Exercises in Classical Mechanics

CUNY GC, Prof. D. Garanin No.2 Solution

1 Rod on the axis

(10 points) Rod of length l and mass M is mounted on an axis at its center.

a) If the angle θ between the rod and the axis is fixed and the rod rotates with the angular velocity $\omega_z = \dot{\varphi}$ around the axis, what is the (i) kinetic energy of the rod; (ii) breaking torque acting from the rod on the axis?

b) Set up the Lagrange equations for the rod in the case where both θ and φ can freely change. Find integrals of motion. If you have access to mathematical software, you can try to produce numerical solutions with particular initial conditions such as $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$, $\varphi(0) = 0$, $\dot{\varphi}(0) = \omega_0$.

c) Consider the motion of this system confined to the vicinity of $\theta = \pi/2$ and try to integrate Lagrange equations analytically

Solution: a) It is convenient to choose the 3 axis along the rod, the 1 axis perpendicular to the rod and to $\mathbf{e}_z$, and the 2 axis perpendicular to 1 and 3, making the angle $\pi/2 - \theta$ with $\mathbf{e}_z$. This corresponds to $\psi = 0$. The kinetic energy of the rod with $\theta = \text{const}$ is given by

$$E = \frac{1}{2} I \omega_2^2. \tag{1}$$

Using $\omega_2 = \omega \sin\theta$ and $I = Ml^2/12$, one obtains

$$E = \frac{1}{24} M l^2 \omega_z^2 \sin^2\theta. \tag{2}$$

The angular momentum is directed along 2 and given by

$$L_2 = I\omega_2 = \frac{1}{12} M l^2 \omega_z \sin\theta. \tag{3}$$

In the laboratory frame, the vector $\mathbf{L}$ is precessing around the z axis and thus it changes with time. Consequently there should be a torque $\mathbf{K}$ acting from the vertical axis on the rod. The same but opposite torque acts from the rod on the axis and tends to break it. The torque on the rod can be found from the Newton's second law for the rotational motion

$$\dot{\mathbf{L}} = \mathbf{K}. \tag{4}$$

On the other hand, any vector that rotates with the angular velocity $\boldsymbol{\omega}$ changes with time accordng to

$$\dot{\mathbf{L}} = [\mathbf{L} \times \boldsymbol{\omega}]. \tag{5}$$

Thus one obtains for the torque

$$\mathbf{K} = [\mathbf{L} \times \boldsymbol{\omega}] \tag{6}$$

and

$$K = L_2 \omega_z \cos\theta = \frac{1}{12} M l^2 \omega_z^2 \sin\theta\cos\theta = \frac{1}{24} M l^2 \omega_z^2 \sin(2\theta). \tag{7}$$

The breaking torque K reaches its maximum at $\theta = \pi/4$.

(b) If θ can freely change, there are two components of the angular velocity

$$\omega_1 = \dot{\theta}, \qquad \omega_2 = \dot{\varphi}\sin\theta. \tag{8}$$

The kinetic energy and thus also the Lagrange function of the rod in terms of the Euler angles θ and φ becomes

$$\mathcal{L} = T = E = \frac{1}{2} I \left(\omega_1^2 + \omega_2^2 \right) = \frac{1}{2} I \left(\dot{\varphi}^2 \sin^2 \theta + \dot{\theta}^2 \right). \tag{9}$$

The Lagrange equations have the form

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0, \qquad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0. \tag{10}$$

As φ is a cyclic coordinate, $\partial \mathcal{L} / \partial \varphi = 0$, one obtains the integral of motion

$$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = I \dot{\varphi} \sin^2 \theta = L_z. \tag{11}$$

This allows to eliminate $\dot{\varphi}$ as

$$\dot{\varphi} = \frac{L_z}{I \sin^2 \theta} \tag{12}$$

and obtain the effective energy in terms of θ

$$E = \frac{1}{2} I \dot{\theta}^2 + U_{\text{eff}}(\theta), \qquad U_{\text{eff}}(\theta) \equiv \frac{L_z^2}{2I \sin^2 \theta}. \tag{13}$$

The equation of motion for θ follows from the first of Eqs. (10) of from the effective energy just above. It can be written in the form

$$I \ddot{\theta} = -\frac{\partial U_{\text{eff}}(\theta)}{\partial \theta} = \frac{L_z^2}{I} \frac{\cos \theta}{\sin^3 \theta}. \tag{14}$$

