# 1 Hamiltonian formalism for the double pendulum

(10 points) Consider a double pendulum that consists of two massless rods of length $l_1$ and $l_2$ with masses $m_1$ and $m_2$ attached to their ends. The first pendulum is attached to a fixed point and can freely swing about it. The second pendulum is attached to the end of the first one and can freely swing, too. The motion of both pendulums is confined to a plane, so that it can be described in terms of their angles with respect to the vertical, $\varphi_1$ and $\varphi_2$.

a) Write down the Lagrange function for this system.

b) Introduce generalized momenta $p_1$ and $p_2$ and change to the Hamiltonian description. Find the transformation matrix that yields the velocities $\dot{\varphi}_1$ and $\dot{\varphi}_2$ in terms of the momenta $p_1$ and $p_2$. Write down the Hamilton function $\mathcal{H}(\varphi_1, p_1, \varphi_2, p_2)$ using the transformation matrix.

c) Obtain the Hamilton equations.

**Solution:** a) Both kinetic and potential energy of the system are the sums of the contributions of the first and second masses:

$$\mathcal{L} = T - U, \qquad T = T_1 + T_2, \qquad U = U_1 + U_2. \tag{1}$$

For the coordinates of the masses 1 and 2 one has

$$x_1 = l_1 \sin\varphi_1, \qquad y_1 = l_1 \cos\varphi_1 \tag{2}$$

and

$$x_2 = l_1 \sin\varphi_1 + l_2 \sin\varphi_2, \qquad y_2 = l_1 \cos\varphi_1 + l_2 \cos\varphi_2, \tag{3}$$

with the $y$-axis directed downwards. For the potential energies one has

$$U_1 = -m_1 g y_1 = -m_1 g l_1 \cos\varphi_1, \qquad U_2 = -m_2 g y_2 = -m_2 g \left(l_1 \cos\varphi_1 + l_2 \cos\varphi_2\right). \tag{4}$$

Kinetic energies are given by

$$T_1 = \frac{m_1}{2} l_1^2 \dot{\varphi}_1^2 \tag{5}$$

and

$$T_2 = \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) = \frac{m_2}{2}\left[l_1^2\dot{\varphi}_1^2 + l_2^2\dot{\varphi}_2^2 + 2 l_1 l_2 \cos\left(\varphi_1 - \varphi_2\right)\dot{\varphi}_1\dot{\varphi}_2\right]. \tag{6}$$

All together yields

$$\begin{aligned} \mathcal{L} &= \frac{m_1 + m_2}{2} l_1^2 \dot{\varphi}_1^2 + \frac{m_2}{2} l_2^2 \dot{\varphi}_2^2 + m_2 l_1 l_2 \cos\left(\varphi_1 - \varphi_2\right)\dot{\varphi}_1\dot{\varphi}_2 \\ &\quad + \left(m_1 + m_2\right) g l_1 \cos\varphi_1 + m_2 g l_2 \cos\varphi_2. \end{aligned} \tag{7}$$

b) The generalized momenta are given by

$$\begin{aligned} p_1 &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_1} = \left(m_1 + m_2\right) l_1^2 \dot{\varphi}_1 + m_2 l_1 l_2 \cos\left(\varphi_1 - \varphi_2\right)\dot{\varphi}_2 \\ p_2 &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_2} = m_2 l_2^2 \dot{\varphi}_2 + m_2 l_1 l_2 \cos\left(\varphi_1 - \varphi_2\right)\dot{\varphi}_1. \end{aligned} \tag{8}$$

This can be written in the matrix form as

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = K \begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix}, \qquad K = \begin{pmatrix} \left(m_1 + m_2\right) l_1^2 & m_2 l_1 l_2 \cos\left(\varphi_1 - \varphi_2\right) \\ m_2 l_1 l_2 \cos\left(\varphi_1 - \varphi_2\right) & m_2 l_2^2 \end{pmatrix}. \tag{9}$$

