COLLOQUIUM AT DARTMOUTH

DEFORMING FLAT MANIFOLDS & FLAT ORBIFOLDS

4/6/2017

THIS IS A "COLLOQUIUM": ASK QUESTIONS!

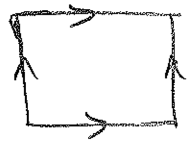
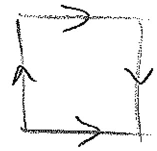
OUTLINE

1. FLAT MANIFOLDS & FLAT ORBIFOLDS
2. DEFORMATIONS $(\mathcal{M}_{flat})$
3. LIMITS $(\partial \mathcal{M}_{flat})$

1. FLAT MANIFOLDS & FLAT ORBIFOLDS

DEF: A FLAT MANIFOLD M IS A METRIC SPACE LOCALLY ISOMETRIC TO $\mathbb{R}^n$.
A FLAT ORBIFOLD $\mathcal{O}$ IS A METRIC SPACE LOCALLY ISOMETRIC TO $\mathbb{R}^n/\Gamma$, WHERE $\Gamma \subset O(n)$ IS A FINITE GROUP OF (LINEAR) ISOMETRIES.

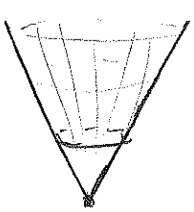
EXAMPLES: 2-DIMENSIONAL FLAT MANIFOLDS:

T^2 (TORUS) , K^2 (KLEIN BOTTLE) 

2-DIMENSIONAL FLAT ORBIFOLDS:

LEMMA ("DA VINCI THM" [ACCORDING TO WEYL]): $\Gamma < O(2)$ FINITE $\Rightarrow \Gamma \cong \mathbb{Z}_k$ (CYCLIC) OR $\Gamma \cong D_k$ (DIHEDRAL)

THUS NEAR SINGULAR POINTS, FLAT ORBIFOLDS ARE ISOMETRIC TO:

$\mathbb{R}^2/\mathbb{Z}_k$ 

CONE W/ VERTEX ANGLE $2\pi/k$, $k \geq 2$

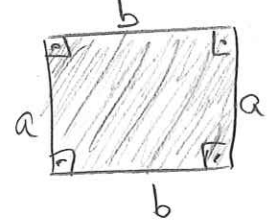
OR

$\mathbb{R}^2/D_k$ 

WEDGE WITH ANGLE π/k, $k \geq 1$

NOTE: BOTH ARE FLAT!

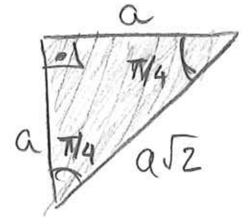
1) START WITH ANGLE $\pi/2$:



$D^2(;2,2,2,2)$ $H_\pi = D_2$ $d=2$

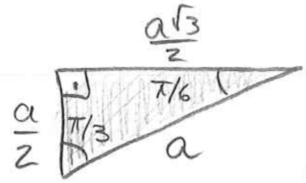
RECTANGLE (SIDES $a,b>0$)

$[\pi = \langle (x,y)\mapsto(x+b,y),\ (x,y)\mapsto(x,y+a),\ (x,y)\mapsto(-x,-y),\ (x,y)\mapsto(x,-y)\rangle]$



$D^2(;2,4,4)$ $H_\pi = D_4$ $d=1$

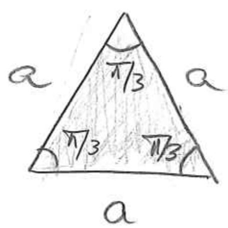
"HALF SQUARE" (BASE $a>0$)



$D^2(;2,3,6)$ (SIDE $a>0$)

"HALF EQUILATERAL TRIANGLE" $H_\pi = D_6$ $d=1$

2) START WITH ANGLE $\pi/3$:



$D^2(;3,3,3)$

EQUILATERAL TRIANGLE (SIDE $a>0$)

