^{n/2}$. The above expression is a superposition of two waves propagating in opposite directions. In the Ohmic case, $n = 1$, the shape of the wave is simply $u(\xi) = 1/(1 + \xi^2)$. In Fig. 3, we present the amplitude of $H_F(t)$, defined via $H_F(t) = H_{F_1} \sigma_1^a \sigma_2^a$, as well as the sum of $H_{\text{int}}$ and $H_F(t)$, for $n = 1$. One can observe that the “onset wave” of considerable amplitude and of shape $u(\xi)$ propagates once between the qubits, “switching on” the interaction. It does not affect the qubits once the interaction has set in.

To understand the dynamics of the qubit system and its entanglement, let us again begin with the analysis of the coherent part in Eq. (3.2). After the interaction, $H_{\text{int}}$, has set in, it will split the system energies into two degenerate pairs $E_0 = E_1 = -H_{\text{int}}$ and $E_2 = E_3 = H_{\text{int}}$. The wave function is then $| \phi(t) \rangle = \exp(-i H_{\text{int}} t) | \phi(0) \rangle$. For the initial “up-up” state, $| \phi(0) \rangle = | \uparrow \uparrow \rangle$, it develops as $| \phi(t) \rangle = | \uparrow \uparrow \rangle \cos H_{\text{int}} t + | \downarrow \downarrow \rangle \sin H_{\text{int}} t$, where $H_{\text{int}} = H_{\text{int}} \sigma_1^a \sigma_2^a$. One can easily notice that at times $t_k = \pi/4 H_{\text{int}}, 3 \pi/4 H_{\text{int}}, \ldots$, maximally entangled (Bell) states are obtained, while at times $t_0 = 0, \pi/2 H_{\text{int}}, \pi/4 H_{\text{int}}, \ldots$, the entanglement vanishes; these special times can also be seen in Fig. 2. The coherent dynamics obtained with the Markovian assumption is the same.

However, the coherent dynamics just described is only approximate, because the bath also induces decoherence that enters via Eq. (3.3). The result for the entanglement is that the decaying envelope function is superimposed on the coherent oscillations described above. The magnitudes of the
first and subsequent peaks of the concurrence are determined only by this function. As temperature increases, the envelope decays faster resulting in lower values of the concurrence; see the inset in Fig. 2. Although in the Markovian approach, presented in Sec. II, the noise is quantitatively different, one expects qualitatively similar results for the dynamics of entanglement.

Note also that nonmonotonic behavior of the entanglement is possible only provided that the initial state is a superposition of the eigenvectors of the induced interaction with more than one eigenvalue (for pure initial states); see Eqs. (3.2)–(3.4). For example, taking the initial state $|\uparrow\downarrow\rangle$ in our case would only lead to the destruction of entanglement, i.e., to a monotonically decreasing concurrence, similar to the results of Refs. 12 and 13.

For the model that allows the exact solution, i.e., for $H_T=0$, one can notice that there is no relaxation by energy transfer between the system and bath. The exponentials in Eqs. (3.2), with (3.3), suppress only the off-diagonal matrix elements, i.e., those with $\lambda \neq \lambda'$. It happens, however, that at large times the $d$ dependence is not important in Eq. (3.3), and $\text{Re} \Delta_{\lambda \lambda'}(t \to \infty)$ vanishes for certain values of $\lambda \neq \lambda'$. In the basis of the qubit-bath interaction, $\sigma_z^x \sigma_z^y$, the limiting $t \to \infty$ density matrix for our initial state $|\uparrow\downarrow\rangle$ is

$$\rho(t \to \infty) \rightarrow \frac{1}{8} \begin{pmatrix} 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 \end{pmatrix}$$

in the basis of states $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, $|\uparrow\downarrow\rangle$, and $|\downarrow\uparrow\rangle$. The significance of such results, see also Ref. 11, is that in the model with $H_T=0$ and nonrelaxerizing bath not all the off-diagonal matrix elements are suppressed by decoherence, even though the concurrence of this mixed state is zero.

The probabilities for the spins to occupy the states $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, $|\uparrow\downarrow\rangle$, and $|\downarrow\uparrow\rangle$ are presented in Fig. 4. For the initial state $|\uparrow\downarrow\rangle$, only the diagonal and inverse-diagonal matrix elements are affected, and the system oscillates between the two states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$, as mentioned earlier in the description of the coherent dynamics, while decoherence dampens these oscillations. In addition, decoherence actually raises the other two diagonal elements to a certain level, see Eq. (4.2). The dynamics of the inverse-diagonal matrix elements is shown in Fig. 5.