Since the energy is conserved, it follows from Eq. (13) that θ changes within a symmetric region around the equator $\theta = \pi/2$ and it cannot reach the poles $\theta = 0, \pi$. If θ becomes close to $\theta = 0, \pi$, the rod begins to rotate very fast according to Eq. (12).

c) Expanding $U_{\text{eff}}(\theta)$ near $\theta = \pi/2$ using the variable

$$\delta\theta \equiv \theta - \pi/2, \tag{15}$$

one obtains the effective energy of the form

$$E = \frac{1}{2} I \left(\delta\dot{\theta} \right)^2 + \frac{1}{2} \left. \frac{\partial^2 U_{\text{eff}}(\theta)}{\partial \theta^2} \right|_{\theta = \pi/2} (\delta\theta)^2. \tag{16}$$

Practically it is easier to write

$$U_{\text{eff}}(\delta\theta) \equiv \frac{L_z^2}{2I \cos^2(\delta\theta)} \cong \frac{L_z^2}{2I \left(1 - (\delta\theta)^2 / 2 \right)^2} \cong \frac{L_z^2}{2I} \left(1 + (\delta\theta)^2 \right), \tag{17}$$

so that

$$\left. \frac{\partial^2 U_{\text{eff}}(\theta)}{\partial \theta^2} \right|_{\theta = \pi/2} = \left. \frac{\partial^2 U_{\text{eff}}(\delta\theta)}{\partial (\delta\theta)^2} \right|_{\delta\theta = 0} = \frac{L_z^2}{I} = I \omega_z^2. \tag{18}$$

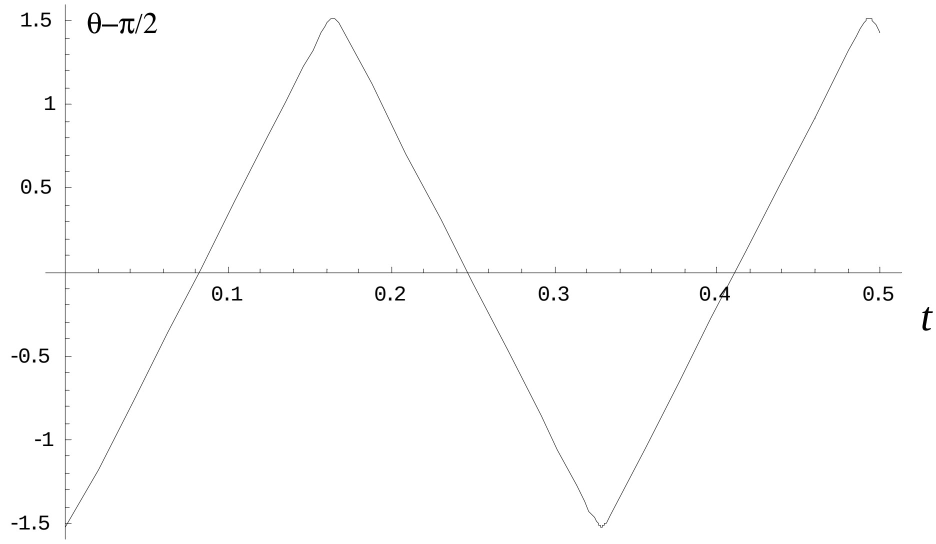
Now the effective energy can be written as

$$E = \frac{1}{2} I \left[\left(\delta\dot{\theta} \right)^2 + \omega_z^2 (\delta\theta)^2 \right]. \tag{19}$$

