SOME NONUNIQUENESS RESULTS FOR THE YAMABE PROBLEM

YAMABE PROBLEM: GIVEN A RIEM. MFLD (M^n, g_0), FIND A COMPLETE METRIC $g \in [g_0]$ WITH CONSTANT SCALAR CURVATURE.

$\Updownarrow$

FIND $u: M \to \mathbb{R}$, $u > 0$, AND $u \nearrow +\infty$ "FAST ENOUGH" SOLVING.

$$4\,\frac{n-1}{n-2}\,\Delta_{g_0} u - \mathrm{scal}_{g_0} \cdot u = \mathrm{scal}_g \cdot u^{\frac{n+2}{n-2}} \rightsquigarrow g = u^{\frac{4}{n-2}} \cdot g_0$$

STATUS

M COMPACT:
- EXISTENCE: YES [YAMABE, TRUDINGER, AUBIN, SCHOEN]
- UNIQUENESS: NO [YES, IF $\mathrm{scal}_{g_0} \leq 0$, OR (M, g_0) IS EINSTEIN BUT NOT $(S^n, g_{\mathrm{round}})$, AND ALSO GENERICALLY AMONG $[g_0]$ WITH $\mathrm{scal}_{g_0} > 0$]

M NONCOMPACT:
- EXISTENCE: NO [JIN], BUT YES IF $M(\infty)$ IS "TAME" ENOUGH [MAZZEO-PACARD]
- UNIQUENESS: NO

RESULTS

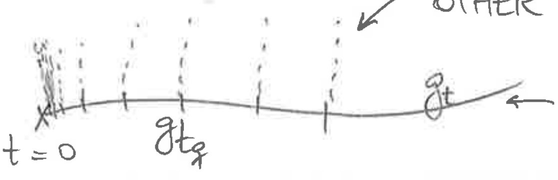
THM 1 (B.-PICCIONE, 2013). LET $H < K < G$ BE COMPACT LIE GROUPS WITH EITHER $H \triangleleft K$ OR $K \triangleleft G$, AND $\mathrm{scal}_{K/H} > 0$. LET g_t BE THE HOMOGENEOUS METRICS ON G/H OBTAINED RESCALING BY t THE FIBERS OF

$$K/H \longrightarrow G/H \longrightarrow G/K$$

THEN $\exists\, t_q \searrow 0$ SEQUENCE OF BIFURCATION INSTANTS FOR g_t.

I.E,

EACH g_{t_q} IS THE LIMIT OF A SEQUENCE OF OTHER METRICS W/ CONSTANT SCALAR CURVATURE (K-INVARIANT)



G-INVARIANT METRICS ON G/H

EXAMPLE: $S^3 \longrightarrow S^{4n+3} \longrightarrow \mathbb{H}P^n$ (BERGER METRICS)

(BUT ALSO $S^7 \longrightarrow S^{15} \longrightarrow S^8(1/2)$)

THM 2 (B.–PICCIONE–SANTORO, 2015; B.–PICCIONE, 2016). THERE EXIST INFINITELY MANY PAIRWISE NONISOMETRIC COMPLETE CONSTANT SCALAR CURVATURE METRICS ON $S^n \setminus S^k$, $0 \le k < \frac{n-2}{2}$ THAT ARE CONFORMAL TO THE ROUND (INCOMPLETE) METRIC.

I.E., SOLUTIONS TO SO-CALLED "SINGULAR YAMABE PROBLEM"!

BIFURCATION INTERMEZZO

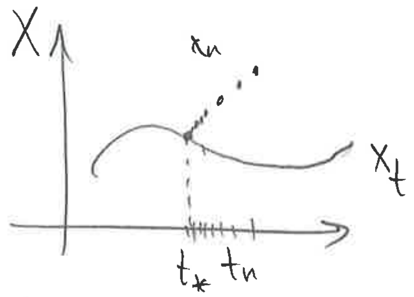
[POINCARÈ, 1885]: "TOPOLOGICAL CHANGE IN THE STRUCTURE OF A DYNAMICAL SYSTEM WHEN A PARAMETER CROSSES A 'BIFURCATION' VALUE"

$f_t : X \to \mathbb{R}$ 1-PARAMETER FAMILY OF FUNCTIONALS

$x_t \in X$ "TRIVIAL" BRANCH OF SOLUTIONS: $df_t(x_t) = 0$

DEF: BIFURCATION OCCURS AT t_* IF $\exists t_n \to t_*$, $\exists x_n \to x_{t_*}$, S.T. $df_{t_n}(x_n) = 0$ AND $x_n \neq x_{t_n}$

I.E., THE IMPLICIT FUNCTION THEOREM "FAILS" AT x_{t_*}.



