3 – Standing waves, overtone series

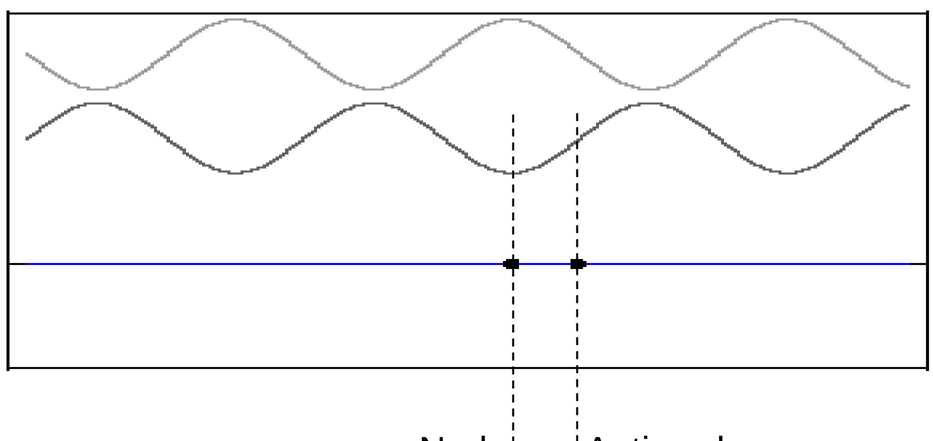
Definition of standing waves

Superposition (sum) of two plane waves with the same frequency, wave length, and amplitude but propagating in different directions forms a so-called standing wave:

$$\underbrace{Q_{\rightarrow}(x,t)+Q_{\leftarrow}(x,t)=A\sin\left(\frac{2\pi}{T}t-\frac{2\pi}{\lambda}x\right)}_{\text{Right-bound}}+\underbrace{A\sin\left(\frac{2\pi}{T}t+\frac{2\pi}{\lambda}x\right)}_{\text{Left-bound}}=\underbrace{2A\cos\left(\frac{2\pi}{\lambda}x\right)\sin\left(\frac{2\pi}{T}t\right)}_{\text{Standing}}$$

Standing wave does not travel anywhere. There are oscillations in time and space, independently. For some values of x the wave quantity W (pressure, media displacement, etc.) turns to zero at all times because of the cos-factor. Such points are called <u>nodal points</u>. The points where cos reaches ±1 correspond to strongest oscillations in time. Such points are called <u>anti-nodal points</u>.

(Animation by Dan Russell)



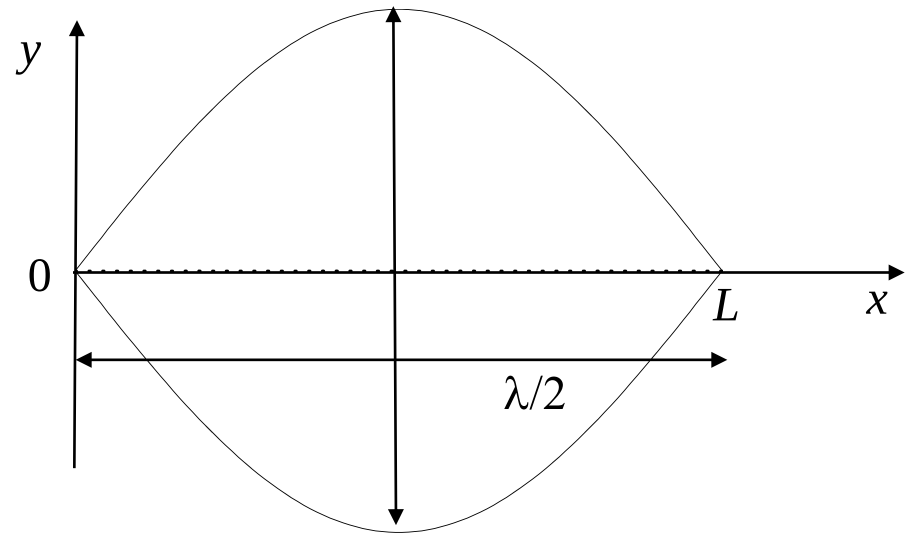
If the frequences or amplitudes of the two travelling waves differ a little, their sum is a slowly moving but fast oscillating wave.