Note that the kinetic energy in Eq. (7) can be written as

$$T = \frac{1}{2}\begin{pmatrix} \dot{\varphi}_1 & \dot{\varphi}_2 \end{pmatrix} K \begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \dot{\varphi}_1 & \dot{\varphi}_2 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \frac{1}{2}\left(\dot{\varphi}_1 p_1 + \dot{\varphi}_2 p_2\right) \tag{10}$$

and that $K^T = K$. The inverse transformation reads

$$\begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} = K^{-1} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \tag{11}$$

where

$$K^{-1} = \frac{1}{m_2 l_1^2 l_2^2 \left[m_1 + m_2 - m_2 \cos^2(\varphi_1 - \varphi_2)\right]} \begin{pmatrix} m_2 l_2^2 & -m_2 l_1 l_2 \cos(\varphi_1 - \varphi_2) \\ -m_2 l_1 l_2 \cos(\varphi_1 - \varphi_2) & (m_1 + m_2) l_1^2 \end{pmatrix}, \tag{12}$$

again $\left(K^{-1}\right)^T = K^{-1}$. Using the transposed relation

$$\begin{pmatrix} \dot{\varphi}_1 & \dot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} \left(K^{-1}\right)^T = \begin{pmatrix} p_1 & p_2 \end{pmatrix} K^{-1}, \tag{13}$$

one can write the kinetic energy, Eq. (10), in the form

$$T = \frac{1}{2} \begin{pmatrix} p_1 & p_2 \end{pmatrix} K^{-1} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \tag{14}$$

Now the Hamilton function becomes,

$$\mathcal{H} = T + U = \frac{1}{2} \begin{pmatrix} p_1 & p_2 \end{pmatrix} K^{-1} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - (m_1 + m_2) g l_1 \cos\varphi_1 - m_2 g l_2 \cos\varphi_2. \tag{15}$$

In the standard form this reads

$$\mathcal{H} = \frac{m_2 l_2^2 p_1^2 + (m_1 + m_2) l_1^2 p_2^2 - 2 m_2 l_1 l_2 \cos(\varphi_1 - \varphi_2) p_1 p_2}{2 m_2 l_1^2 l_2^2 \left[m_1 + m_2 - m_2 \cos^2(\varphi_1 - \varphi_2)\right]} - (m_1 + m_2) g l_1 \cos\varphi_1 - m_2 g l_2 \cos\varphi_2. \tag{16}$$

c) The Hamilton equations read

$$\begin{aligned} \dot{\varphi}_1 &= \frac{\partial \mathcal{H}}{\partial p_1}, \qquad \dot{p}_1 = -\frac{\partial \mathcal{H}}{\partial \varphi_1} \\ \dot{\varphi}_2 &= \frac{\partial \mathcal{H}}{\partial p_2}, \qquad \dot{p}_2 = -\frac{\partial \mathcal{H}}{\partial \varphi_2}. \end{aligned} \tag{17}$$

The task to work out these equations is left to the reader. The equations for $\dot{p}_1$ and $\dot{p}_2$ are pretty cumbersome since one has to differentiate the denominator. It is best to do with a mathematical software. The whole system of Hamiltonian equations for the double pendulum is much more cumbersome than the system of Lagrange equations. The only purpose to consider the Hamilton equations here is to show that the Hamiltonian formalism is not well suited for engineering-type problems with constraints.

## 2 Canonical transformations

(10 points)

a) The canonical transformations between two sets of variables are

$$Q = \ln\left(1 + \sqrt{q}\cos p\right), \qquad P = 2\left(1 + \sqrt{q}\cos p\right)\sqrt{q}\sin p. \tag{18}$$

Show directly that this transformation is canonical. Show that

$$F_{pQ}(p, Q) = -\left(e^Q - 1\right)^2 \tan p$$

is the generating function of this transformation.

b) For what values of $\alpha$ and $\beta$ do the equations

$$Q = q^\alpha \cos(\beta p), \qquad P = q^\alpha \sin(\beta p)$$

represent a canonical transformation? What is the form of the generating function $F_{pQ}(p, Q)$ in this case?

