WEITZENBÖCK FORMULAE AND SECTIONAL CURVATURE

4/2017

OUTLINE:

1. "TEXT BOOK" BOCHNER TECHNIQUE
2. GENERAL WEITZENBÖCK FORMULAE
3. ALGEBRAIC CHARACTERIZATION OF $sec \geq 0$
4. 4-MANIFOLDS WITH $sec > 0$

1. "TEXTBOOK" BOCHNER TECHNIQUE

THM (BOCHNER, 1946). LET (M,g) BE A CLOSED RIEM. MFLD.

(1) IF $Ric_g > 0$, THEN $b_1(M;\mathbb{R}) = 0$

(2) IF $Ric_g < 0$, THEN $Iso(M,g)$ IS FINITE.

NOTE: BONNET-MYERS: $Ric_g \geq (n-1)K > 0 \Rightarrow diam(\tilde{M},\tilde{g}) \leq \frac{\pi}{\sqrt{K}}$

$$\Rightarrow |\pi_1(M)| < \infty$$

$$H^1(M) \cong \frac{\pi_1(M)}{[\pi_1(M),\pi_1(M)]} \Rightarrow b_1(M;\mathbb{R}) = 0.$$

HISTORIC REMARK:
BOCHNER WAS ORTHODOX JEW, ESCAPED FROM AUSTRIA/HUNGARY TO USA.
WEITZENBÖCK: WAS A NAZI ARMY OFFICER!

PROOF: (1) WEITZENBÖCK FORMULA FOR 1-FORMS $\omega \in \Omega^1(M)$:

$$\Delta\omega = \nabla^*\nabla\omega + 2\,Ric(\omega)$$

HODGE LAPLACIAN

CONNECTION LAPLACIAN $\left(\nabla^*\nabla = -\sum_i \nabla_{E_i}\nabla_{E_i}\right)$

RICCI ENDOMORPHISM APPLIED TO VECTOR FIELD DUAL TO ω, IDENTIFY $T_pM^* \cong T_pM$.

TAKE INNER PRODUCT WITH ω AND INTEGRATE ON (M,g):

$$\int_M \langle \Delta\omega, \omega\rangle = \int_M |\nabla\omega|^2 + 2\,\mathrm{Ric}(\omega,\omega)$$

IF ω IS <u>HARMONIC</u> $(\Delta\omega = 0)$ AND $\mathrm{Ric}_g > 0$, THEN $\omega = 0$:

$$0 = \int_M \underbrace{|\nabla\omega|^2}_{\geq 0} + \underbrace{2\,\mathrm{Ric}(\omega,\omega)}_{>0 \text{ UNLESS } \omega\equiv 0}$$

<u>THUS</u>: THERE ARE NO NONZERO HARMONIC 1-FORMS
$(\nexists\, \omega\in\Omega^1(M)\setminus\{0\},\ \Delta\omega = 0)$

$$\xRightarrow{\text{HODGE THEORY}} b_1(M;\mathbb{R}) = \dim\{\omega\in\Omega^1(M) : \Delta\omega = 0\} = 0.$$

"<u>VANISHING THEOREM</u>"

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(2) WEITZENBÖCK FORMULA FOR VECTOR FIELDS $X\in\mathfrak{X}(M)$:

$$\Delta X = \nabla^*\nabla X - 2\,\mathrm{Ric}(X)$$

DEFINED BY THIS FORMULA

CONNECTION LAPLACIAN
$\left(\nabla^*\nabla = -\sum_i \nabla_{E_i}\nabla_{E_i}\right)$

RICCI ENDOMORPHISM

IF X IS <u>HARMONIC</u> $(\Delta X = 0)$ AND $\mathrm{Ric}_g < 0$, THEN $X = 0$:

$$0 = \int_M \underbrace{|\nabla X|^2}_{\geq 0} - \underbrace{2\,\mathrm{Ric}_g(X,X)}_{>0 \text{ UNLESS } X=0.}$$

<u>THUS</u>: THERE ARE NO NONZERO HARMONIC VECTOR FIELDS.