Role of the boundary conditions in formation of standing waves

Superposition of two waves propagating in different directions and having the same amplitude is a practically important case. In many cases, for instance, in music instruments, waves are confined within a closed space and subject to boundary conditions. Boundaries can be fixation points of strings in piano or guitar, close or open ends of organ pipes, etc. Physical conditions at the boundaries allow existence of standing waves with particular wave lengths λ_n, $n=1,2,\ldots$, that satisfy these boundary conditions.

Case 1: Nodes at both ends (node-node)

Guitar or piano string fixed at $x=0$ and $x=L$:



This is the fundamental mode of standing waves in the string with the fundamental wave length

$$\lambda_1 = 2L$$

The wave length of the fundamental mode is maximal of all allowed modes (see below). The corresponding fundamental frequency

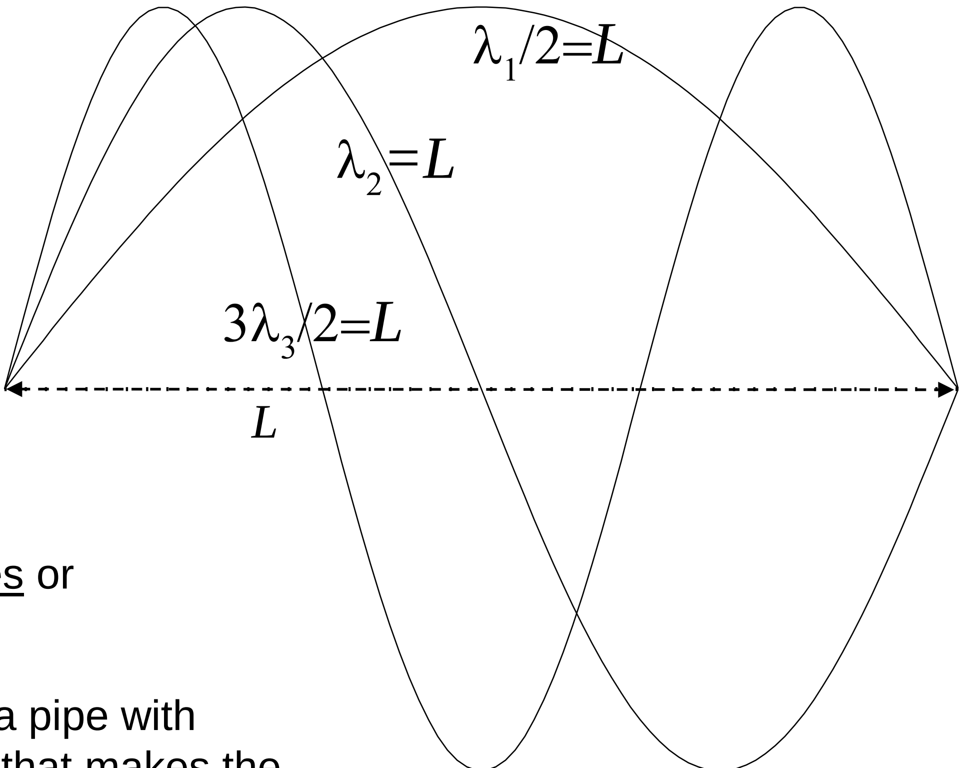
$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$$

is the minimal frequency of all allowed modes.

First (fundamental), second, and third modes in the node-node case:

The wave lengths of all allowed modes are

$$\lambda_n = \frac{\lambda_1}{n}, \quad n = 1,2,3,...$$



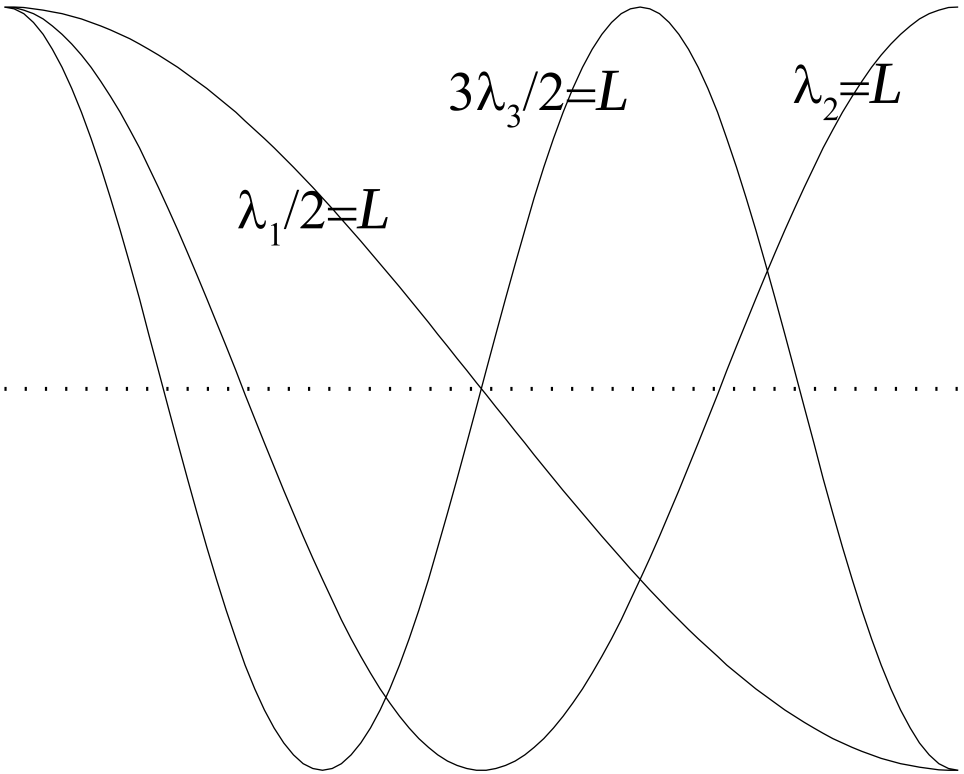
Correspondingly, all allowed frequencies are multiples of the fundamental frequency f_1:

$$f_n = \frac{v}{\lambda_n} = \frac{nv}{\lambda_1} = nf_1, \quad n = 1,2,3,...$$

Similar is realized in a metal rod with both ends clamped. Modes with n=2,3,.. are called overtones or harmonics.

Yet another realization of the node-node case is a pipe with both end open. In this case the physical quantity that makes the wave motion is the pressure of the air P. Inside the pipe there are standing sound waves. The pressure due to these sound waves should (approximately!) coincide with the constant athmospheric pressure outside the pipe. That is, the excessive pressure due to the sound wave should turn to zero at the ends of the pipe. This is completely similar to the case of a string with both ends fixed, only the physical quantities describing waves are different in the two cases. Thus for a pipe with both ends open, the allowed wave lengths and frequencies are the same as for a string with both ends fixed.

Case 2: Anti-nodes at both ends (antinode-antinode)