V. OHMIC AND SUPER-OHMIC BATH MODELS IN GENERAL DIMENSION

Let us generalize the results of the previous sections obtained primarily for the Ohmic bath model and 1D geometry. In Sec. II, we considered the 1D case with Ohmic dissipation with the Markovian approach. In the general case, let us consider the Markovain model and, again, assume that $\Delta/\omega_c$ is small. We will also assume that the absolute square of the $m$th component of the spin-boson coupling, when multiplied by the density of states, can be modeled by $\phi_m^m \omega^m \exp(-\omega/\omega_c)$; see Eq. (2.12). The integration in

![FIG. 4. Dynamics of the occupation probabilities for the states $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, $|\uparrow\downarrow\rangle$, and $|\downarrow\uparrow\rangle$. The parameters are the same as in Fig. 2. The inset shows the structure of the reduced density matrix (the nonshaded entries are all zero).](image)

![FIG. 5. Dynamics of the inverse-diagonal matrix elements for the same system as in Fig. 4. Note that Im $\rho_{\uparrow\downarrow,\uparrow\downarrow}=0$.](image)
amplitude in Eq. (2.17) vs Eq. (2.16) for \( n_m = 1 \), also suffices
for an approximate estimate for the super-Ohmic case, as
long as \( \eta(k) \) does not dominate in \( \dot{M} \). In Fig. 6, we plot the
amplitudes of Eq. (5.1), i.e., \( \mathcal{H}_{\text{int}} = \text{Tr} [ H_m \sigma \sigma_{\epsilon m} ] / 4 \), for different values of \( n_m \).
Figure 7 shows the magnitude of the noise super-operator. Figures 6 and 7 suggest that the noise
amplitudes can be made sufficiently small with respect to the
induced interaction, at small values of \( \Delta / \omega_c \), for any \( n_m \).
We note that for quantum computing applications one usually
assumes the regime \( k_B T \ll \Delta \ll \omega_c \). The temperature dependence
in Fig. 7 becomes insignificant for \( k_B T / \Delta \ll 1 \), which is approximately to the right of \( \Delta / \omega_c \sim 0.01 \). To the left of \( \Delta / \omega_c \sim 0.01 \), by reducing the temperature one can further reduce the values of the noise amplitudes even for small \( \Delta \).

In higher dimensions the structure of \( g_{\text{eff}}^{\text{eff}} \) in the \( k \) space becomes important. Provided \( \omega_{\epsilon m} \) is nearly isotropic, the integrals entering Eqs. (2.8)–(2.11) will include (in 3D) a factor \( f_{ij}^{\text{eff}}(\omega, k | d) \), which can be written as \( f_{ij}^{\text{eff}}(\omega, k | d) \sim \sin \theta g_{\text{eff}}^{\text{eff}}(\omega, k | d) \), e.g., Eqs. (2.2) and (B3) in Appendix B. When the dependence of \( f_{ij}^{\text{eff}}(\omega, k | d) \) on \( k | d \) is negligible, the interaction is simply \( \mathcal{H}_{\text{int}} \sim \sum_{\{n_\ell \} \rightarrow \{n_\ell \}} \langle \sigma \cdot \hat{d} \rangle \mathcal{H}_{\text{int}} \), where \( a \) and \( b \) are sets of three integers representing the \( \omega \) dependence of \( f_{ij}^{\text{eff}}(\omega, k | d) \). Alternatively, a more complicated dependence on \( | d \rangle \) is expected. The noise superoperator can be treated similarly. As a result one can see that the form of the interaction depends more on the structure of \( g_{\text{eff}}^{\text{eff}} \) than on the dimensionality via the phonon density of states. However, reduced dimensionality in the \( k \) space might allow for better control over the magnitude of the interaction by external potentials that modify the spin-orbit coupling.

VI. PHONON INDUCED COHERENT SPIN-SPIN
INTERACTION VS NOISE FOR P-DONOR ELECTRONS
IN Si AND Ge

As a specific example, let us consider a model of two
localized impurity-electron spins of phosphorus donors in a
Si-Ge type semiconductor, coupled to acoustic phonon
modes by spin-orbit interaction. In what follows, we first
obtain the coupling constants \( g_{\text{eff}}^{\text{eff}} \) entering Eq. (1.3), which
define the interaction and noise amplitudes. A brief discussion
is then offered on the possibility of having an Ohmic
bath model realized in 1D channels. The rest of the present
section is devoted to calculations of the induced interaction
and noise in 3D bulk material. A comparison with the dipole-dipole
spin interaction is given.