**Solution:** a) To check whether a transformation is canonical, one can show that the fundamental Poisson brackets are invariant:

$$\{Q,P\}_{q,p} = \{Q,P\}_{Q,P} = 1. \tag{19}$$

Explicit calculation is below:

$$\begin{aligned}
\{Q,P\}_{q,p} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\
&= \frac{1}{1+\sqrt{q}\cos p}\frac{1}{2\sqrt{q}}\cos p \times 2\left[-\sqrt{q}\sin p\sqrt{q}\sin p + \left(1+\sqrt{q}\cos p\right)\sqrt{q}\cos p\right] \\
&\quad + \frac{1}{1+\sqrt{q}\cos p}\sqrt{q}\sin p \times \left[\frac{1}{\sqrt{q}}\sin p + 2\cos p\sin p\right] \\
&= \frac{1}{1+\sqrt{q}\cos p}\left[-\sqrt{q}\cos p\sin^2 p + \cos^2 p + \sqrt{q}\cos^3 p + \sin^2 p + 2\sqrt{q}\cos p\sin^2 p\right] \\
&= \frac{1}{1+\sqrt{q}\cos p}\left[1 + \sqrt{q}\cos p\sin^2 p + \sqrt{q}\cos^3 p\right] \\
&= \frac{1}{1+\sqrt{q}\cos p}\left[1 + \sqrt{q}\cos p\left(\sin^2 p + \cos^2 p\right)\right] = 1.
\end{aligned} \tag{20}$$

Now check that

$$F_{pQ}(p,Q) = -\left(e^Q - 1\right)^2 \tan p$$

is the generating function of this transformation. that is, one has to check the relations

$$q = -\frac{\partial F_{pQ}}{\partial p}, \qquad P = -\frac{\partial F_{pQ}}{\partial Q}. \tag{21}$$

One obtains

$$-\frac{\partial F_{pQ}}{\partial p} = \left(e^Q - 1\right)^2 \frac{1}{\cos^2 p}. \tag{22}$$

On the other hand, from Eq. (18) follows

$$e^Q - 1 = \sqrt{q}\cos p, \tag{23}$$

so that indeed

$$-\frac{\partial F_{pQ}}{\partial p} = q. \tag{24}$$

Further with the help of Eq. (23) one obtains

$$\begin{aligned}
-\frac{\partial F_{pQ}}{\partial Q} &= 2\left(e^Q - 1\right)e^Q \tan p = \left(2\sqrt{q}\cos p\right)\left(1 + 2\sqrt{q}\cos p\right)\tan p \\
&= 2\left(1 + 2\sqrt{q}\cos p\right)\sqrt{q}\sin p = P,
\end{aligned} \tag{25}$$

as it should be.

b) Let us calculate the Poisson brackets

$$\begin{aligned}
\{Q,P\}_{q,p} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\
&= \alpha q^{\alpha-1}\cos(\beta p) \times q^\alpha \beta\cos(\beta p) + q^\alpha\beta\sin(\beta p)\,\alpha q^{\alpha-1}\sin(\beta p) \\
&= \alpha\beta q^{2\alpha-1}\left[\cos^2(\beta p) + \sin^2(\beta p)\right] = \alpha\beta q^{2\alpha-1}.
\end{aligned} \tag{26}$$