• DEGENERACY OF x_{t_*} $\left(\mathrm{Ker}\, d^2 f_{t_*}(x_{t_*}) \neq \{0\}\right)$ IS NECESSARY, BUT NOT SUFFICIENT.

THM (KRASNOSEL'SKII). ASSUME x_t IS SUCH THAT $df_t(x_t) = 0$ AND

(1) $\exists\, a < b$ S.T. x_a AND x_b ARE NONDEGENERATE CRITICAL POINTS WITH $i_{Morse}(x_a) \neq i_{Morse}(x_b)$;

(2) $d^2 f_t$ IS A FREDHOLM OPERATOR OF INDEX ZERO,

THEN $\exists\, t_* \in (a,b)$ A BIFURCATION INSTANT FOR x_t.

SILLY EXAMPLE:

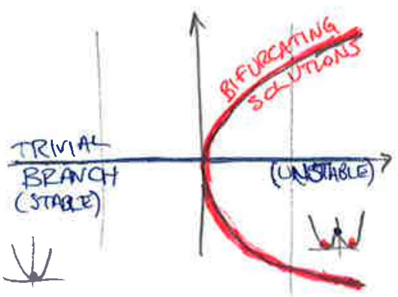
$f_t : \mathbb{R}^2 \to \mathbb{R}$, $f_t(x,y) = x^2 + y^4 - t y^2$

$df_t(x,y) = (2x,\ 4y^3 - 2ty)$

TRIVIAL BRANCH: $(x_t, y_t) = (0,0)$

"BIFURCATING BRANCH" $(x_t, y_t) = \left(0, \pm\sqrt{\frac{t}{2}}\right)$

$$i_{Morse}(0,0) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \qquad d^2 f_t(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2t \end{pmatrix}$$



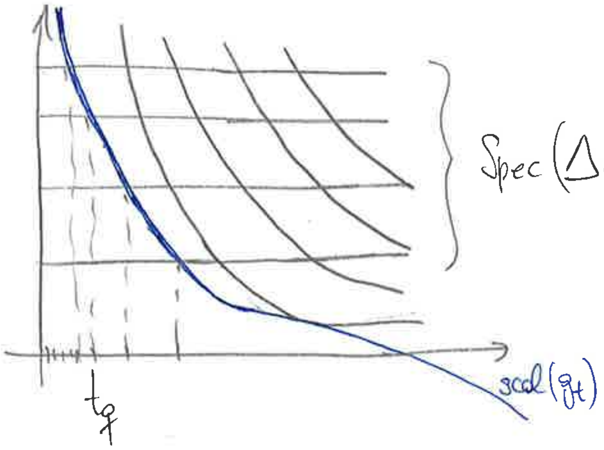
YAMABE PROBLEM VARIATIONAL SETUP

(M, g_0) COMPACT

- $\mathcal{A} : [g_0]_1 \longrightarrow \mathbb{R}, \quad \mathcal{A}(g) = \int_M \mathrm{scal}_g \, \mathrm{vol}_g$
- $d\mathcal{A}(g) = 0 \iff g \in [g_0]$ HAS CONSTANT SCALAR CURVATURE
- $d^2\mathcal{A}(g)(\psi, \psi) = \int_M \left(\Delta_g \psi - \frac{\mathrm{scal}}{n-1} \psi \right) \psi$

$$i_{\mathrm{Morse}}(g) = \# \, \mathrm{Spec}(\Delta_g) \cap \left(-\infty, \frac{\mathrm{scal}_g}{n-1} \right)$$

SKETCH OF PROOF OF THM 1



$K/H \longrightarrow (G/H, g_t) \longrightarrow G/K$ HAS TOTALLY GEODESIC FIBERS, SO

$$\underbrace{\mathrm{Spec}(\Delta_{G/K})}_{\text{INDEPENDENT OF } t} \subset \mathrm{Spec}(\Delta_{g_t})$$

AND $\mathrm{scal}(g_t) \nearrow +\infty$ AS $t \searrow 0$.