Averaging the spin orbit Hamiltonian over the localized
electron’s wave function, one obtains the spin-orbit coupling
in the presence of magnetic field \( B = (H_x, H_y, H_z) \) in the form

\[
H_{\text{SO}} = \mu_B \sum_{m, \lambda=x,y,z} \sigma_m g_{\text{eff}}^{\text{eff}} \lambda H_\lambda,
\]

where \( \mu_B \) is the Bohr magneton. Here the tensor \( g_{\text{eff}}^{\text{eff}} \) is sensitive to lattice deformations. It was shown\(^{25} \) that for the
donor state which has tetrahedral symmetry, the Hamiltonian
(6.1) yields the spin-deformation interaction of the form

\[
H = A\mu_B [\bar{\varepsilon}_{x\delta} \sigma_x H_x + \bar{\varepsilon}_{y\delta} \sigma_y H_y + \bar{\varepsilon}_{z\delta} \sigma_z H_z + \bar{\Delta} (\sigma \cdot H) / 3 ]
+ B\mu_B [\bar{\varepsilon}_{x\delta} \sigma_x H_y + \bar{\varepsilon}_{y\delta} \sigma_y H_x + \bar{\varepsilon}_{z\delta} \sigma_z H_z + c.p.].
\]

Here c.p. denotes cyclic permutations and \( \bar{\Delta} \) is the effective
dilatation. The tensor \( \bar{\varepsilon}_{ij} \) already includes averaging of the
strain with the gradient of the potential over the donor
ground state wave function.

As before, let us assume that the separation \( d \), as well as the
magnetic field, are directed along the \( z \) axis, for an illustrative
calculation. Then the spin-deformation interaction
Hamiltonian simplifies to

\[
H = A\mu_B \bar{\varepsilon}_{z\delta} \sigma_z H_z + B\mu_B [\bar{\varepsilon}_{x\delta} \sigma_x H_y + \bar{\varepsilon}_{y\delta} \sigma_y H_x + \bar{\varepsilon}_{z\delta} \sigma_z H_z].
\]

In terms of the quantized phonon field, we have\(^{1,41} \)

\[
\bar{\varepsilon}_{ij} = \sum_{k, \xi} f(k) \sqrt{\frac{\hbar}{8\rho V \omega_{\xi k}}} (\xi_{k, i} \xi_{k, j}^* + \xi_{k, j} \xi_{k, i}^*),
\]

where in the spherical donor ground state approximation\(^{34,41} \)

\[
f(k) = \frac{1}{(1 + \alpha_B^2 k^2)^2}.
\]

Here \( \alpha_B \) is half the effective Bohr radius of the donor ground
state wave function. In an actual Si or Ge crystal, donor
states are more complicated and include corrections due to
the symmetry of the crystal matrix including the fast Bloch-function
oscillations. However, the wave function of the donor
electrons in our case is spread over several atomic
dimensions (see below). Therefore, it suffices to consider
“envelope” quantities. Thus, the spin-phonon Hamiltonian
(1.2) and (1.3) coupling constants will be taken in the form
\[ g_k^m = \frac{D_m}{(1 + a_k^2 k^2)^2} \sqrt{\frac{\hbar}{8 \rho V_{\text{ohm}}}} (\delta_k \cdot k_m + \delta_k \cdot k_n), \]  \hspace{1cm} (6.6)

where \( D_m = D_n = B \mu_B H_2 \) and \( D_n = A \mu_B H_2 \).

Let us first consider a 1D channel geometry along the \( z \) direction. This will give an example of an Ohmic bath model discussed towards the end of Sec. II. In 1D channel the boundaries\(^2\) can approximately quantize the spectrum of phonons along \( x \) and \( y \), depleting the density of states except at certain resonant values. Therefore, the low-frequency effects, including the induced coupling and quantum noise, will become effectively one dimensional, especially if the effective gap due to the confinement is of the order of \( \omega_0 \). As mentioned earlier, this frequency cutoff comes from Eq. (6.5), namely, it is due to the bound electron wave function localization. A channel of width comparable to \( \sim a_B \) will be required. This, however, may be difficult to achieve in bulk Si or Ge with the present-day technology. Other systems may offer more immediately available 1D geometries for testing similar theories, for instance, carbon nanotubes, chains of ionized atoms suspended in ion traps,\(^2\) etc. In our case, the longitudinal acoustic (LA, \( \parallel \)) phonons of the one-dimensional spectrum will account for the \( g^m_k \), component of the coupling, whereas the transverse acoustic (TA, \( \perp \)) phonons will affect only the \( x \) and \( y \) spin projections.