<u>LEMMA</u>: $\Delta X = 0 \Leftrightarrow \mathcal{L}_X g = 0 \Leftrightarrow X$ IS KILLING

$$\Longrightarrow \dim \mathrm{Iso}(M,g) = \dim\{X\in\mathfrak{X}(M) : X \text{ KILLING}\} = 0. \qquad \square$$

REMARK: "RIGIDITY CASES"

THUS ω IS DETERMINED BY ITS VALUE AT A POINT

$\boxed{Ric_g \geq 0}$: $\Delta\omega = 0 \Longrightarrow \nabla\omega = 0$ (ω IS PARALLEL)

$\Longrightarrow b_1(M) \leq \dim M$ NOTE: EQUALITY HOLDS $\Leftrightarrow (M,g) \underset{isom.}{=} T^n$

$\boxed{Ric_g \leq 0}$: $\Delta X = 0 \Longrightarrow \nabla X = 0$ (X IS PARALLEL)

$\Longrightarrow \dim Iso(M,g) \leq \dim M$ NOTE: $\tilde{M} \underset{isom.}{=} \mathbb{R}^d \times N$, $d = \dim Iso(M,g)$

NOTE: CAN REPLACE $Ric_g > 0$ BY "QUASI-POSITIVE": $Ric_g \geq 0$ AND $Ric_p > 0$.
(ANALOGOUS FOR $Ric_g < 0$).

2. GENERAL WEITZENBÖCK FORMULAE

HITCHIN: $\rho : O(n) \to O(E)$ ORTHOGONAL REPRESENTATION

$$\begin{array}{ccccc} O(n) \longrightarrow Fr\,M & & E \longrightarrow \mathcal{E}_\rho \\ \downarrow & \xrightarrow{\text{ASSOCIATED BUNDLE}} & \downarrow \\ M & & M \end{array}$$

FRAME BUNDLE

EXAMPLES: $\rho : O(n) \curvearrowright \mathbb{R}^n \Longrightarrow \mathcal{E}_\rho = TM$ VECTOR FIELDS

$\rho : O(n) \curvearrowright Sym^p(\mathbb{R}^n) \Longrightarrow \mathcal{E}_\rho = Sym^p(M)$ SYMMETRIC p-TENSORS

$\rho : O(n) \curvearrowright \Lambda^p(\mathbb{R}^n) \Longrightarrow \mathcal{E}_\rho = \Lambda^p(M)$ p-FORMS

WEITZENBÖCK FORMULA FOR SECTIONS OF $\mathcal{E}_\rho$:

$$\Delta = \nabla^*\nabla + t\,K(R,\rho)$$

CONNECTION LAPLACIAN $\left(\nabla^*\nabla = -\sum_i \nabla_{E_i}\nabla_{E_i}\right)$

$t \in \mathbb{R}$

"CURVATURE TERM"

Q: HOW TO COMPUTE $K(R,\rho)$?

$\{X_a\}$ O.N. BASIS OF $\Lambda^2\mathbb{R}^n \cong \mathfrak{so}(n)$

$$R = \sum_{a,b} R_{ab}\, X_a \otimes X_b \in \mathrm{Sym}^2(\Lambda^2\mathbb{R}^n)$$

$$\rho : O(n) \longrightarrow O(E)$$
$$d\rho : \mathfrak{o}(n) \longrightarrow \mathfrak{o}(E)$$

$$\boxed{K(R,\rho) = -\sum_{a,b} R_{ab}\, d\rho(X_a)\circ d\rho(X_b)}$$

PROPERTIES:

- $\mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) \ni R \longmapsto K(R,\rho) \in \mathrm{Sym}^2(E)$ IS $O(n)$-EQUIVARIANT
- $K(R,\rho_1\oplus\rho_2) = K(R,\rho_1)\oplus K(R,\rho_2)$
- $K(R,\rho) = 0$ IF ρ IS TRIVIAL REPRESENTATION
- $K(R,\rho) > 0$ IF $R > 0$ AND ρ HAS NO TRIVIAL FACTORS.

$$\boxed{\text{THM (HITCHIN).}\ R \geq 0 \iff K(R,\rho) \geq 0,\ \forall \rho : O(n) \to O(E).}$$

3. ALGEBRAIC CHARACTERIZATION OF $\sec \geq 0$

$$\boxed{\text{THM (B.-MENDES).}\ \sec_R \geq 0 \iff K(R, \mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0}$$

$\sec_R \geq k$

$\Updownarrow$

$K(R - k\,\mathrm{Id}, \mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0$

$\sec_R \leq k$

$\Updownarrow$

$K(R - k\,\mathrm{Id}, \mathrm{Sym}^p_0(\mathbb{R}^n)) \leq 0$

RECALL:

$$\mathrm{Sym}^p(\mathbb{R}^n) \cong \{\varphi : \mathbb{R}^n \to \mathbb{R} \text{ HOMOG. POLY. OF DEGREE } p\}$$

$$\mathrm{Sym}^p_0(\mathbb{R}^n) \cong \{\varphi \in \mathrm{Sym}^p(\mathbb{R}^n) : \Delta\varphi = 0\}$$

IRREDUCIBLE AS $O(n)$-REPRESENTATION

THEOREM OF TARSKI, $\exists$ FINITELY MANY POLYNOMIAL CONDITIONS ON R EQUIVALENT TO $\sec_R \geq 0$.

SKETCH OF PROOF:

- A DIRECT COMPUTATION GIVES:

$$\langle K(R, \mathrm{Sym}^p_0(\mathbb{R}^n))\varphi, \varphi\rangle = \int_{S^{n-1}} R(X, \nabla\varphi, X, \nabla\varphi)\, dX$$

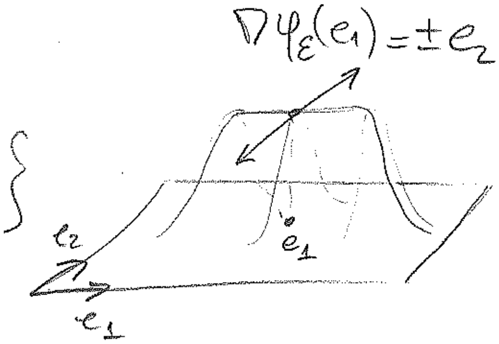
THUS $\sec_R \geq 0 \implies K(R, \mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0, \ \forall p \geq 2.$

- FOR ($\Leftarrow$), ARGUE BY CONTRADICTION:

SUPPOSE $\sigma = e_1 \wedge e_2 \in Gr_2(\mathbb{R}^n)$ HAS $\sec_R(\sigma) < 0$.

LET $\varphi_\varepsilon : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\quad X = (x_1, \dots, x_n)$

$$\varphi_\varepsilon(X) = \max\{0, \varepsilon^2 - |x_2| - \|X - e_1\|^2\}$$



SO: $\nabla\varphi_\varepsilon(X) = \pm e_2 - 2(X - e_1)$

$\left(\text{"} \nabla\varphi_\varepsilon(e_1) = \pm e_2 \text{"}\right)$

$\mathrm{supp}\, \varphi_\varepsilon \subset B_\varepsilon(e_1)$

$$\sec_R(e_1 \wedge e_2) = R(e_1, e_2, e_1, e_2) < 0 \implies \int_{S^{n-1}} R(X, \nabla\varphi_\varepsilon, X, \nabla\varphi_\varepsilon)\, dX < 0$$