$H_\pi = D_3$ $d=1$

3) "DOUBLES": GLUE 2 COPIES OF THE ABOVE ALONG BOUNDARY:

$H_\pi = D_2$ $d=3$ $S^2(2,2,2,2;)$

$H_\pi = D_4$ $d=1$ $S^2(2,4,4;)$

$H_\pi = \mathbb{Z}_6$ $d=1$ $S^2(2,3,6;)$

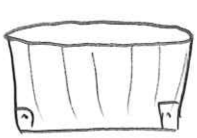
$H_\pi = \mathbb{Z}_3$ $d=1$ $S^2(3,3,3;)$

"PILLOWCASE"

$[\pi = \langle a,b,c,d \mid a^2=b^2=c^2=d^2=abcd=1\rangle]$

"TURNOVERS"

4) QUOTIENTS OF THE ABOVE:



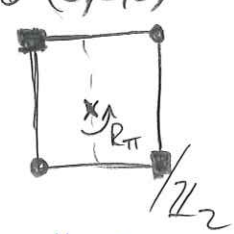
$H_\pi = D_2$ $d=2$ $D^2(2,2;)$

"HALF PILLOWCASE" (REFLECT ABOUT EQUATOR)

$\mathbb{RP}^2(2,2;)$ $H_\pi = D_2$ $d=2$

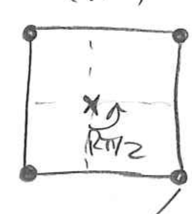
"PROJECTIVE PILLOWCASE" (ANTIPODAL MAP)

$D^2(2;2,2)$



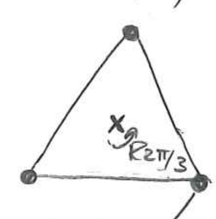
$H_\pi = D_2$ $d=2$

$D^2(4;2)$



$H_\pi = D_4$ $d=1$

$D^2(3;3)$



$H_\pi = D_3$ $d=1$

5) "BORING" EXAMPLES: T^2 (2-TORUS), $H_\pi = 1$ $d=3$

K^2 (KLEIN BOTTLE) $H_\pi = \mathbb{Z}_2$ $d=2$

$S^1 \times I$ (CYLINDER/ ANNULUS) $H_\pi = \mathbb{Z}_2$ $d=2$

M^2 (MÖBIUS BAND) $H_\pi = \mathbb{Z}_2$ $d=2$

COUNTING THE ABOVE, WE SEE <u>17</u> DIFFERENT FLAT 2-ORBIFOLDS!

<u>THM</u>: M FLAT MANIFOLD $\iff$ $M = \mathbb{R}^n/\pi$, π <u>BIEBERBACH</u>

$\mathcal{O}$ FLAT ORBIFOLD $\iff$ $\mathcal{O} = \mathbb{R}^n/\pi$, π <u>CRYSTALLOGRAPHIC</u>

GLOBAL!

$\pi \subset \mathrm{Iso}(\mathbb{R}^n) \cong O(n) \ltimes \mathbb{R}^n$ IS <u>CRYSTALLOGRAPHIC</u> IF IT IS DISCRETE / COCOMPACT

" <u>BIEBERBACH</u> IF IT IS CRYSTALLOGRAPHIC & <u>TORSION FREE</u>

→ MENTION PRESENTATION OF π IN SOME <u>EXAMPLES</u>.

<u>REMARK</u>: ABOVE 17 FLAT 2-ORBIFOLDS CORRESPOND TO THE 17 "WALLPAPER GROUPS" (CRYSTALLOGRAPHIC GROUPS IN DIM=2).

$$0 \longrightarrow \underset{\text{(TRANSLATIONS)}}{L_\pi} \longrightarrow \pi \xrightarrow{r} \underset{\text{(ROTATIONS)}}{H_\pi} \longrightarrow 0, \qquad r: \mathrm{Iso}(\mathbb{R}^n) \to O(n) \text{ "ROTATIONAL PART"}$$

<u>BIEBERBACH THEOREMS (1911-1912)</u>:

I. L_π IS AN n-DIM. LATTICE (SO $\mathbb{R}^n/L_\pi$ IS A <u>TORUS</u>)

$H_\pi \subset O(n)$ IS FINITE (<u>HOLONOMY</u> GROUP OF $\mathcal{O} = \mathbb{R}^n/\pi$).

