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POSITIVE BIORTHOGONAL CURVATURE IN DIMENSION 4

1. $\sec > 0$ AND $\sec^\perp > 0$
2. METRICS ON $S^2 \times S^2$
3. CLASSIFICATION RESULT

1. $\sec > 0$ AND $\sec^\perp > 0$

$\boxed{\text{HOPF PROBLEM. DOES } S^2 \times S^2 \text{ ADMIT A METRIC W/ } \sec > 0 \text{?}}$

CONJECTURALLY, IF M^4 HAS $\sec > 0$, $\pi_1 M = 0$, THEN $M = S^4$ OR $M = \mathbb{C}P^2$.

TRUE, UNDER SYMMETRY HYPOTHESIS (CONTINUOUS ISOMETRY GROUP) [HSIANG-KLEINER]

WEAKER CURVATURE POSITIVITY CONDITION: ON (M^4, g),

$\forall \sigma \in Gr_2 T_pM$, $\exists! \sigma^\perp \in Gr_2 T_pM$, $T_pM = \sigma \oplus \sigma^\perp$

g-ORTHOGONAL DIRECT SUM.

$$\sec^\perp(\sigma) = \frac{1}{2}\left(\sec(\sigma) + \sec(\sigma^\perp)\right)$$ "BIORTHOGONAL CURVATURE"

NOTE: $\sec > 0 \implies \sec^\perp > 0 \implies \mathrm{scal} > 0$

THM (B. 2015). LET M^4 BE A SMOOTHABLE CLOSED SIMPLY-CONNECTED TOPOLOGICAL 4-MANIFOLD. UP TO ENDOWING M WITH DIFFERENT SMOOTH STRUCTURES, THE FOLLOWING ARE EQUIVALENT:

(i) M ADMITS METRICS WITH $\sec^{\perp} > 0$

(ii) M ADMITS METRICS WITH $\mathrm{Ric} > 0$

(iii) M ADMITS METRICS WITH $\mathrm{scal} > 0$.

(i) $\Rightarrow$ (iii) OBVIOUS

(ii) $\Leftrightarrow$ (iii) SHA-YANG 1990, 1991

$\star$ (iii) $\Rightarrow$ (i) CONSTRUCT METRICS W/ $\sec^{\perp} > 0$ ON 4-MANIFOLDS WITH $\mathrm{scal} > 0$.

2. Metrics on $S^2 \times S^2$

$$\Big(S^2 \times S^2, \underbrace{g_{\mathrm{round}} \oplus g_{\mathrm{round}}}_{g_0}\Big) \quad \text{STANDARD PRODUCT METRIC}$$

$$\sec_{g_0}(X \wedge Y) = |X_1 \wedge Y_1|^2 + |X_2 \wedge Y_2|^2 \geq 0$$

$= 0 \Leftrightarrow X \wedge Y$ IS A MIXED PLANE, I.E.

$$\begin{cases} X_1 = 0 \\ Y_2 = 0 \end{cases} \text{ OR } \begin{cases} X_2 = 0 \\ Y_1 = 0. \end{cases}$$

$$\sec_{g_0} : Gr_2 \mathbb{R}^4 \cong S^2 \times S^2 \longrightarrow \mathbb{R}$$

NOTE: $Gr_2 \mathbb{R}^4 = \dfrac{SO(4)}{SO(2) \times SO(2)} \cong S^2 \times S^2$

$\Lambda^2 \mathbb{R}^4 = \Lambda^2_+ \mathbb{R}^4 \oplus \Lambda^2_- \mathbb{R}^4$ EIGENSPACES OF $*$

$\sigma \in Gr_2 \mathbb{R}^4 \subset \Lambda^2 \mathbb{R}^4 \iff \sigma \wedge \sigma = 0$

$\sigma = \sigma_+ + \sigma_- \Rightarrow 0 = \sigma \wedge \sigma = \left(|\sigma_+|^2 - |\sigma_-|^2 \right) \mathrm{vol}$

So $Gr_2 \mathbb{R}^4 = \left\{ (\sigma_+, \sigma_-) \in \Lambda^2_+ \mathbb{R}^4 \oplus \Lambda^2_- \mathbb{R}^4 ; |\sigma_+|^2 = |\sigma_-|^2 = \frac{1}{2} \right\}$

$\sec_{g_0}(\sigma) = \frac{1}{2} \langle \sigma_+ , \omega_+ \rangle^2 + \frac{1}{2} \langle \sigma_- , \omega_- \rangle^2$

$\omega_\pm = \mathrm{vol}_1 \pm \mathrm{vol}_2 \in \Lambda^2_\pm \mathbb{R}^4$.