- APPROXIMATING $\varphi_\varepsilon|_{S^{n-1}}$ WITH (HOMOGENEOUS) POLYNOMIALS, SENDING $\varepsilon \searrow 0$, GET A CONTRADICTION WITH $K(R, \mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0$. $\square$

REMARK: $K(R, \mathrm{Sym}^2_0(\mathbb{R}^n)) \geq 0 \overset{\text{BERGER}}{\Longrightarrow} K(R, \underbrace{\mathrm{Sym}^1(\mathbb{R}^n)}_{\cong \mathbb{R}^n}) = \mathrm{Ric}_R \geq 0.$

HOWEVER, NO EVIDENCE THAT $K(R, \mathrm{Sym}^{p+1}_0(\mathbb{R}^n)) \geq 0 \Rightarrow K(R, \mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0$ FOR LARGER $p > 2$...

HAD THIS BEEN TRUE, WOULD GET $\sec_R \geq 0 \iff$ "$\lim_{p\to\infty} K(R, \mathrm{Sym}^p_0(\mathbb{R}^n))$" ≥ 0.

4. 4-MANIFOLDS WITH $\sec > 0$

CONJECTURALLY, S^4 AND $\mathbb{C}P^2$ ARE THE ONLY 4-MANIFOLDS WITH $\sec > 0$ AND $\pi_1 \cong \{1\}$. [<u>HOPF QUESTION</u>: $S^2 \times S^2$ HAS $\sec > 0$?]

THM (B.–MENDES) LET (M^4, g) BE A CLOSED RIEM. MFLD. WITH $\pi_1(M) = \{1\}$, INDEFINITE INTERSECTION FORM, AND $\sec > 0$. THEN THE SET $M \setminus \{p \in M : R_p > 0\}$ HAS AT LEAST 2 CONNECTED COMPONENTS.

COR: $\nexists$ CURVATURE-HOMOGENEOUS METRICS WITH $\sec > 0$ ON SUCH M.

SKETCH OF PROOF: (M^4, g) HAS $\sec_g > 0 \overset{\text{THORPE}}{\iff} \exists f : M \to \mathbb{R}$ S.T. $(R + f *) > 0$

- NOTE: $R_p > 0$, $\forall p \in f^{-1}(0)$. SO SUFFICES TO SHOW $\exists p_\pm \in M$ S.T. $f(p_-) < 0 < f(p_+)$ AND $R_{p_\pm}$ NOT POSITIVE-DEFINITE.
- IF $\nexists p_-$, THEN $R_p > 0$ WHENEVER $f(p) < 0$

  $\Rightarrow$ CAN REPLACE f WITH $f_+ = \max\{f, 0\} \geq 0$: $(R + f_+ *) > 0$
- M INDEFINITE $\xRightarrow{\text{HODGE THEORY}}$ $\exists \alpha_\pm \in \Omega^2_\pm(M) \setminus \{0\}$, $\Delta \alpha_\pm = 0$, $*\alpha_\pm = \pm \alpha_\pm$

$$0 = \int_M \langle \Delta\alpha, \alpha \rangle = \int_M \|\nabla\alpha\|^2 + \langle K(R, \Lambda^2\mathbb{R}^4)\alpha, \alpha \rangle$$

$$= \int_M \underbrace{\|\nabla\alpha\|^2}_{\geq 0} + \underbrace{\langle K(\overbrace{R + f_+ *}^{>0}, \Lambda^2\mathbb{R}^4)\alpha, \alpha \rangle}_{>0} \underbrace{- f_+ \langle K(*, \Lambda^2\mathbb{R}^4)\alpha, \alpha \rangle}_{= 4\int_M \underbrace{f_+|\alpha|^2}_{\geq 0}}$$

$\alpha = \alpha_-$ ANTI-SELF-DUAL

$\Rightarrow$ CONTRADICTION! HENCE $\exists p_- \in M$ AS ABOVE.

ANALOGOUSLY FOR $\exists p_+ \in M$ (USE $\alpha = \alpha_+$). $\square$