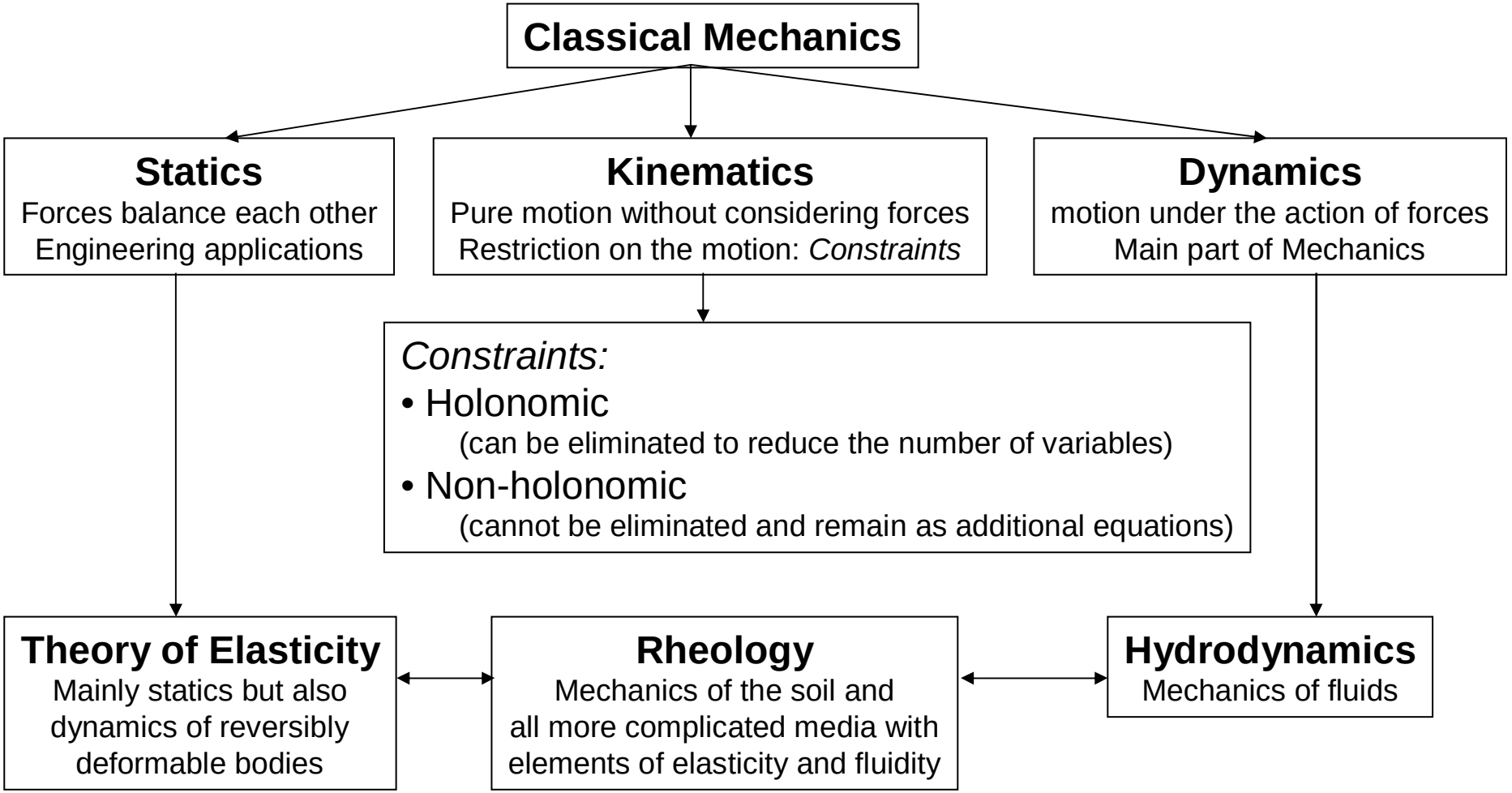
Overview of Mechanics and general comments

- **Classical Mechanics** (v<<c, macroscopic objects)
- **Relativistic Mechanics** (v~c, macroscopic objects)
- **Quantum Mechanics** (microscopic objects)



Theoretical formalisms in Classical Mechanics

1. Newtonian mechanics
   (Basic)
2. Lagrangian mechanics
   (Efficient for more complicated systems with holonomic constraints, mechanisms)
3. Hamiltonian mechanics
   (Inefficient for mechanisms, good for some special problems.
   Fundamental importance in providing a connection to Quantum Mechanics)

All three formalisms lead to the same resuts for physical quantities. For the basic system of point masses with interaction and no constraints all three formalisms lead to explicitly the same equations.

Newtonian Mechanics

Newton‘s second law for a system of *N* point masses (particles):

$$m_i \dot{\mathbf{v}}_i \equiv m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i, \qquad i = 1 \ldots N$$

With the initial conditions $\mathbf{r}_i(0)$ and $\mathbf{v}_i(0)$ specified, the system of equations above fully specifies the motion of the system.