For the transformation to be canonical, this Poisson bracket should be identically equal to 1 that requires

$$\alpha = 1/2, \qquad \beta = 2, \tag{27}$$

i.e.,

$$Q = q^{1/2}\cos(2p), \qquad P = q^{1/2}\sin(2p). \tag{28}$$

The generating function $F_{pQ}(p,Q)$ should satisfy Eq. (21). To use Eq. (21) to find $F_{pQ}(p,Q)$, one should first express $q$ and $P$ via the arguments $p,Q$. From Eq. (28) one obtains

$$q = -\frac{\partial F_{pQ}}{\partial p}, \qquad P = -\frac{\partial F_{pQ}}{\partial Q}. \tag{29}$$

$$q = \frac{Q^2}{\cos^2(2p)}, \qquad P = Q\tan(2p).$$

Now integrating the equation

$$P = Q\tan(2p) = -\frac{\partial F_{pQ}}{\partial Q} \tag{30}$$

on $Q$ one obtains

$$F_{pQ} = -\frac{Q^2}{2}\tan(2p) + f(p). \tag{31}$$

Here the integration $Q$-constant $f(p)$ can be obtained from another relation

$$q = \frac{Q^2}{\cos^2(2p)} = -\frac{\partial F_{pQ}}{\partial p} = \frac{Q^2}{\cos^2(2p)} + \frac{df(p)}{dp}. \tag{32}$$

This yields

$$\frac{df(p)}{dp} = 0 \qquad \Rightarrow \qquad f(p) = \text{const}, \tag{33}$$

an irrelevant constant that can be dropped. Thus $F_{pQ}$ is given by Eq. (31) with $f(p) = 0$.

# 3 Vortex dynamics

(10 points) Consider the equations of motions describing vortices of strength $\gamma_i$ with positions $r_i = (x_i, y_i)$ in the plane

$$\dot{x}_i = -\sum_{j\neq i}\gamma_j \frac{y_i - y_j}{|\mathbf{r}_i - \mathbf{r}_j|^2}, \qquad \dot{y}_i = +\sum_{j\neq i}\gamma_j \frac{x_i - x_j}{|\mathbf{r}_i - \mathbf{r}_j|^2}. \tag{34}$$

Consider the Hamiltonian $\mathcal{H}$ and the following Poisson brackets:

$$\mathcal{H} = -\frac{1}{2}\sum_{j\neq i}\gamma_i\gamma_j \ln|\mathbf{r}_i - \mathbf{r}_j|, \qquad \{f,g\} = \sum_i^n \frac{1}{\gamma_i}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}\right). \tag{35}$$

(a) Check that Hamilton equations

$$\dot{x}_i = \{x_i, \mathcal{H}\}, \qquad \dot{y}_i = \{y_i, \mathcal{H}\} \tag{36}$$

reproduce the equations of motions. (Check that the standard Hamilton equations can be written in this form, too)

(b) Show that the following quantities are conserved:

$$P_x = \sum_i \gamma_i y_i, \qquad P_y = -\sum \gamma_i x_i. \tag{37}$$

(c) Find the Poisson brackets $\{P_x, \mathcal{H}\}$, $\{P_y, \mathcal{H}\}$, and $\{P_x, P_y\}$, as defined above.

(d) Find the solution of the equations of motion for a system of two vortices.

**Solution:** a) The Poisson brackets yield the Hamiltonian equations

$$\begin{aligned}
\dot{x}_k &= \{x_k, \mathcal{H}\} = \sum_j \frac{1}{\gamma_j}\left(\frac{\partial x_k}{\partial x_j}\frac{\partial \mathcal{H}}{\partial y_j} - \frac{\partial x_k}{\partial y_j}\frac{\partial \mathcal{H}}{\partial x_j}\right) = \frac{1}{\gamma_k}\frac{\partial \mathcal{H}}{\partial y_k} \\
\dot{y}_k &= \{y_k, \mathcal{H}\} = \sum_j \frac{1}{\gamma_j}\left(\frac{\partial y_k}{\partial x_j}\frac{\partial \mathcal{H}}{\partial y_j} - \frac{\partial y_k}{\partial y_j}\frac{\partial \mathcal{H}}{\partial x_j}\right) = -\frac{1}{\gamma_k}\frac{\partial \mathcal{H}}{\partial x_k}.
\end{aligned} \tag{38}$$

