BIFURCATION THEORY IN GEOMETRY

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BIFURCATION:

- BUCKLING OF RODS (EULER)
  (FAILURE OF STRUCTURES DUE TO COMPRESSIVE STRESS)
- TAYLOR VORTICES
  (TURBULENCE IN FLUID MECHANICS)
- ONSET OF OSCILLATIONS IN ELECTRIC CIRCUITS.

POINCARÈ (1885): TOPOLOGICAL CHANGE IN THE STRUCTURE OF A DYNAMICAL SYSTEM WHEN A PARAMETER CROSSES A "BIFURCATION" VALUE

TODAY:
- VARIATIONAL BIFURCATION
- APPLICATIONS TO YAMABE PROBLEM

VARIATIONAL BIFURCATION:

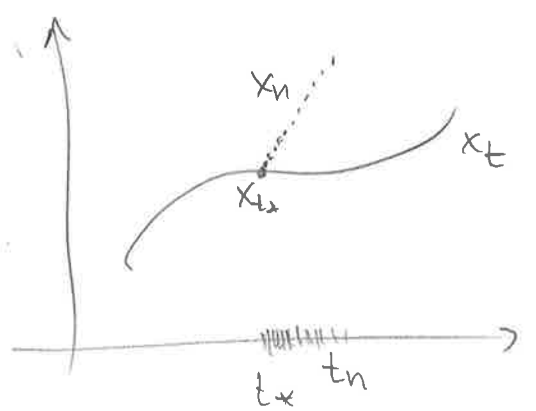
$f_t : X \to \mathbb{R}$ 1-PARAMETER FAMILY OF FUNCTIONALS

$x_t \in X$ "TRIVIAL" BRANCH OF SOLUTIONS: $df_t(x_t) = 0$.

DEF: BIFURCATION OCCURS AT t_* IF $\exists\, t_n \to t_*$, $\exists\, x_n \to x_{t_*}$ S.T.

- $df_{t_n}(x_n) = 0$
- $x_n \neq x_{t_n}$

I.E., THE IMPLICIT FUNCTION THEOREM FAILS AT x_{t_*}.



- "NEW" SOLUTIONS x_n TYPICALLY HAVE FEWER SYMMETRIES ("SYMMETRY-BREAKING BIFURCATION")
- DEGENERACY OF x_{t_*} $\left(\ker d^2 f_{t_*}(x_{t_*}) \neq \{0\}\right)$ IS NECESSARY FOR BIFURCATION, BUT NOT SUFFICIENT.

THM (KRASNOSEL'SKII). ASSUME x_t IS A FAMILY OF SOLUTIONS TO $df_t(x_t) = 0$ AND

(i) $\exists\, a<b$ S.T. x_a AND x_b ARE NONDEGENERATE CRITICAL POINTS WITH $i_{Morse}(x_a) \neq i_{Morse}(x_b)$;

ii) $d^2 f_t$ IS A FREDHOLM OPERATOR OF INDEX ZERO

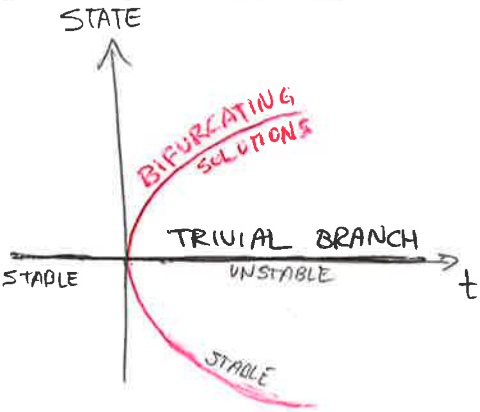
THEN $\exists\, t_* \in (a,b)$ A BIFURCATION INSTANT FOR x_t.