Newtonian mechanics is most flexible since forces and constraints of different kinds can be easily added. In the Lagrangian mechanics the latter is not the case.

Conservative and non-conservative systems

If in a mechanical system all forces are potential forces,

$$\mathbf{F}_i = -\frac{\partial U(\{\mathbf{r}_i\})}{\partial \mathbf{r}_i}$$

with the potential energy U independent of time, the sum of kinetic and potential energies is conserved,

$$E = T + U = const$$

A special case is the Lorentz force acting on charged particles moving in a magnetic field that is non-potential but, nevertheless, conserves the energy.

If there are forces of other kinds than above (such as friction forces), the mechanical energy is dissipated. Such systems are called non-conservative.

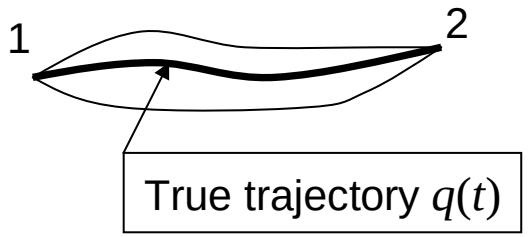
In fact, any microscopically formulated mechanical problem is conservative. Non-conservation arises as the result of reduced description (coarse graning) of large systems. For instance, motion of a body on a surface with friction can be described by the (non-conservative) friction force. This is a reduced description of the problem, the energy of the moving body is converted into the non-mechanical thermal energy. The full description would include the motion of individual atoms of both the moving body and the large body forming the surface. As the kinetic energy of the moving body decreases, the energy of the motion of individual atoms increases. In this picture, the system is conservative.

Least-action principle and Lagrangian formalism

Whereas the Newtonian mechanics is formulated in terms of N vector equations of motion, the Lagrangian formalism is based on a single generating function, the Lagrange function or Lagrangian, from which the equations of motion can be obtained. In addition, the form of these equations follows from a principle that can be chosen as the basis of mechanics, the Hamilton principle or the least action principle

$$\mathcal{S} = \int_{t_1}^{t_2} dt \mathcal{L}(q, \dot{q}, t) \Rightarrow \min \qquad q \equiv \{q_i\}$$

Action — Lagrangian — Generalized coordinates and velocities



- The true trajectory between the fixed boundary points $q(t_1) = q_1$ and $q(t_2) = q_2$ minimizes $\mathcal{S}$.

Varying $\mathcal{S}$ with respect to an arbitrary small deviation $\delta q(t)$ from the true trajectory, one obtains

$$0 = \delta\mathcal{S} = \int_{t_1}^{t_2} dt \left[\mathcal{L}(q+\delta q, \dot{q}+\delta\dot{q}, t) - \mathcal{L}(q, \dot{q}, t)\right] \cong \int_{t_1}^{t_2} dt \sum_i \left[\frac{\partial \mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta\dot{q}_i\right]$$

$$= \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta q_i \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \sum_i \left[\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right]\delta q_i \qquad \Rightarrow \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

0 — 0 — Lagrange equations

Adding a full time derivative

$$\frac{d}{dt} f(q,t) = \sum_i \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial t}$$

to $\mathcal{L}$ does not change physics since this term makes a zero contribution into $\delta\mathcal{S}$. Thus the Lagrange equations remain the same that can also be checked directly.

To make the Lagrangian formalism usable, one has to find the Lagrangian $\mathcal{L}$. Practically the best way (implicitly used by Landau and Lifschitz) is to postulate the form of $\mathcal{L}$. For systems with holonomic constraints and potential forces the Lagrangian has the form

$$\mathcal{L}(q,\dot{q},t) = T(q,\dot{q},t) - U(q,t)$$

The justification of this choice for the basic system of particles with interactions in a Descarte frame

$$\mathcal{L} = \sum_i \frac{m}{2} \mathbf{v}_i^2 - \frac{1}{2} \sum_{ij} V(|\mathbf{r}_i - \mathbf{r}_j|)$$

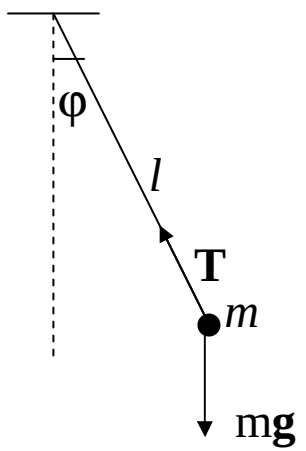
is that the resulting Lagrange equations coinside with the Newton‘s equations. In the case of holonomic constraints and non-Descarte generalized coordinates one obtains, again, correct Lagrange equations that can be derived from Newton‘s equations by changing to the generalized coordinates and resolving constraints. The latter can be done with the help of the d‘Alembert principle (e.g., Goldstein) or Lagrange multipliers (e.g., David Tong).

For systems with holonomic constraints, the Lagrangian formalism is more efficient than the Newtonian description since it directly leads to eqiations of motions with constraints already eliminated. To the contrast, within the Newton formalism one has to work on elimination of the reaction forces and resolving constraints.