$\sec_{g_0}^{-1}(0) = \underbrace{S^1 \times S^1}_{\text{2-TORUS OF MIXED PLANES}} \subset S^2 \times S^2 = Gr_2 \mathbb{R}^4$.

DEFORMATIONS?

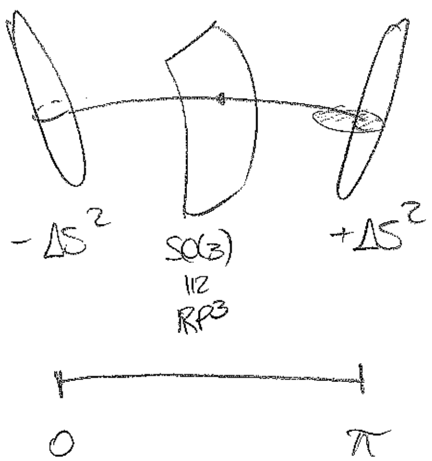
LOCAL HOPF PROBLEM: ARE THERE SEQUENCES OF METRICS ON $S^2 \times S^2$ WITH $\sec > 0$ THAT ACCUMULATE ON g_0?

THM (B. 2014). THERE ARE METRICS ON $S^2 \times S^2$ WITH $\sec^\perp > 0$, ARBITRARILY CLOSE TO g_0.

SKETCH OF PROOF:

STEP 1: CHEEGER DEFORMATION

$$SO(3) \curvearrowright S^2 \times S^2 \subset \mathbb{R}^3 \oplus \mathbb{R}^3 \qquad A(p,q) = (Ap, Aq)$$



COHOMOGENEITY 1 ACTION

$$g_t = g|_{\mathcal{H}} \oplus \frac{1}{1+t}\, g|_{\mathcal{V}}$$

$g|_{\mathcal{H}}$: 1-dim

$g|_{\mathcal{V}}$: 3-dim at (p,q), $q \neq \pm p$; 2-dim at $(p, \pm p)$.

(MÜTER '87). $\forall t > 0$, $\sec_{g_t} \geq 0$ AND

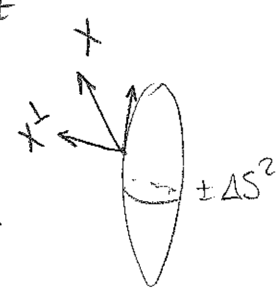
- $(p,q) \notin \pm\Delta S^2$: $\exists!\, \sigma \in Gr_2 T_{(p,q)}(S^2 \times S^2)$, $\sec_{g_t}(\sigma) = 0$
- $(p,q) \in \pm\Delta S^2$; $\sec_{g_t}^{-1}(0) = S^1 \subset Gr_2 T_{(p,\pm p)}(S^2 \times S^2)$

S^1: $\boxed{\text{"CIRCLE-WORTH" OF FLAT PLANES}}$ ⊛

NOTE: $\sec^{\perp} > 0$ ON OPEN DENSE SET $S^2 \times S^2 \setminus (\pm\Delta S^2)$!

STEP 2: LOCAL CONFORMAL CHANGE: $g_{s,t} = (1 + s\phi)\, g_t$

$$\phi = \underset{\substack{\uparrow \\ \text{CUTOFF}}}{\chi} \cdot \operatorname{dist}(\cdot, \pm\Delta S^2)^2$$



1st VARIATION ARGUMENT: $\operatorname{Hess}\phi\,(X,X) = 2|X^{\perp}|^2$

CRUCIAL FACT: FLAT PLANES ⊛ NEVER TANGENT TO $\pm\Delta S^2$ □

3. CLASSIFICATION RESULT

DONALDSON-FREEDMAN:

$$M^4 \underset{\text{homeo}}{\cong} N^4 \iff Q_M \cong Q_N$$

INTERSECTION FORM

$Q_M : H_2(M,\mathbb{Z}) \times H_2(M,\mathbb{Z}) \longrightarrow \mathbb{Z}$

$Q_M(\alpha,\beta) = \langle \alpha \cup \beta, [M] \rangle$

UNIMODULAR SYMMETRIC BILINEAR FORM

UPSHOT: IF M^4 IS A SMOOTHABLE CLOSED SIMPLY-CONNECTED TOPOLOGICAL 4-MANIFOLD, THEN IT IS HOMEOMORPHIC TO ONE OF:

$$S^4, \quad \underbrace{\#^k \mathbb{CP}^2 \#^\ell \overline{\mathbb{CP}}^2}_{\substack{\text{NON SPIN} \\ \text{sign} = k-\ell}}, \quad \underbrace{\#^k S^2 \times S^2 \#^\ell M_{E_8}}_{\substack{\text{SPIN} \\ \text{sign} = \pm 8\ell}}$$

NOT ALL OF THESE ARE SMOOTH

RECALL: $S^2 \times S^2 \# \mathbb{CP}^2 \cong \#^2 \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$

THM (LICHNEROWICZ). M^4 SPIN, $\mathrm{scal} > 0 \Rightarrow \hat{A}(M) = 0$.

THM (HIRZEBRUCH). $\hat{A}(M) = -\frac{1}{8} \operatorname{sign} Q_M$.

ALTOGETHER,

THM: IF M^4 ADMITS A METRIC WITH $\mathrm{scal} > 0$, THEN IT IS HOMEOMORPHIC TO ONE OF: S^4, $\#^k \mathbb{CP}^2 \#^\ell \overline{\mathbb{CP}}^2$, OR $\#^k S^2 \times S^2$

ALL OF THESE ARE KNOWN TO CARRY METRICS W/ $\mathrm{scal} > 0$.

"BUILDING BLOCKS" FOR $\sec^\perp > 0$:

- S^4 ✓
- $\mathbb{CP}^2$ ✓
- $S^2 \times S^2$ ✓

PROP: IF M^4 AND N^4 ADMIT METRICS WITH $\sec^{\perp}>0$, THEN SO DOES $M\# N$.

SKETCH OF PROOF: BY HOELZEL'S SURGERY STABILITY CRITERION [2013], SUFFICES TO SHOW THAT THE CONE OF CURVATURE OPERATORS WITH $\sec^{\perp}>0$,

$$C_{\sec^{\perp}>0}=\left\{R:\Lambda^2\mathbb{R}^4\to\Lambda^2\mathbb{R}^4:\ \langle R(\sigma),\sigma\rangle+\langle R(\sigma^{\perp}),\sigma^{\perp}\rangle>0\ \ \forall\sigma\in Gr_2\mathbb{R}^4\right\}$$

IS AN OPEN $O(4)$-INVARIANT CONVEX CONE S.T.

$R_{S^3\times\mathbb{R}}\in C_{\sec^{\perp}>0}$. INDEED, IF $\langle R_{S^3\times\mathbb{R}}(\sigma),\sigma\rangle=0$, THEN $\sigma^{\perp}\subset T_pS^3$, SO $\langle R_{S^3\times\mathbb{R}}(\sigma^{\perp}),\sigma^{\perp}\rangle=1$, SO $R_{S^3\times\mathbb{R}}\in C_{\sec^{\perp}>0}$. $\square$

REMARKS: ① ABOVE METRICS CAN BE CONSTRUCTED S^1-INVARIANT.

② NO CLASSIFICATION UP TO DIFFEOMORPHISMS, $\exists M^4$ WITH SMOOTH STRUCTURES THAT ADMIT AND DO NOT ADMIT $scal>0$, AND SEIBERG-WITTEN INVARIANTS VANISH IF $\sec^{\perp}>0$.

BONUS: USING DIFFERENT TECHNIQUES (STRONGLY POSITIVE CURVATURE)

THM (B.-MENDES 2015). IF $S^2\times S^2$ ADMITS A METRIC g WITH $\sec>0$, THEN THERE EXISTS A SMOOTH 2-SIDED HYPERSURFACE $N\subset S^2\times S^2$ SUCH THAT $R|_N$ IS POSITIVE-DEFINITE.

THM (B.-MENDES 2015). IF (M^4,g) HAS $\sec>0$, $\pi_1M=0$, AND $\forall p$, $R_p:\Lambda^2T_pM\to\Lambda^2T_pM$ IS NOT POS.-DEF., THEN $M\cong\#^k\mathbb{CP}^2$, $k\in\mathbb{Z}$.