VERY SILLY EXAMPLE:

$f_t : \mathbb{R}^2 \to \mathbb{R}$

$$f_t(x,y) = x^2 - y^4 - t y^2$$
$$df_t(x,y) = (2x,\ 4y^3 - 2ty)$$
$$d^2 f_t(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 12y^2 - 2t \end{pmatrix}$$

CRITICAL POINTS:

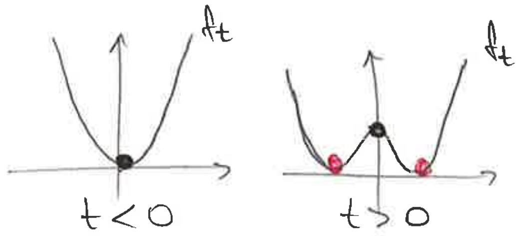


- TRIVIAL BRANCH: $(x_t, y_t) = (0,0)$
- JUMP IN MORSE INDEX AT $t=0$:

$$i_{Morse}(0,0) = \begin{cases} 0 & \text{IF } t<0 \\ 1 & \text{IF } t>0 \end{cases}$$

- "NEW" CRITICAL POINTS

$$\left(0, \pm\sqrt{\tfrac{t}{2}}\right)$$



- "PITCHFORK BIFURCATION": GROUND STATE GOES FROM STABLE TO UNSTABLE, 2 BRANCHES OF STABLE SOLUTIONS ARE CREATED.

APPLICATIONS TO YAMABE PROBLEM

YAMABE PROBLEM: GIVEN A RIEM. MFLD. (M^n, g_0), FIND A COMPLETE METRIC $g \in [g_0]$ WITH CONSTANT SCALAR CURVATURE.

$$g = \varphi^{\frac{4}{n-2}} g_0, \qquad 4\frac{n-1}{n-2}\Delta_{g_0}\varphi + \mathrm{scal}_{g_0}\varphi = \mathrm{scal}_g \cdot \varphi^{\frac{n+2}{n-2}}$$

M COMPACT: EXISTENCE ✓ (YAMABE '60, TRUDINGER '68, AUBIN '76, SCHOEN '84)

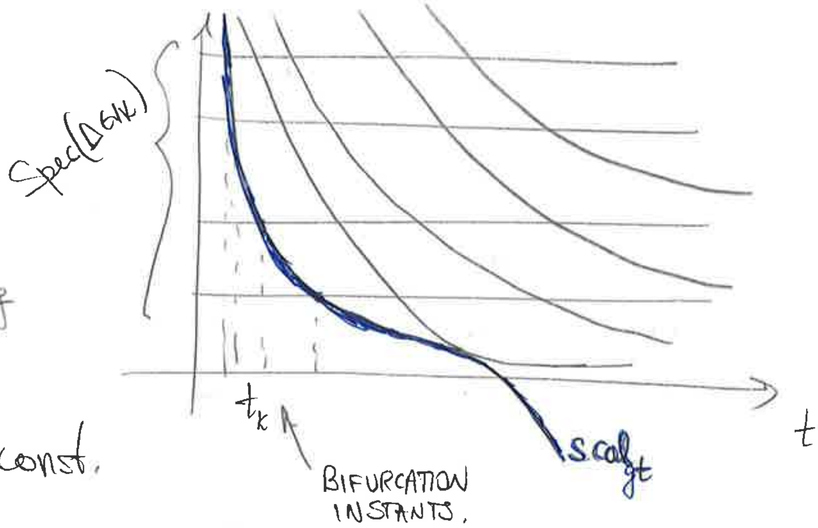
UNIQUENESS ✗ (..., BIFURCATION!)

THM (B. - PICCIONE, '13) LET $H < K < G$ BE S.T. $H \triangleleft K$ OR $K \triangleleft G$ AND $\mathrm{scal}(K/H) > 0$. THEN $(G/H, g_t)$ OBTAINED SHRINKING THE FIBERS OF $K/H \to G/H \to G/K$ HAS INFINITELY MANY BIFURCATIONS AS $t \searrow 0$. [g_t HOMOGENEOUS $\rightsquigarrow$ "TRIVIAL" FAMILY W/ $\mathrm{scal}_{g(t)} = \mathrm{const}$.]

E.G., $S^3 \to S^{4n+3} \to \mathbb{H}P^n$, (S^{4n+3}, g_t) BERGER SPHERES.