One can check that the contributions of the cross-products of coupling constants, \( g_k^m (g_k^m)^* \) with \( m \neq m' \), to quantities of interest vanish. The diagonal combinations are

\[ |g_k^m|^2 = \frac{A^2 \mu_B^2 H_2^2}{4 \rho V_{\text{ohm}} (1 + a_k^2 k^2)^3}, \]

\[ |g_k^m|^2 = \frac{A^2 \mu_B^2 H_2^2}{4 \rho V_{\text{ohm}} (1 + a_k^2 k^2)^3}. \]  \hspace{1cm} (6.7)

With our usual assumption for the low-frequency dispersion relations \( \omega_{x \perp} = \omega_{x \perp} = c_k \), and \( \omega_{y \perp} = \omega_{y \perp} = c_k \), the expressions (6.7) lead to the Ohmic bath model discussed at the end of Sec. II.

The shape of the frequency cutoff resulting from Eq. (6.7) is not exponential. However, to estimate the magnitude of the interaction and noise one can utilize the results obtained earlier for the 1D Ohmic bath model with exponential cutoff. The coupling constants in Eqs. (2.17) and (2.16) should, then, be taken as

\[ \alpha_i = \frac{A^2 \mu_B^2 H_2^2}{8 \rho S_c \gamma_i}, \]

\[ \alpha_i' = \alpha_i. \]  \hspace{1cm} (6.8)

where \( S \) is a cross section of the channel, and the cutoff is \( \omega_x = c_k/a_B \) for the \( z \) component, and \( \omega_y = c_k/a_B \) for the \( x \) and \( y \) components. Considering Si as an example, we arrive at an approximately adiabatic Hamiltonian \( \alpha_i' \approx \alpha_i' \), with Ohmic-type coupling. The dynamics of the concurrence, then, is qualitatively similar to the one shown on Fig. 2, with the peak entanglement \( \approx 0.4 \) as well. The coupling constant \( \alpha_i' \), however, is significantly smaller due to the weakness of the spin-orbit coupling of P-impurity electrons in Si, which results in low magnitude of the induced interaction (and the noise due to the same environment), and slightly modifies the shape of the concurrence.

In the 3D geometry, let us consider for simplicity only the LA phonon branch, \( \xi \rightarrow k \), and assume an isotropic dispersion \( \omega_{k,z} = \omega_k |k| \). The expression for the coupling constants is then

\[ g_k^m = D_m \frac{k_k m}{(1 + a_k^2 k^2)^2} \sqrt{\frac{\hbar}{2 \rho V_{z} k^3}}. \]  \hspace{1cm} (6.9)

The cross terms, with \( m \neq n \), of the correlation functions \( \langle \hat{X}_m(t) \hat{X}_n(t) \rangle \), see Eq. (A3) in Appendix A, depend on the combination \( g_k^m (g_k^m)^* \), which is always an odd function of one of the projections of the wave vector. The nondiagonal terms thus vanish, as mentioned in Appendix A.

Integrating Eqs. (2.8) and (2.9) with Eq. (6.9), see Appendix B, one can demonstrate that decoherence is dominated by the individual noise terms for each spin, with the typical amplitude

\[ \eta^\alpha_i(0) = C_B \frac{2 \pi^2}{15} \frac{b^3}{(1 + b^2)^2} \frac{\Delta}{2 k_B T}. \]  \hspace{1cm} (6.10)

where \( b = \Delta a_B/c \), and \( C_B = B^2 \mu_B^2 H_2^2/(16 \pi^2 \rho^3 a_B^3 c^2) \). The interaction amplitude \( \chi^\alpha_i(d) \) and, therefore, the induced spin-spin interaction, has inverse-square-power-law asymptotic form for the \( x \) and \( y \) spin components, with a superimposed oscillation, and inverse-fifth-power-law decay for the \( z \) spin components

\[ \begin{align*}
H_{\text{int}} &= \sum \chi^\alpha_i(d) a_{\alpha m}^i a_{\alpha m}^i \\
&\quad - 4 \pi^2 C_B \frac{2b}{(1 + b^2)^2} \frac{\sin b r}{r^2} (a_{\alpha m}^i a_{\alpha m}^i + a_{\alpha m}^i a_{\alpha m}^i) \\
&\quad + 384 \pi^2 C_A \frac{2}{r^3} a_{\alpha m}^i a_{\alpha m}^i. \end{align*} \]  \hspace{1cm} (6.11)