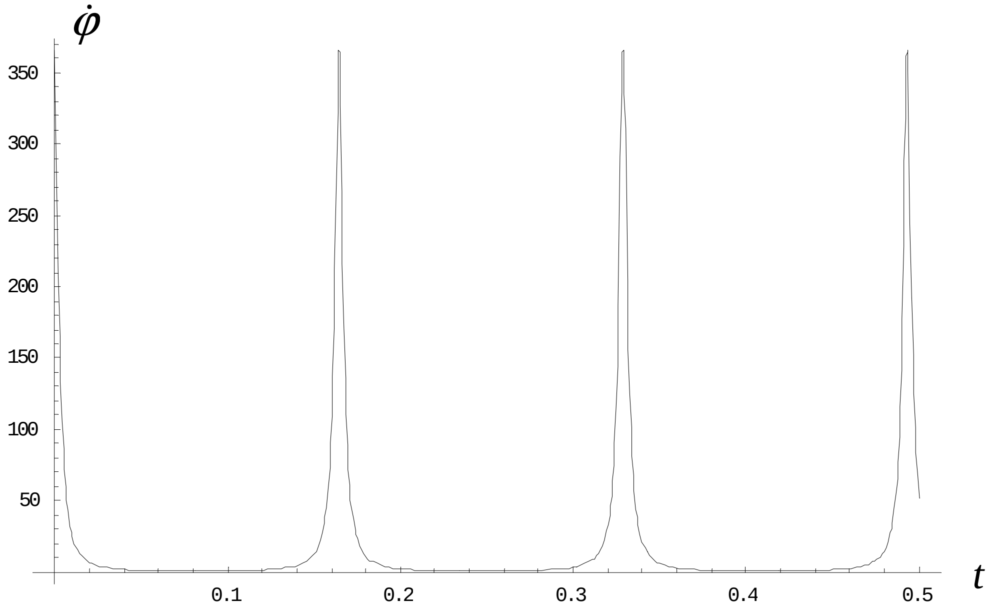
This is the energy of a harmonic oscillator with the frequency

$$\omega_0 = \omega_z. \tag{20}$$

d) Here are the results of the numerical solution of Eq. (14) for $I = 1$, $L_z = 1$, $\dot{\theta}(0) = 0$ and $\theta(0) = 3°$.



The anharmonicity of the θ oscillations is very strong, and the period is much smaller than $2\pi/\omega_0$. Apparently the time dependence $\theta(t)$ is piecewise linear, and it would be interesting to search for the analytical mechanism of this numerical finding. Below is shown the time dependence $\dot{\varphi}(t)$:



2 Symmetric top with gravity

(10 points) Consider a symmetric top with moments of inertia $I_1 = I_2 \neq I_3$ that can freely rotate around a point that is at the distance a from its center of mass. Take into account the gravity force.

a) Set up the Lagrange equations for this top, find integrals of motion;

b) Eliminate ψ and φ to obtain an effective energy for θ; Is the motion of the top with $\theta = \text{const}$ possible and what is the condition for this?

c) Consider the case of a top that very fast rotates around its 3-axis and obtain the Larmor equation for the presession of the angular momentum $\dot{\mathbf{L}} = [\mathbf{L} \times \mathbf{\Omega}]$ from the above formalism using the Euler angles.

Solution: a) The kinetic energy of a symmetric top is given by (see, e.g., Landau & Lifshitz, vol.1)

$$T = \frac{1}{2} I_1' \left(\dot{\varphi}^2 \sin^2 \theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\varphi} \cos \theta + \dot{\psi} \right)^2 , \tag{21}$$

where

$$I_1' = I_2' = I_1 + ma^2 = I_2 + ma^2. \tag{22}$$

With the potential energy

$$U = mga \cos \theta \tag{23}$$

the Lagrange function becomes

$$\mathcal{L} = T - U = \frac{1}{2} I_1' \left(\dot{\varphi}^2 \sin^2 \theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\varphi} \cos \theta + \dot{\psi} \right)^2 - mga \cos \theta. \tag{24}$$

There are two cyclic variables, φ and ψ, that results in two integrals of motion:

$$\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = p_\psi = I_3 \left(\dot{\varphi} \cos \theta + \dot{\psi} \right) = L_3 \tag{25}$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = p_\varphi = I_1' \dot{\varphi} \sin^2 \theta + I_3 \left(\dot{\varphi} \cos \theta + \dot{\psi} \right) \cos \theta = L_z. \tag{26}$$

(b) These integrals of motion can be use to eliminate $\dot{\varphi}$ and $\dot{\psi}$. Multiplying Eq. (25) by $\cos \theta$ and subtracting it from Eq. (26) one obtains

$$I_1' \dot{\varphi} \sin^2 \theta = L_z - L_3 \cos \theta \tag{27}$$

and

$$\dot{\varphi} = \frac{L_z - L_3 \cos \theta}{I_1' \sin^2 \theta}, \qquad \dot{\psi} = \frac{L_3}{I_3} - \dot{\varphi} \cos \theta = \frac{L_3}{I_3} - \frac{L_z - L_3 \cos \theta}{I_1' \sin^2 \theta} \cos \theta. \tag{28}$$

The Lagrange equation for θ has the form

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \tag{29}$$

or

$$I_1' \ddot{\theta} = I_1' \dot{\varphi}^2 \sin \theta \cos \theta - I_3 \left(\dot{\varphi} \cos \theta + \dot{\psi} \right) \sin \theta - \frac{\partial U}{\partial \theta}. \tag{30}$$

Plugging here $\dot{\varphi}$ and $\dot{\psi}$ from Eq. (28) one obtains an isolated equation of motion for θ.