Using Eq. (35) one obtains

$$\dot{x}_k = \frac{1}{\gamma_k}\frac{\partial \mathcal{H}}{\partial y_k} = -\frac{1}{2}\sum_{j\neq i}\frac{\gamma_i\gamma_j}{\gamma_k}\frac{\partial \ln|\mathbf{r}_i-\mathbf{r}_j|}{\partial y_k} = -\frac{1}{2}\sum_{j\neq i}\frac{\gamma_i\gamma_j}{\gamma_k}\frac{1}{|\mathbf{r}_i-\mathbf{r}_j|}\frac{\partial|\mathbf{r}_i-\mathbf{r}_j|}{\partial y_k} \tag{39}$$

and

$$\dot{y}_k = -\frac{1}{\gamma_k}\frac{\partial \mathcal{H}}{\partial x_k} = \frac{1}{2}\sum_{j\neq i}\frac{\gamma_i\gamma_j}{\gamma_k}\frac{\partial \ln|\mathbf{r}_i-\mathbf{r}_j|}{\partial x_k} = \frac{1}{2}\sum_{j\neq i}\frac{\gamma_i\gamma_j}{\gamma_k}\frac{1}{|\mathbf{r}_i-\mathbf{r}_j|}\frac{\partial|\mathbf{r}_i-\mathbf{r}_j|}{\partial x_k} \tag{40}$$

In these expressions

$$\frac{\partial|\mathbf{r}_i-\mathbf{r}_j|}{\partial y_k} = \frac{\partial\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}}{\partial y_k} = \frac{y_i-y_j}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}}\frac{\partial\,(y_i-y_j)}{\partial y_k} = \frac{y_i-y_j}{|\mathbf{r}_i-\mathbf{r}_j|}\,(\delta_{ik}-\delta_{jk}) \tag{41}$$

and

$$\frac{\partial|\mathbf{r}_i-\mathbf{r}_j|}{\partial x_k} = \frac{x_i-x_j}{|\mathbf{r}_i-\mathbf{r}_j|}\,(\delta_{ik}-\delta_{jk})\,. \tag{42}$$

Substituting these results in the equations above one obtains

$$\dot{x}_k = -\frac{1}{2}\sum_{j\neq i}\frac{\gamma_i\gamma_j}{\gamma_k}\frac{y_i-y_j}{|\mathbf{r}_i-\mathbf{r}_j|^2}\,(\delta_{ik}-\delta_{jk}) = -\frac{1}{2}\sum_{j\neq k}\gamma_j\frac{y_k-y_j}{|\mathbf{r}_k-\mathbf{r}_j|^2} + \frac{1}{2}\sum_{k\neq i}\gamma_i\frac{y_i-y_k}{|\mathbf{r}_i-\mathbf{r}_k|^2} \tag{43}$$

that is,

$$\dot{x}_k = -\sum_{i\neq k}\gamma_i\frac{y_k-y_i}{|\mathbf{r}_k-\mathbf{r}_i|^2} \tag{44}$$

and, similarly,

$$\dot{y}_k = \sum_{i\neq k}\gamma_i\frac{x_k-x_i}{|\mathbf{r}_k-\mathbf{r}_i|^2}. \tag{45}$$

These equations indeed coincide with those given in the formulation of the problem. Note that $\gamma_i$ is the strenght or vorticity of the $i$th vortex.

b) One has

$$\dot{P}_x = \{P_x,\mathcal{H}\}, \qquad \dot{P}_y = \{P_y,\mathcal{H}\}, \tag{46}$$

so that calculating Poisson brackets with $\mathcal{H}$ amounts to calculating time derivatives. This can be proven using Eq. (38). It is more direct to calculate these derivatives using the equations of motion:

$$\dot{P}_x = \sum_k \gamma_k\dot{y}_k = \sum_{i\neq k}\gamma_i\gamma_k\frac{x_k-x_i}{|\mathbf{r}_k-\mathbf{r}_i|^2} = 0. \tag{47}$$

It is easy to see that the result is zero since the summand is antisymmetric in $i$ and $k$. Similarly one can prove that $\dot{P}_y = 0$, too. Thus $P_x$ and $P_y$ are integrals of motion.