Here \( r = |d|/a_B \) and \( C_A = A^2 C_B / B^2 \). At small distances the interaction is regular and the amplitudes converge to constant values, see Fig. 8. The complete expressions for \( \chi^\alpha_i(d) \) and \( \eta^\alpha_i(d) \) are given in Appendix B.

In Fig. 8, we plot the amplitudes of the induced spin-spin interaction (6.11), (B4), and (B7) and noise for different values of the spin-spin separation and \( b \), for electron impurity spins in 3D Si-Ge type structures. The value of \( b \) can be controlled via the applied magnetic field, \( b = \mu_B H_g \alpha_B/a_B \). The temperature dependence of the noise is insignificant provided \( 2k_B T/\Delta \ll 1 \).

A typical value\(^2\) of the effective Bohr radius in Si for the P-donor-electron ground state wave function is \( 2a_B = 2.0 \) nm. The crystal lattice density is \( \rho = 2.3 \times 10^3 \) kg/m\(^3\), and the \( g \) factor \( g = 1.98 \). For an order-of-magnitude estimate, we take a typical value of the phonon group velocity, \( c_g = 0.93 \times 10^3 \) m/s. The spin-orbit coupling constants in Si are\(^2\) \( A^2 \approx 10^2 \) and \( B^2 \approx 10^{-4} \). The resulting interaction constants in Eqs. (6.10) and (6.11) are

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the level of noise, for P-donor electron spins in Ge. It transpires that the induced interaction can be considerable as compared to the electromagnetic spin-spin coupling. However the overall coherence-to-noise ratio is quite poor for Ge. In Si, the level of noise is lower as compared to the induced interaction. However, the overall amplitude of the induced terms compares less favorably with the electromagnetic coupling.

In conclusion, we have studied the induced indirect exchange interaction due to a bosonic bath which also introduces quantum noise. We demonstrated that it can create substantial two-spin entanglement. For an appropriate choice of the system parameters, specifically, the spin-spin separation, for low enough temperatures this entanglement can be maintained and the system can evolve approximately coherently for many cycles of its internal dynamics. For larger times, the quantum noise effects will eventually dominate and the entanglement will be erased.

Estimates for P-impurity electron spins in Si and Ge structures have demonstrated that the induced interaction can be comparable to the dipole-dipole spin interaction. One can also infer that this phonon-mediated interaction in the bulk (3D) Ge is not very effective to be used for quantum computing purposes, i.e., to entangle qubits. This is due to poor coherence to noise ratio. Therefore, the use of Si may be favored, despite the fact that it has weaker spin-phonon coupling. Indeed, the noise amplitudes for Si are significantly smaller then the induced exchange interaction, where the latter is dominated by the adiabatic term. The situation is expected to be further improved for systems with reduced dimensionality for phonon propagation.

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APPENDIX A: DERIVATION STEPS FOR THE INDUCED INTERACTION AND NOISE

An important assumption required for the validity of the Markovian approach concerns the time scale of the decay of the bath correlations introduced in Eqs. (A2) and (A3) below. By constantly resetting the bath to thermal, one implies that this time scale is significantly shorter than the dynamical system time scales of interest. The Markovian treatment, considered here, is, therefore, valid at all but very short times. The short-time dynamics, for the time scales down to order \(1/\omega_b\), requires a different approach. \(^{31-33}\) Note that one usually assumes that \(\omega_b \gg \Delta\).