The latter can be obtained in an alternative way using Newtonean mechanics. One writes the energy of the top with $\dot{\varphi}$ and $\dot{\psi}$ eliminated with the help of Eq. (28)

$$E = T + U = \frac{1}{2} I_1' \dot{\theta}^2 + U_{\text{eff}}(\theta) \tag{31}$$

where the effective potential energy is defined by

$$\begin{aligned} U_{\text{eff}}(\theta) &= \frac{1}{2} I_1' \dot{\varphi}^2 \sin^2 \theta + \frac{1}{2} I_3 \left(\dot{\varphi} \cos \theta + \dot{\psi} \right)^2 + U(\theta) \\ &= \frac{1}{2} I_1' \left(\frac{L_z - L_3 \cos \theta}{I_1' \sin^2 \theta} \right)^2 \sin^2 \theta + \frac{1}{2} I_3 \left(\frac{L_3}{I_3} \right)^2 + U(\theta) \\ &= \frac{1}{2 I_1'} \frac{(L_z - L_3 \cos \theta)^2}{\sin^2 \theta} + U(\theta) + \text{const.} \end{aligned} \tag{32}$$

The equation of motion for θ corresponding to the energy given by Eq. (31) has the form

$$I_1' \ddot{\theta} = -\frac{\partial U_{\text{eff}}}{\partial \theta}, \tag{33}$$

and it can be checked that this equation coincides with Eq. (30). The energy of the top is conserved, so there is another integral of motion

$$E = \frac{1}{2} I_1' \dot{\theta}^2 + U_{\text{eff}}(\theta) = \text{const}. \tag{34}$$

This equation can be resolved to give

$$\dot{\theta} = \pm \sqrt{\frac{2\left[E - U_{\text{eff}}(\theta)\right]}{I_1'}}. \tag{35}$$

Motion with $\theta = \text{const} = \theta_0$ can be realized if, as the initial condition, the top is set into the state with $\dot{\theta} = 0$ and θ_0 corresponding to the minimum of U_{eff}, so that it follows from Eq. (33) that $\ddot{\theta} = 0$ and this state is stable. The stationary states correspond to

$$0 = \frac{dU_{\text{eff}}(\theta)}{d\theta} = \frac{1}{I_1'} \frac{(L_z - L_3 \cos\theta)\, L_3}{\sin\theta} - \frac{1}{I_1'} \frac{(L_z - L_3 \cos\theta)^2}{\sin^3\theta} \cos\theta - mga \sin\theta. \tag{36}$$

One of the stationary states is $\theta = 0$, so that one has to set $L_z = L_3$. We are not going to analyze this state here. Stationary states with $\theta \neq 0$ satisfy the equation

$$0 = (L_z - L_3 \cos\theta)\, L_3 \sin^2\theta - (L_z - L_3 \cos\theta)^2 \cos\theta - mga I_1' \sin^4\theta. \tag{37}$$

If we require $\theta = \text{const}$, this equation defines the relation between L_3 and L_z. It is more convenient to use $\dot{\varphi}$ and $\dot{\psi}$ instead of L_3 and L_z. Using Eqs. (27) and (25), one can rewrite Eq. (37) as

$$0 = I_3 \dot{\varphi} \left(\dot{\psi} + \dot{\varphi} \cos\theta \right) - I_1' \dot{\varphi}^2 \cos\theta - mga. \tag{38}$$

This sets the relation between $\dot{\varphi}$ and $\dot{\psi}$. It is mostly convenient to require that $\dot{\varphi}$ has a given value and find the corresponding value of $\dot{\psi}$. The latter is given by

$$\dot{\psi} = -\dot{\varphi} \cos\theta + \frac{I_1' \dot{\varphi}^2 \cos\theta + mga}{I_3 \dot{\varphi}} = \left(\frac{I_1'}{I_3} - 1 \right) \dot{\varphi} \cos\theta + \frac{mga}{I_3 \dot{\varphi}}. \tag{39}$$