IDEA OF PROOF:

$$A : [g_0]_1 \to \mathbb{R}$$
$$g \longmapsto \int_M \mathrm{scal}_g \, \mathrm{vol}_g$$



$$dA(g) = 0 \iff \mathrm{scal}_g = \mathrm{const}.$$

$$d^2A(g)(\psi,\psi) = \int_M \left(\Delta_g \psi - \frac{\mathrm{scal}_g}{n-1}\psi\right)\psi$$

$$i_{Morse}(g_t) = \# \, \mathrm{Spec}(\Delta_{g_t}) \cap \left(-\infty, \frac{\mathrm{scal}_{g_t}}{n-1}\right) \nearrow +\infty \text{ AS } t \searrow 0. \quad \square$$

(ACTUALLY, NEED EQUIVARIANT VERSION TO RULE OUT COMPENSATION) 2

$\boxed{\text{COR: } \exists\, \mathcal{T} \subset (0,\infty) \text{ ACCUMULATING AT } 0, \text{ S.T. } \forall t \in \mathcal{T}, \text{ THERE ARE AT LEAST 3 SOLUTIONS IN } [g_t].}$

$\to \underbrace{g_t, \text{ BIFURCATION SOL.}}_{i_{Morse} \gg 1}, \text{ \& } \underbrace{\text{YAMABE SOL.}}_{\text{MINIMUM}}$

M NONCOMPACT: EXISTENCE X (COUNTER-EXAMPLES BY JIN '88)

UNIQUENESS X

"SINGULAR YAMABE PROBLEM": $(M, g_0) = (\overline{M}, g_0) \setminus \Lambda$

INCOMPLETE — CLOSED MFLD — CLOSED SUBMFLD

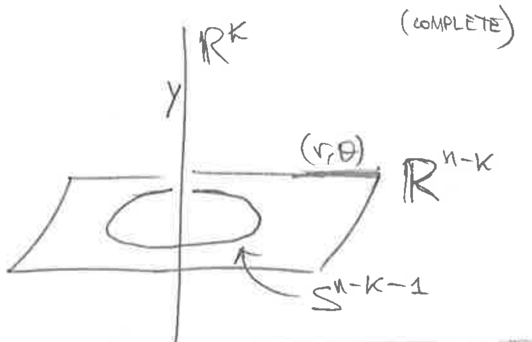
E.G.,

$$S^n \setminus S^k \xrightarrow{\cong} \mathbb{R}^n \setminus \mathbb{R}^k \xrightarrow{\cong} S^{n-k-1} \times \mathbb{H}^{k+1}$$

$$\underset{\text{(INCOMPLETE)}}{g_{round}} \xrightarrow[\text{projection}]{\text{stereographic}} dr^2 + r^2 d\theta^2 + dy^2 \xrightarrow[\cdot \frac{1}{r^2}]{} d\theta^2 + \frac{dr^2 + dy^2}{r^2} = \underset{\text{(COMPLETE)}}{g_{prod}}$$

THUS, OBTAIN "TRIVIAL SOLUTION" W/

$$\mathrm{scal}_{n,k} = (n-k-1)(n-k-2) - (k+1)k$$
$$= (n-2k-2)(n-1) > 0 \iff k < \frac{n-2}{2}$$



MAXIMAL POSSIBLE RANGE (n) OF EXISTENCE BY SCHOEN-YAU

$\boxed{\text{THM (B.-PICCIONE-SANTORO, '15). THERE EXIST UNCOUNTABLY MANY BIFURCATING BRANCHES OF PERIODIC SOLUTIONS TO THE S.Y.P. ON } S^n \setminus S^1 \text{ FOR ALL } n \geq 5, \text{ WITH } \mathrm{scal} \approx \underbrace{(n-4)(n-1)}_{\mathrm{scal}_{n,1}}.}$

GET BIFURCATION FROM WHAT "TRIVIAL" BRANCH?

IDEA OF PROOF:

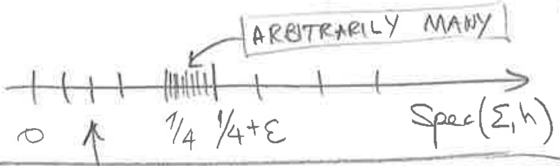
- FOR $k=1$, $S^{n-2} \times \mathbb{H}^2 \longrightarrow S^{n-2} \times \Sigma^2$, $\Sigma^2_t = \mathbb{H}^2 / \Gamma_t$ HYP. SURFACES, Γ_t KLEINIAN GROUP

$\mathcal{H}(\Sigma) = \{ h \in \mathrm{Met}(\Sigma) : \sec_h = -1 \}$ SPACE OF HYPERBOLIC METRICS