Here we review some of the steps that lead from the Markovian equation for the density matrix (2.2), to the expressions for the induced interaction and quantum noise. Substituting Eqs. (1.2) and (1.3) in Eq. (2.2), one can represent the integrand, \(\text{Tr}_{j}[H_{SB}^{ij}][H_{SB}(t'-\tau)\rho_{BB}(\tau)]\rho_{BB}(\tau),\) as a summation over \(i,j,m,n\,\text{of the following expression:}
\[
\begin{align*}
\text{Tr}[X'_{m,n}(t-t')c_{\rho B}]{\sigma}'_m{\sigma}'_n(t-t')c_{\rho B}, \\
- \text{Tr}[X'_{m}(t-t')\rho B X'_{n}(t-t')c_{\rho B}]\sigma'_m c_{\rho B}, \\
- \text{Tr}[X'_{m}(t-t')\rho B X'_{m}(t-t')c_{\rho B}]\sigma'_m c_{\rho B}, \\
+ \text{Tr}[\rho B X'_{m}(t-t')X'_{m}(t-t')\sigma'_m c_{\rho B}].
\end{align*}
\]

(A1)

Here \(\sigma'_m(t) = e^{it\mathcal{H}}\sigma_c e^{-it\mathcal{H}}\). All the terms in Eq. (A1) involve the correlation functions

\[
C_{mn}[(1 - \delta_{ij})d,t] = \text{Tr}[X'_{m}(t)\rho B],
\]

(A2)

where \(X'_{m}(t) = e^{it\mathcal{H}}X_{m} e^{-it\mathcal{H}}\). The explicit expression for these functions can then be obtained from Eqs. (1.3) and (A2),

\[
C_{mn}(d,t) = \frac{V}{(2\pi)} \sum_{k} \int_{-\infty}^{\infty} dk |g^m_{k}\rangle \langle g^{n}_{k}| \left[ i \sin \omega_{k} \xi + \text{coth} \frac{\omega_{k}}{2k_BT} \cos \omega_{k} \xi \right] \cos(k \cdot d + \phi_{k,n}^m - \phi_{k,n}^m),
\]

(A3)

where \(\phi_{k,n}^m\) is a possible phase of the coupling constants \(g^m_{k}\), which is not present in most cases. The coupling constants are examined in detail in Sec. V and VI, and explicit model expressions are given. Presently we comment that in many cases the resulting matrix of the correlation functions \(C_{mn}(d,t)\) is diagonal, which simplifies calculations, as illustrated in Sec. VI.

The summation of Eq. (A1) over \(i,j,m,n\) is further simplified by noting that \(C_{mn}(d,t) = C_{nm}(d,-t)\), and writing \(\sigma'_m(t)\) explicitly as \(\sigma'_z\) for \(n = z\), and \(\sigma'_z \cos \Delta t + \frac{1}{2}[\sigma'_x, \sigma'_y]\sin \Delta t\) for \(n \neq z\). For a diagonal \(C_{nn}\), we then get the amplitude expressions

\[
\begin{align*}
\int_{-\infty}^{0} dt' \text{Im} C_{mn}(d,t')\cos(\Delta t') &= \chi_{c,m}^{n}(d), \\
\int_{-\infty}^{0} dt' \text{Im} C_{mn}(d,t')\sin(\Delta t') &= \chi_{s,m}^{n}(d), \\
\int_{-\infty}^{0} dt' \text{Re} C_{mn}(d,t')\cos(\Delta t') &= \eta_{m}^{n}(d), \\
\int_{-\infty}^{0} dt' \text{Re} C_{mn}(d,t')\sin(\Delta t') &= \eta_{s,m}^{n}(d).
\end{align*}
\]

(A4)

(A5)

(A6)

(A7)

This finally leads to Eqs. (2.4) and (2.5).

**APPENDIX B: INTERACTION AND NOISE AMPLITUDES FOR Si-Ge TYPE SPIN-ORBIT COUPLING**

In Eqs. (2.8) and (2.9), with Eq. (6.9), there is a common angular part

\[
I_{m}(kd) = 2\pi \int_{0}^{\pi} d\psi \int_{0}^{\pi} \sin \theta d\theta \frac{k^2 l^2}{k^2} \cos(kd \cos \theta),
\]

(B1)

which gives

\[
I_{z,3}(kd) = 4\pi \text{Re} \left[ \frac{12 - k^2 d^2}{k^2 d^2} e^{i\Delta kd} - 4\pi \text{Re} i \frac{5k^2 d^2 - 12}{k^2 d^2} e^{i\Delta kd} \right].
\]

(B2)

and

\[
I_{z}(kd) = 16\pi \text{Re} \left[ \frac{k^2 d^2 - 6}{k^2 d^2} e^{i\Delta kd} - 4\pi \text{Re} i \frac{k^2 d^2 - 12k^2 d^2 + 24}{k^2 d^2} e^{i\Delta kd} \right].
\]