In the absence of gravity, $g = 0$, one obtains the relation

$$\dot{\psi} = \left(\frac{I_1'}{I_3} - 1 \right) \dot{\varphi} \cos\theta \tag{40}$$

that is similar to the well-known relation

$$\dot{\psi} = \left(\frac{I_1}{I_3} - 1 \right) \dot{\varphi} \cos\theta \tag{41}$$

for a free top that rotates around its center of mass. If the precession frequency $\dot{\varphi}$ is required to be very small, the second term dominates the rhs of Eq. (39), so that one obtains the relation

$$\dot{\psi} \cong \frac{mga}{I_3 \dot{\varphi}} \tag{42}$$

that yields large values of $\dot{\psi}$ that are independent of θ. On the other hand, if one requires that $\dot{\psi}$ corresponding to a given $\theta = \text{const}$ is very large, one obtains a small precession frequency of the top that is given by

$$\dot{\varphi} \cong \frac{mga}{I_3 \dot{\psi}}, \tag{43}$$

also independent of θ.

(c) Eq. (43) yields a solution for a top that is fast rotating around its 3-axis that corresponds to $\theta = \text{const}$. One can see that the top is slowly precessing around the z axis in this case,

$$\dot{\mathbf{e}}_3 \cong [\Omega \times \mathbf{e}_3], \qquad \Omega = \frac{mga}{I_3 \dot{\psi}} \mathbf{e}_z. \tag{44}$$

In this case, $\dot{\varphi} \ll \dot{\psi}$, the angular momentum of the top is mainly due to $\dot{\psi}$ and thus it is directed nearly along the 3-axis. Thus the vector $\mathbf{L}$ obeys the same equation as $\mathbf{e}_3$:

$$\dot{\mathbf{L}} = [\Omega \times \mathbf{L}] = -[\mathbf{L} \times \Omega] = \left[\mathbf{L} \times \frac{m\mathbf{g}a}{L}\right] = \mathbf{K}, \tag{45}$$

where the torque due to the gravity force is given by

$$\mathbf{K} = [\mathbf{r} \times \mathbf{F}] = \left[a\frac{\mathbf{L}}{L}\times m\mathbf{g}\right] = [a\mathbf{e}_3\times m\mathbf{g}]\,. \tag{46}$$

One can obtain the same results for a top fast rotating around the 3-axis without requiring that $\theta = \text{const}$. The result $\theta \cong \text{const}$ follows from the equations of motion in this case. The key observation is that here, in the effective potential energy $U_{\text{eff}}(\theta)$, the difference $L_z - L_3\cos\theta \propto \dot{\varphi}$ is much smaller than both L_z and L_3. One can choose the initial condition

$$\theta(0) = \theta_0, \qquad \dot{\theta}(0) = 0, \qquad \dot{\varphi}(0) = 0 \tag{47}$$

and see how the top will behave with time. As both L_z and L_3 are constants, one can eliminate L_z using the initial condition, $L_z = L_3\cos\theta_0$. Substituting this into Eq. (32), one obtains

$$U_{\text{eff}}(\theta) = \frac{1}{2I_1'}\frac{L_3^2(\cos\theta_0 - \cos\theta)^2}{\sin^2\theta} + U(\theta). \tag{48}$$

Since $L_3 \cong I_3\dot{\psi}$ is very large, it follows from the conservation of the energy that θ remains in a close vicinity of θ_0 for all times as θ fast oscillates around the minimum of $U_{\text{eff}}(\theta)$. One can expand $U_{\text{eff}}(\theta)$ in small deviations

$$\delta\theta \equiv \theta - \theta_0 \ll 1 \tag{49}$$

as follows:

$$\begin{aligned} U_{\text{eff}}(\delta\theta) &\cong \frac{1}{2I_1'}\frac{L_3^2\left[\cos\theta_0 - \cos(\theta_0+\delta\theta)\right]^2}{\sin^2\theta_0} + mga\cos(\theta_0+\delta\theta) \\ &\cong \frac{L_3^2}{2I_1'}(\delta\theta)^2 - mga\sin\theta_0\,\delta\theta. \end{aligned} \tag{50}$$

One can see that $U_{\text{eff}}(\delta\theta)$ is a parabolic potential well and the minimum of $U_{\text{eff}}(\delta\theta)$ follows from

$$0 = \frac{dU_{\text{eff}}(\delta\theta)}{d\delta\theta} = \frac{L_3^2}{I_1'}\delta\theta - mga\sin\theta_0. \tag{51}$$