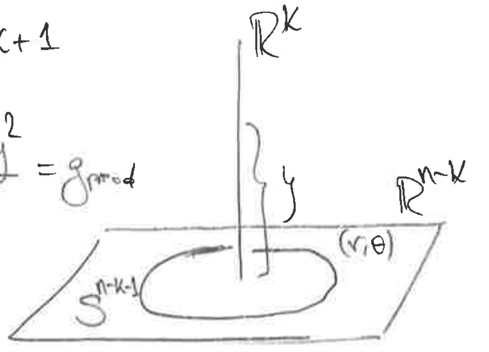
THUS, $i_{\mathrm{Morse}}(g_t) \nearrow +\infty$ AS $t \searrow 0$, AND GET BIFURCATION INSTANTS. $\square$

⚠ "COMPENSATION ISSUE": USE EQUIVARIANT BIFURCATION AND NORMALITY ASSUMPTIONS

SKETCH OF PROOF OF THM 2

$$S^n \setminus S^k \longrightarrow \mathbb{R}^n \setminus \mathbb{R}^k \longrightarrow S^{n-k-1} \times H^{k+1}$$

$$g_{\mathrm{round}} \xrightarrow[\text{PROJECTION}]{\text{STEREOGRAPHIC}} \underbrace{dr^2 + r^2 d\theta^2 + dy^2}_{g_{\mathrm{flat}}} \xrightarrow{\cdot \frac{1}{r^2}} d\theta^2 + \frac{dr^2 + dy^2}{r^2} = g_{\mathrm{prod}}$$



PULLING BACK BY ABOVE CONFORMAL EQUIV., GET "TRIVIAL SOLUTION" (ONLY ONE PREVIOUSLY KNOWN), WITH:

$$\mathrm{scal} = (n-k-1)(n-k-2) - (k+1)k = (n-2k-2)(n-1) > 0 \iff 0 \leq k < \frac{n-2}{2} \cdot 2$$

$\boxed{[\text{BPS}] \quad k = 1}$: $S^n \setminus S^1 \cong S^{n-2} \times \mathbb{H}^2$

BIFURCATE DEFORMING HYPERBOLIC STRUCTURE ON COMPACT QUOTIENT!

$$\downarrow$$

$\Sigma^2_t = \mathbb{H}^2 / \Gamma_t$ HYP. SURFACES $\qquad S^{n-2} \times \Sigma^2_t$

$$i_{Morse}(g_{round} \oplus h_t) = \#\{\lambda_i(S^{n-2}, g_{round}) + \underbrace{\lambda_j(\Sigma^2, h_t)}_{\text{ONLY PART THAT CHANGES W/ } t.} < \underset{\underset{\frac{scal}{n-1}}{\shortparallel}}{n-4}\}$$

$\boxed{\text{THM (BUSER '77). } \forall \varepsilon > 0,\ j \in \mathbb{N},\ \exists h \in \mathcal{H}(\Sigma) \text{ WITH } \lambda_j(\Sigma, h) < \tfrac{1}{4} + \varepsilon.}$

$0 \quad 1/4 \quad 1/4+\varepsilon \quad \to \operatorname{Spec}(\Sigma, h)$

ARBITRARILY MANY!

SO CAN MAKE $i_{Morse} \nearrow \infty$, GET INFINITELY MANY SOLUTIONS

DIFFERENT CONFORMAL CLASSES IN THE QUOTIENT, BUT PULL BACK TO SAME CONFORMAL CLASS ON $S^n \setminus S^1$.

⚠ NONDEGENERACY ISSUE; NEED A DEEP RESULT BY WOLPERT (SUGGESTED BY S. MONDAL)

FOR $k \geq 2$, CANNOT DO THIS BY MOSTOW RIGIDITY!