(B3)

Here \(d = |d|\). The remaining integral in Eq. (2.8),

\[
\chi_{c,y}^{z,m}(d) = - \frac{D_{m}^2}{2\rho(2\pi)^{2}\epsilon_{s}} \int_{-\infty}^{\infty} \frac{I_{c,y}(kd)}{\epsilon_{s} (1 + a_{k}^{2})^{4}} \frac{c_{k} d k}{c_{k}^{2} - \Delta^2},
\]

(B4)

can be evaluated along a contour in the upper complex plane. The integration contour includes two simple poles, at \(k = \pm \Delta/c_{s}\), which have to be taken with weight 1/2 due to principal value integration. Also included is the pole at \(k = i/\alpha_{B}\), of order four, and a simple pole at \(k = 0\) (with weight 1/2). The latter pole is for the second term in Eq. (B2) only.

The pole at \(k = i/\alpha_{B}\) yields exponentially decaying terms \(\sim \text{exp}(-d/\alpha_{B})\). At large \(d\), the asymptotic behavior is controlled by the poles at \(k = \pm \Delta/c_{s}\),

\[
\chi_{c,y}^{z,m}(d) \xrightarrow{r \rightarrow 1} - 4\pi^2 \epsilon_{s} \frac{b_0}{(1 + b_0)^{4}} \frac{\sin br}{r^2},
\]

(B5)

where \(b = \Delta/a_{B}/c_{s}\). The complete expression can be easily obtained from Eq. (B4). One can also note that at \(d \rightarrow 0\), Eq. (B4) is

\[
\chi_{c,y}^{z,m}(d) = - 4\pi^2 \epsilon_{s} \frac{1 + 9b^2 - 9b^3 + b^2}{24(1 + b_0)^{4}} + O(r^2).
\]

(B6)

Along the same contour, the \(z\) component of Eq. (2.8),

\[
\chi_{c}^{y}(d) = - \frac{D_{m}^2}{2\rho(2\pi)^{2}\epsilon_{s}} \int_{-\infty}^{\infty} \frac{I_{c}(kd)}{\epsilon_{s} (1 + a_{k}^{2})^{4}}
\]

(B7)

has only two poles, at \(k = 0\) [order 2 for the first term, and also of order 1 or 3 for the second term in \(I_{c}(kd)\)], and at \(k = i/\alpha_{B}\) (of order 4). The pole at \(k = 0\) is to be taken with weight 1/2 and gives the 1/r asymptotic,

\[
\chi_{c}^{y}(d) \xrightarrow{r \rightarrow 1} 384\pi^2 \epsilon_{s} \frac{1}{r_5},
\]

(B8)

while for \(r \rightarrow 0\) one obtains

\[
\chi_{c}^{y}(d) = - C_{\Lambda} \frac{\pi^2}{20} + O(r^2).
\]

(B9)

Substituting Eqs. (B5) and (B8) in \(H_{\text{int}}\) i.e., in the second term in Eq. (2.4), one obtains Eq. (6.11).

By using Eqs. (B1) and (B2), for Eq. (2.9) we obtain

\[
\eta_{c}^{y}(d) = - \frac{\pi}{2} \frac{b_3^3}{(1 + b_0^3)} I_{c,y}(rb) \text{coth} \frac{\Delta}{2k_BT}.
\]

(B10)

The decoherence processes are dominated by the local noise
terms, e.g., $\eta_{v}^{y}(0)$. Noting that $I_{v,2}(0)=\frac{17}{15}$, one obtains Eq. (6.10). One also finds that $\eta_{v}^{y}(d)=0$. For low temperatures, $\Delta_{2,4,\rho} \gg 1$, one has $\coth \frac{\Delta_{2,4,\rho}}{k_{B}T} \approx 1$ and, therefore, $\chi_{\rho}^{y}(d) = \eta_{v}^{y}(d)$, see Eqs. (2.10) and (2.9). Note that the expressions involving $\chi_{\rho}^{y}(d)$ can be neglected in Eq. (2.4), since the corrections they introduce to the induced interaction have the same magnitude as the noise amplitudes. The function $\eta_{v}^{y}(d)$ is often comparable to $\eta_{v}^{y}(d)$ for Eq. (6.9) with the parameters used in Sec. VI. This amplitude is calculated numerically in Fig. 8.

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