One obtains

$$\delta\theta_{\text{min}} = \frac{mgaI_1'\sin\theta_0}{L_3^2}. \tag{52}$$

Then $U_{\text{eff}}(\delta\theta)$ above can be rewritten in the form

$$U_{\text{eff}}(\delta\theta) \cong \frac{L_3^2}{2I_1'}(\delta\theta - \delta\theta_{\text{min}})^2 + \text{const} = \frac{L_3^2}{2I_1'}(\theta - \theta_{\text{min}})^2 + \text{const}, \tag{53}$$

so that the total energy E of Eq. (34) takes the form

$$E = \frac{1}{2}I_1'\dot{\theta}^2 + \frac{L_3^2}{2I_1'}(\theta - \theta_{\text{min}})^2 = \frac{1}{2}I_1'\left[\dot{\theta}^2 + \omega_{\text{nut}}^2(\theta - \theta_{\text{min}})^2\right], \tag{54}$$

where

$$\omega_{\text{nut}} = \frac{L_3}{I_1'} \cong \frac{I_3}{I_1'}\dot{\psi} \tag{55}$$

is the frequency of nutations that is large.

Let us now find the average of $\dot{\varphi}$ over the period of nutations. From Eq. (27) follows

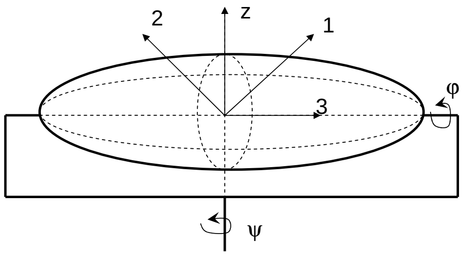
$$\dot{\varphi} = \frac{L_z - L_3\cos\theta}{I_1'\sin^2\theta} = \frac{L_3\left[\cos\theta_0 - \cos(\theta_0+\delta\theta)\right]}{I_1'\sin^2\theta} \cong \frac{L_3\sin\theta_0\,\delta\theta}{I_1'\sin^2\theta_0} = \frac{L_3\,\delta\theta}{I_1'\sin\theta_0}. \tag{56}$$

The average of $\dot{\varphi}$ is obtained by replacing $\delta\theta \Rightarrow \delta\theta_{\text{min}}$ of Eq. (52):

$$\langle\dot{\varphi}\rangle_{\text{nut}} = \frac{L_3\,\delta\theta_{\text{min}}}{I_1'\sin\theta_0} = \frac{L_3}{I_1'\sin\theta_0}\frac{mgaI_1'\sin\theta_0}{L_3^2} = \frac{mga}{L_3} = \frac{mga}{I_3\dot{\psi}} \tag{57}$$

This result coincides with that of Eq. (43) and it describes a slow precession of $\mathbf{e}_3$ and thus of $\mathbf{L}$, see Eqs. (44) – (46).

3 Asymmetric top with the $\theta = 0$ holder



(10 points) Consider an asymmetric top with moments of inertia $I_1 < I_2$ supported by a holder that allows the top to freely rotate changing its Euler angles φ and ψ while preserving $\theta = \pi/2$, see Fig. The axes of the holder cross at the center of mass of the top.

a) Set up the Lagrange equations for this top, find integrals of motion;

b) Eliminate φ to obtain an effective energy for ψ. What kinds of motion for ψ are possible? Analyze the behavior of ψ near the minimum of the effective potential energy.

c) If you have access to mathematical software, you can try to produce numerical solutions with particular initial conditions.

Solution: The potential energy of the top is zero, so that the Lagrange function is just its kinetic energy:

$$\mathcal{L} = T = E = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2 \tag{58}$$

The projections of $\boldsymbol{\omega}$ on the principal axes 1, 2, and 3 should be expressed via the Euler angles. In our particular case $\theta = \pi/2$ one has

$$\omega_1 = \dot{\varphi} \sin \psi, \qquad \omega_2 = \dot{\varphi} \cos \psi, \qquad \omega_3 = \dot{\psi}, \tag{59}$$

so that

$$\mathcal{L} = T = E = \frac{1}{2} \left(I_1 \sin^2 \psi + I_2 \cos^2 \psi \right) \dot{\varphi}^2 + \frac{1}{2} I_3 \dot{\psi}^2. \tag{60}$$

Here φ is a cyclic coordinate, so that the φ-Lagrange equation yields the integral of motion that is the projection of the angular momentum L_z:

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{\partial T}{\partial \dot{\varphi}} = L_\varphi \equiv L_z = \left(I_1 \sin^2 \psi + I_2 \cos^2 \psi \right) \dot{\varphi} = \text{const.} \tag{61}$$

The ψ-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\psi}} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \tag{62}$$

has the form

$$I_3 \ddot{\psi} = (I_1 - I_2) \sin \psi \cos \psi \, \dot{\varphi}^2 = \frac{(I_1 - I_2) L_z^2 \sin \psi \cos \psi}{\left(I_1 \sin^2 \psi + I_2 \cos^2 \psi \right)^2}, \tag{63}$$

where we have used Eq. (61). This is an autonomous equation for ψ.

Eliminating $\dot{\varphi}^2$ in Eq. (60) using

$$\dot{\varphi} = \frac{L_z}{I_1 \sin^2 \psi + I_2 \cos^2 \psi} \tag{64}$$

that follows from Eq. (61), one obtains

$$E = \frac{1}{2} I_3 \dot{\psi}^2 + \frac{1}{2} \frac{L_z^2}{I_1 \sin^2 \psi + I_2 \cos^2 \psi} = \frac{1}{2} I_3 \dot{\psi}^2 + U_{\text{eff}}(\psi), \tag{65}$$

where we have introduced the effective potential energy for ψ

$$U_{\mathrm{eff}}(\psi)=\frac{1}{2}\frac{L_z^2}{I_1\sin^2\psi+I_2\cos^2\psi}. \tag{66}$$

One can see that Eq. (63) can be written in the Newtonean form

$$I_3\ddot{\psi}=-\frac{dU_{\mathrm{eff}}(\psi)}{d\psi}. \tag{67}$$

Since $I_1<I_2$, the minimum of $U_{\mathrm{eff}}(\psi)$ corresponds to $\sin\psi=0$ (e_2 collinear with e_z) and the maximum corresponds to $\cos\psi=0$ (e_1 collinear with e_z)

$$U_{\mathrm{eff,min}}=\frac{1}{2}\frac{L_z^2}{I_2},\qquad U_{\mathrm{eff,max}}=\frac{1}{2}\frac{L_z^2}{I_1}. \tag{68}$$

Evidently $E\geq U_{\mathrm{eff,min}}$. In the range $U_{\mathrm{eff,min}}<E<U_{\mathrm{eff,max}}$ the angle ψ will be oscillating, whereas for $U_{\mathrm{eff,max}}<E$ the angle ψ will monotonically increase. The motion of ψ resembles that of the pendulum, both in the bounded and unbounded regimes.

To the contrast, the angle φ changes monotonically, according to Eq. (64). Near the maximum of $U_{\mathrm{eff}}(\psi)$ one has $L_z=I_1\dot{\varphi}$ thus $\dot{\varphi}=L_z/I_1$. This is the maximum of $\dot{\varphi}$. Near the minimum $U_{\mathrm{eff}}(\psi)$ one has $L_z=I_2\dot{\varphi}$ thus $\dot{\varphi}=L_z/I_2$ that is smaller than that near the maximum of $U_{\mathrm{eff}}(\psi)$.

Near the minimum of $U_{\mathrm{eff}}(\psi)$ the angle ψ performs small harmonic oscillations. To see this, we expand $U_{\mathrm{eff}}(\psi)$ near $\psi=0$ using

$$\sin\psi\cong\psi,\qquad\cos\psi\cong1-\frac{1}{2}\psi^2, \tag{69}$$

so that

$$\begin{aligned}U_{\mathrm{eff}}(\psi) &\cong \frac{1}{2}\frac{L_z^2}{I_1\psi^2+I_2\left(1-\psi^2\right)}=\frac{1}{2}\frac{L_z^2}{I_2-\left(I_2-I_1\right)\psi^2}\\ &= U_{\mathrm{eff,min}}\frac{1}{1-\left(1-I_1/I_2\right)\psi^2}\cong U_{\mathrm{eff,min}}+U_{\mathrm{eff,min}}\left(1-\frac{I_1}{I_2}\right)\psi^2.\end{aligned} \tag{70}$$