Example: The point-mass pendulum

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\varphi}^2, \qquad U = -mgl\cos\varphi$$

$$\mathcal{L} = T - U = \frac{1}{2}ml^2\dot{\varphi}^2 + mgl\cos\varphi$$



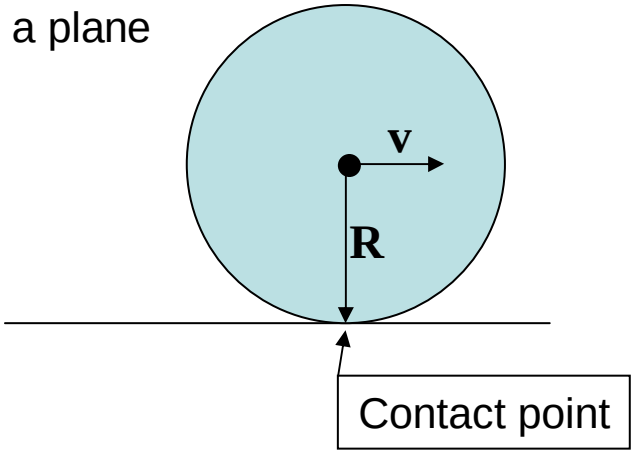
The Lagrange equation is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} - \frac{\partial\mathcal{L}}{\partial\varphi} = 0 \quad \Rightarrow \quad ml^2\ddot{\varphi} + mgl\sin\varphi = 0$$

In the Newton formalism setting up the equation of motion for the pendulum is more cumbersome as it requires elimination of the reaction (tension) force **T** by projecting on the direction perpendicular to the rod.

Non-holonomic constraints

Example: Sphere rolling on a plane



Constraint - The velocity of the contact point is zero -

$$\mathbf{v}+[\boldsymbol{\omega}\times\mathbf{R}]=0$$

cannot be integrated and eliminated.

To the contrast, for a cylinder the constraint

$$v+\omega R=0 \quad\Rightarrow\quad \dot{x}+\dot{\varphi}R=0 \quad\Rightarrow\quad x+\varphi R=const$$

is holonomic as the time derivatives can be integrated out and then one of the variables x or φ can be eliminated.

For systems with non-holonomic constraints the Lagrange formalism loses its elegance as these constrants cannot be eliminated. The general form of a non-holonomic constraint for velocities and thus for the trajectory variations is

$$\sum_i c_{\alpha i}(q)\dot{q}_i=0, \quad\Rightarrow\quad \sum_i c_{\alpha i}(q)\delta q_i=0, \qquad \alpha=1,2,...$$

The least-action principle with constraint reads

$$0=\delta\mathcal{S}'=\delta\mathcal{S}+\sum_\alpha\lambda_\alpha\int_{t_1}^{t_2}dt\sum_i c_{\alpha i}\delta q_i, \qquad \lambda_\alpha \text{ - Lagrange multipliers}$$

The full system of the Lagrange and constraint equations has the form

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}-\frac{\partial\mathcal{L}}{\partial q_i}-\sum_\alpha\lambda_\alpha c_{\alpha i}=0, \qquad \sum_i c_{\alpha i}(q)\dot{q}_i=0$$

Using the Lagrangian formalism with non-holonomic constraints only makes sense is there are both holonomic and non-holonomic constraints. In the case of one non-holonomic constraint it is better to use the Newtonian formalism, where the physical meaning of the reaction forces (the same as the constraint terms in the Lagrange equations!) is transparent.

Important special case: Charged particle in an electromagnetic field

The Newtonian equation of motion has the form

$$m\dot{\mathbf{v}} = q\mathbf{E} + \frac{q}{c}[\mathbf{v}\times\mathbf{B}] \qquad (1)$$

where the electric and magnetic fields can be expressed via the scalar and vector potentials

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \frac{\partial \varphi}{\partial \mathbf{r}}, \qquad \mathbf{B} = \mathrm{rot}\mathbf{A} \qquad (2)$$

Let us show that the Lagrangian can be written as

$$\mathcal{L} = \frac{mv^2}{2} + \frac{q}{c}\mathbf{v}\bullet\mathbf{A} - q\varphi$$

Indeed, the Lagrange equation has the form

$$0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \mathbf{v}} - \frac{\partial \mathcal{L}}{\partial \mathbf{r}} = \frac{d}{dt}\left(m\mathbf{v} + \frac{q}{c}\mathbf{A}\right) + q\frac{\partial \varphi}{\partial \mathbf{r}} - \frac{q}{c}\frac{\partial}{\partial \mathbf{r}}(\mathbf{v}\bullet\mathbf{A})$$

$$= m\dot{\mathbf{v}} + \frac{q}{c}\frac{d\mathbf{A}}{dt} + q\frac{\partial \varphi}{\partial \mathbf{r}} - \frac{q}{c}\left\{\left(\mathbf{v}\bullet\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{A} + [\mathbf{v}\times\mathrm{rot}\mathbf{A}]\right\}$$

Now using

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \left(\mathbf{v}\bullet\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{A}$$

and definitions in Eq.(2), one obtains Eq.(1). This model is not potential but still conservative.