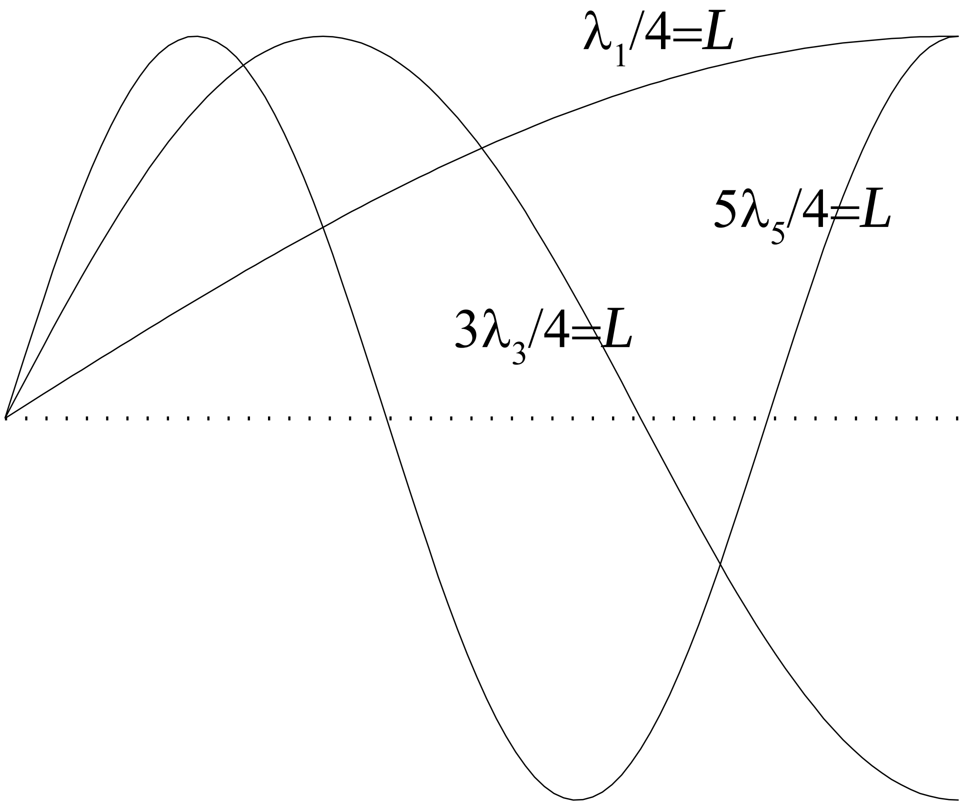


For a string such a case would be unrealistic but it is realized in a metal rod with both free (unclamped) ends and in an organ pipe with both ends closed. In the latter case, the media displacement (the displacement of the air particles) should have nodes at the ends of the pipe. Then from the relation between the media displacement and the pressure in a longitudinal wave follows that the pressure should have antinodes at the boundaries (see book). In the antinode-antinode case, the allowed wave lengths and frequencies have exactly the same form as in the node-node case:

$$\lambda_n = \frac{\lambda_1}{n}, \quad n = 1,2,3,\ldots \qquad \lambda_1 = 2L$$

$$f_n = \frac{v}{\lambda_n} = \frac{nv}{\lambda_1} = nf_1, \quad n = 1,2,3,\ldots \qquad f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$$

Case 3: Node at one end and antinode at the other end (node-antinode)



A practically important case is that of node at one end (say, left end) and antinode at the other end (say, right end). One example is a metallic rod clamped at its left end and free at its right end (such as one side of a tuning fork). Another example is an organ pipe with the left side open and the right side closed. In the node-antinode case, the fundamental wave length is two times larger and the fundamental frequency is two times smaller than that in the both previous cases:

$$\lambda_1 = 4L \qquad f_1 = \frac{v}{\lambda_1} = \frac{v}{4L}$$

In the node-antinode case only the <u>odd</u> overtones with n=3,5,7,… are allowed, whereas the even overtones with n=2,4,6,… do not occur. Thus, several lowest allowed overtones are

$$f_3 = 3f_1, \qquad f_5 = 5f_1, \qquad f_7 = 7f_1, \quad \text{etc.}$$

Air tubes

Open end:
Node for pressure (approximate),
antinode for air displacement

Closed end:
Antinode for pressure (exact),
Node for air displacement

Closed-closed ends (flute): $f_1 = \frac{v}{2L}$, even and odd overtones

Open-closed ends (clarinet etc.): $f_1 = \frac{v}{4L}$, only odd overtones

Phase relations between the incident and reflected waves

After analyzing the role of boundary conditions in the formation of standing waves, we can recall that any standing wave is a superposition of a travelling wave incident on a boundary and the travelling wave reflected from the boundary. If there is a node at the boundary, this means that the incident and reflected waves, just at the boundary, have phases differing by 180° (the reflected wave is antiphase to the incident wave). On the contrary, if there is an antinode at the boundary, this means that the incident and reflected waves are in phase just at the boundary or that the wave reflects without a phase change.

Mersenne's law for a string

Considerations show (see book) that in a string attached at both ends there is a special kind of transverse waves with the speed

$$v = \sqrt{\frac{F}{W}},$$

where F is the tension force in the string, measured in Newton, and W is the mass per unit length, in kg/m. (One can check that the unit of the speed above is just m/s, as it should be.) It should be stressed that these waves have nothing to do with transverse elastic waves in solids whose speed is much higher. The result above is independent of the elastic properties of the material. That is, this is not the elastic force that plays the role of the restoring force here. Here the restoring force arises from the tension of the string F. Now, according to the general result above, the fundamental frequency of the string is given by

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L} = \frac{1}{2L}\sqrt{\frac{F}{W}}$$

This the so-called Mersenne‘s law in ist general form. Its particular consequences, also referred to as Mersenne‘s laws, are

$$f_1 \propto \frac{1}{L}, \qquad f_1 \propto \sqrt{F}, \qquad f_1 \propto \frac{1}{\sqrt{W}}$$