c) As argued above, $\{P_x,\mathcal{H}\} = \{P_y,\mathcal{H}\} = 0$. It remains thus to calculate

$$\begin{aligned}\{P_x,P_y\} &= \sum_k\frac{1}{\gamma_k}\left(\frac{\partial P_x}{\partial x_k}\frac{\partial P_y}{\partial y_k} - \frac{\partial P_x}{\partial y_k}\frac{\partial P_y}{\partial x_k}\right) = \sum_{ijk}\frac{\gamma_i\gamma_j}{\gamma_k}\left(-\frac{\partial y_i}{\partial x_k}\frac{\partial x_j}{\partial y_k} + \frac{\partial y_i}{\partial y_k}\frac{\partial x_j}{\partial x_k}\right) \\ &= \sum_{ijk}\frac{\gamma_i\gamma_j}{\gamma_k}\delta_{ik}\delta_{jk} = \sum_k\gamma_k.\end{aligned} \tag{48}$$

From the Jacoby identity for the Poisson brackets follows that if $P_x$ and $P_y$ are integrals of motion, then $\{P_x,P_y\}$ is also an integral of motion. In our case, however, this new integral of motion is trivial.

d) For the system of two vortices the equations of motion read

$$\begin{aligned}\dot{x}_1 &= -\gamma_2\frac{y_1-y_2}{|\mathbf{r}_1-\mathbf{r}_2|^2}, \qquad \dot{x}_2 = -\gamma_1\frac{y_2-y_1}{|\mathbf{r}_2-\mathbf{r}_1|^2} \\ \dot{y}_1 &= \gamma_2\frac{x_1-x_2}{|\mathbf{r}_1-\mathbf{r}_2|^2}, \qquad \dot{y}_2 = \gamma_1\frac{x_2-x_1}{|\mathbf{r}_2-\mathbf{r}_1|^2}.\end{aligned} \tag{49}$$

The integrals of motion of Eq. (37) for two vortices take the form

$$P_x = \gamma_1 y_1 + \gamma_2 y_2, \qquad P_y = -\gamma_1 x_1 - \gamma_2 x_2. \tag{50}$$

They can be interpreted as coordinates of the center of mass of the two vortices. The relative motion of the vortices is described by the variables

$$X \equiv x_1 - x_2, \qquad Y \equiv y_1 - y_2. \tag{51}$$

For $X$ and $Y$ one obtains the equations of motion

$$\dot{X} = -\left(\gamma_1 + \gamma_2\right) \frac{Y}{R^2}, \qquad \dot{Y} = \left(\gamma_1 + \gamma_2\right) \frac{X}{R^2} \tag{52}$$

with

$$R^2 = X^2 + Y^2. \tag{53}$$

Note that the distance between the vortices $R$ is a constant of motion:

$$\frac{dR^2}{dt} = 2X\dot{X} + 2Y\dot{Y} = -2\left(\gamma_1 + \gamma_2\right) \frac{XY}{R^2} + 2\left(\gamma_1 + \gamma_2\right) \frac{YX}{R^2} = 0. \tag{54}$$

Thus Eq. (52) can be written in the form

$$\dot{X} = -\omega_0 Y, \qquad \dot{Y} = \omega_0 X, \tag{55}$$

where

$$\omega_0 = \frac{\gamma_1 + \gamma_2}{R^2} \tag{56}$$

is an angular velocity. This is the angular velocity with which the two vortices are rotating around each other. Indeed, the solution of Eq. (55) can be represented in the form

$$X = \operatorname{Re} Z, \qquad Y = \operatorname{Im} Z, \tag{57}$$

where

$$Z = X + iY \tag{58}$$

satisfies the equation

$$\dot{Z} = i\omega_0 Z \tag{59}$$

and has the form

$$Z = Re^{i\omega_0 t + \varphi_0}. \tag{60}$$

It would be interesting to investigate the behavior of systems of more than two vortices.