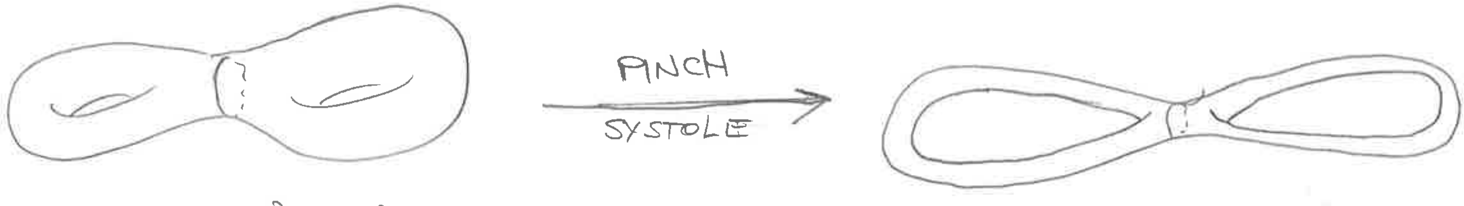
- $i_{Morse}(g_{round} \oplus h_t) = \# \{ \lambda_i(S^{n-2}, g_{round}) + \lambda_j(\Sigma^2, h_t) < n-4 \}$ ONLY MOVING PART

THM (BUSER, '77). $\forall \varepsilon > 0$, $j \in \mathbb{N}$, $\exists h \in \mathcal{H}(\Sigma)$ WITH

$$\lambda_j(\Sigma, h) < \frac{1}{4} + \varepsilon$$



• GEOMETRICALLY, (Σ^2, h_t) LOOK LIKE:



THIS MAKES $i_{Morse}(g_{round} \oplus h_t)$ JUMP!

• SLIGHTLY MORE DELICATE PROBLEM: NONDEGENERACY OF $g_{round} \oplus h_t$

⇝ "AVOIDING" INTEGERS IN $Spec(\Sigma, h_t)$; "MIDDLE C EMBARASSMENT" [WOLPERT]

□

NOTE: CANNOT APPLY THIS IDEA FOR $k > 1$ BY MOSTOW RIGIDITY, BUT ALTERNATIVE ARGUMENT (W/ COVERS/$\pi_1(\Sigma)$ RES. FINITE) GIVES DESIRED MULTIPLICITY.

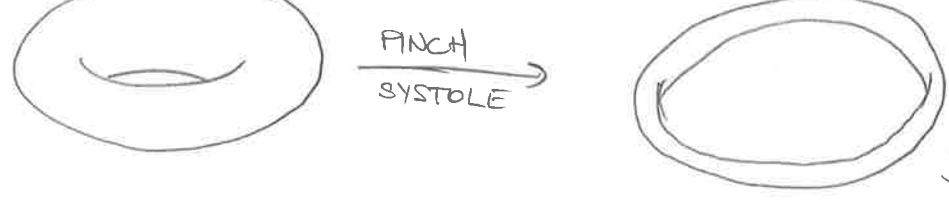
VARIATIONS. $\Sigma_t^2 = \mathbb{H}^2/\Gamma_t$ HYPERBOLIC SURFACE ⇝ $\Sigma_t^k = \mathbb{R}^k/\pi_t$ FLAT MFLDS ("BIEBERBACH MFLD")

INFINITELY MANY "PERIODIC" SOLUTIONS ON $S^n \times \mathbb{R}^k$

$\underbrace{\text{RAMÍREZ-OSPINA, '14}}_{\text{FLAT TORI } (\mathbb{Z}^d \subset Iso(\mathbb{R}^d))}$; $\underbrace{\text{B.-PICCIONE, '16}}_{\text{BIEBERBACH MFLDS } (\pi \subset Iso(\mathbb{R}^d))}$

THM (RAMÍREZ-OSPINA, '14; B.-PICCIONE, '16). LET (M, g) BE A CLOSED MFLD WITH $scal_g = const. > 0$ AND $\pi \subset Iso(\mathbb{R}^d)$ BE ANY BIEBERBACH GROUP, $d \geq 2$. THEN THERE EXIST INFINITELY MANY BRANCHES OF π-PERIODIC SOLUTIONS TO THE YAMABE PROBLEM ON $(M \times \mathbb{R}^d, g \oplus g_{flat})$.

IDEA OF PROOF: CHOOSE COLLAPSING FAMILY $(\mathbb{R}^d/\pi, h_t)$ OF FLAT METRICS



$i_{Morse}(g \oplus h_t) \nearrow +\infty$

B/C SPECTRAL BEHAVIOR IS ANALOGOUS. □