Now Eq. (67) can be written as

$$\ddot{\psi}+\omega_0^2\psi=0, \tag{71}$$

where ω_0 is the frequency of the ψ oscillations given by

$$\omega_0=\sqrt{\frac{1}{I_3\psi}\frac{dU_{\mathrm{eff}}(\psi)}{d\psi}}=\sqrt{\frac{1}{I_3}U_{\mathrm{eff,min}}\left(1-\frac{I_1}{I_2}\right)}. \tag{72}$$

Note that is the case of small oscillations of ψ around zero one obtains $\dot{\varphi}=\omega_z$ that is nearly constant:

$$\omega_z\cong\frac{L_z}{I_2}=\mathrm{const} \tag{73}$$

that yields

$$L_z\cong\omega_z I_2. \tag{74}$$

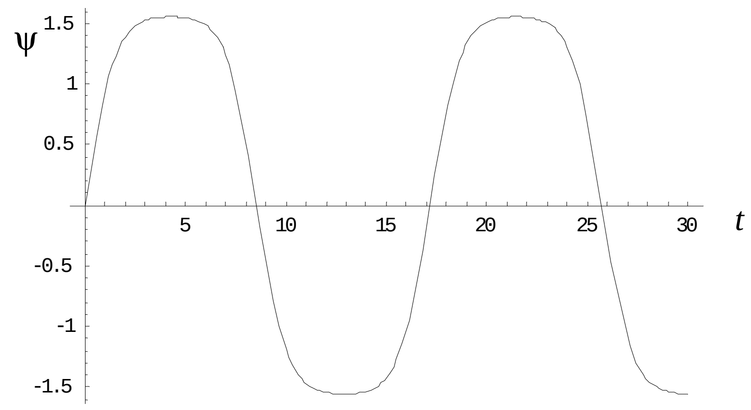
To simplify Eq. (72), one can plug it into Eq. (68) and simplify Eq. (72) to

$$\omega_0=\sqrt{\frac{1}{I_3}\frac{I_2\omega_z^2}{2}\left(1-\frac{I_1}{I_2}\right)}=\omega_z\sqrt{\frac{I_2-I_1}{I_3}}. \tag{75}$$

The general solution of Eq. (67) can be expressed through elliptic integrals. For numerical solution, the most interesting region is that of E close to $U_{\mathrm{eff,max}}$ so that the motion of ψ is strongly nonlinear, as ψ spends much time in the vicinity of $\pm\pi/2$ that correspond to $U_{\mathrm{eff,max}}$. The numerical solution for $\psi(t)$ for E slightly below $U_{\mathrm{eff,max}}$ is shown below:

$I_1 = 1, \quad I_2 = 2, \quad I_3 = 3, \quad \psi(0) = 0, \quad \dot{\psi}(0) = 1$

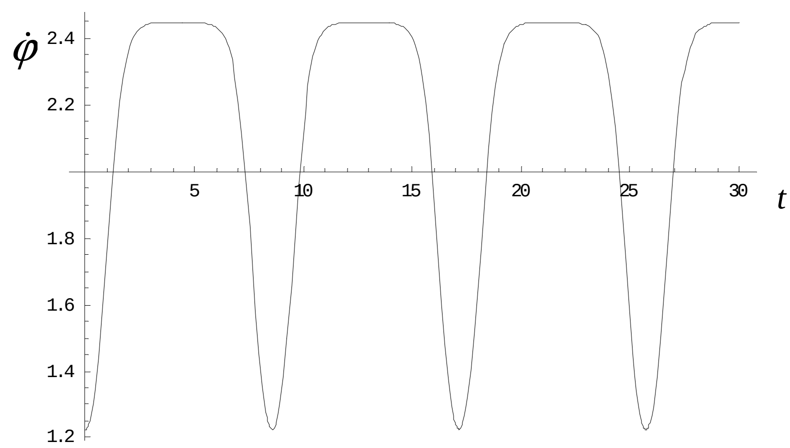
$L_z = 2.45$ so that $U_{\text{eff,min}} = 1.50063, \quad E = 3.00063, \quad U_{\text{eff,max}} = 3.00125$



Here is the corresponding time dependence of $\dot{\varphi}$:

$I_1 = 1, \quad I_2 = 2, \quad I_3 = 3, \quad \psi(0) = 0, \quad \dot{\psi}(0) = 1$

$L_z = 2.45$ so that $U_{\text{eff,min}} = 1.50063, \quad E = 3.00063, \quad U_{\text{eff,max}} = 3.00125$



As was said above, $\dot{\varphi}$ reaches its maximum when $U_{\text{eff}}(\psi)$ is close to its maximal value.