$\boxed{[\text{BP}] \quad k < \frac{n-2}{2}}$: $S^n \setminus S^k \cong S^{n-k-1} \times \mathbb{H}^{k+1}$

USE CHAIN OF FINITE COVERINGS AND AUBIN'S INEQUALITY!

$$\downarrow$$

$$S^{n-k-1} \times \Sigma^{k+1} \quad \text{(RIGID)}$$

FACT: $\pi_1(\Sigma)$ IS INFINITE AND RESIDUALLY FINITE, SO THERE IS

$$\Sigma = \Sigma_1 \leftarrow \Sigma_2 \leftarrow \cdots \leftarrow \Sigma_j \leftarrow \cdots \leftarrow \mathbb{H}^{k+1}$$

INFINITE CHAIN OF FINITE-SHEETED REGULAR COVERINGS, AND

$$\operatorname{Vol}(\Sigma_j) = \underbrace{s(j)}_{\#\text{ SHEETS}} \cdot \operatorname{Vol}(\Sigma).$$

WHEN SCALAR CURVATURE IS CONSTANT, VALUE OF THE FUNCTIONAL IS:

$$\mathcal{A}(M, g) = \operatorname{Vol}(M, g)^{\frac{2}{n}} \cdot scal_g$$

IN PARTICULAR,

$$\mathcal{A}(S^{n-k-1} \times \Sigma^{k+1}_j, g_{prod}) = \underbrace{s(j)^{\frac{2}{n}}}_{\text{GETS ARBITRARILY LARGE AS } j \nearrow +\infty.} \operatorname{Vol}(S^{n-k-1})^{\frac{2}{n}} \operatorname{Vol}(\Sigma^{k+1})^{\frac{2}{n}} (n-2k-2)(n-1)$$

RECALL: $Y(M,[g_0]) = \inf_{g\in[g_0]_1} \mathcal{A}(g)$ "YAMABE INVARIANT"

THM. THE ABOVE inf IS ATTAINED AT $g_Y \in [g_0]$, CALLED A YAMABE METRIC, WHICH HAS $\mathrm{scal}(g_Y) = \mathrm{const.}$, AND MOREOVER

$$\boxed{Y(M,[g_0]) \leq Y(S^n,[g_{round}])}$$

WITH EQUALITY IFF $(M,g_0) \underset{conf}{\cong} (S^n, g_{round})$

SO, GOING UP THE CHAIN OF FINITE COVERINGS OF $S^{n-k-1} \times \Sigma^{k+1}$, GET NEW INFINITELY MANY NEW SOLUTIONS (YAMABE METRICS AT EACH LEVEL WHERE $\mathcal{A} > Y(S^n,[g_{round}])$). $\square$

REMARKS:

1. NEW SOLUTIONS ARE NOT ISOMETRIC TO g_{prod} B/C:
$$\mathrm{Conf}(S^m \times H^d, g_{prod}) = \mathrm{Iso}(S^m \times H^d, g_{prod})$$
$\rightsquigarrow$ PAIRWISE NONISOMETRIC ???

2. BY CAFFARELLI-GIDAS-SPRUCK ASYMPTOTIC SYMMETRY METHOD, NEW SOLUTIONS DO NOT DEPEND ON THE S^{n-k-1} VARIABLE...
$\rightsquigarrow$ SUBCRITICAL PROBLEM!

3. CAN PROVE MORE GENERAL MULTIPLICITY RESULT ON $(M \times N, g \oplus h)$, WHERE (M,g) IS COMPACT AND HAS $\mathrm{scal} = \mathrm{const} > 0$, AND (N,h) IS A SIMPLY-CONNECTED SYMMETRIC SPACE OF NONCOMPACT OR EUCLIDEAN TYPE.
(OR, EVEN MORE GENERALLY, ON $M \times \tilde{\Sigma}$, WHERE (Σ, h) IS COMPACT, HAS CONSTANT SCALAR CURVATURE, AND $\pi_1(\Sigma)$ HAS INFINITE PROFINITE COMPLETION)