II. $\pi, \pi' \subset \mathrm{Iso}(\mathbb{R}^n)$ CRYSTALLOGRAPHIC, $\pi \underset{\text{isom.}}{\cong} \pi' \iff \pi, \pi'$ CONJUGATE IN $\mathrm{Aff}(\mathbb{R}^n) = GL(n) \ltimes \mathbb{R}^n$

($\mathcal{O}, \mathcal{O}'$ FLAT n-ORBIFOLDS, $\mathcal{O} \underset{\text{Aff. equiv}}{\cong} \mathcal{O}' \iff \pi_1^{orb}(\mathcal{O}) \cong \pi_1^{orb}(\mathcal{O}')$).

III. $\forall n$, $\exists$ ONLY <u>FINITELY MANY</u> ISOM. TYPES OF $\pi \subset \mathrm{Iso}(\mathbb{R}^n)$ CRYSTALL. GROUPS

($\forall n$, $\exists$ ONLY <u>FINITELY MANY</u> AFFINE TYPES OF FLAT n-ORBIFOLDS).

[SOLUTION OF HILBERT'S 18th PROBLEM]

n	# BIEBERBACH GROUPS	#CRYSTALLOGRAPHIC GROUPS	
2	2	17	
3	10	219	
4	74	4,783	↓ COMPUTER ASSISTED!
5	1,060	222,018	
6	38,746	28,927,922	

Q1: CAN WE "DEFORM" FLAT ORBIFOLDS / FLAT MANIFOLDS WHILE KEEPING THEM FLAT?

Q2: IF YES, WHAT ARE THE POSSIBLE "LIMITS"?

A1: ON EXAMPLES, YES FOR TORI, RECTANGLE, ..., BUT NO ON EQUILATERAL TRIANGLE, ...

2. DEFORMATIONS

LET $\mathcal{M}_{flat}(\mathcal{O}) := \{\text{FLAT METRICS ON } \mathcal{O}\} / \text{ISOMETRIES}$ "MODULI SPACE"

$\therefore$ Q1 $\iff \dim \mathcal{M}_{flat}(\mathcal{O}) \geq 2$? (NOTE: $\dim \mathcal{M}_{flat}(\mathcal{O}) \geq 1$ BECAUSE CAN ALWAYS RESCALE!)

THM (B.–DERDZINSKI–PICCIONE). THERE EXISTS A "TEICHMÜLLER SPACE" $\mathcal{T}_{flat}(\mathcal{O}) \cong \mathbb{R}^d$ SUCH THAT $\mathcal{M}_{flat}(\mathcal{O}) \cong \mathcal{T}_{flat}(\mathcal{O}) / N_\pi$, WHERE N_π IS DISCRETE. IF M IS A FLAT MANIFOLD, THEN $\dim \mathcal{M}_{flat}(M) = d \geq 2$.

USES WORK OF WOLF, HISS-SZCZEPAŃSKI

I.E., CAN ALWAYS DEFORM FLAT MANIFOLDS! (NOT NECESSARILY FOR FLAT ORBIFOLDS — CAN BE RIGID...)

FOR THE EXPERTS:

$$H_\pi \curvearrowright \mathbb{R}^n = \bigoplus_{i=1}^{\ell} W_i$$

ISOTYPIC COMPONENTS OF TYPE $K_i = \mathbb{R}, \mathbb{C}, \mathbb{H}$, MULTIPLICITY $m_i \geq 1$

COPIES OF THE SAME IRREDUCIBLE

$$\mathcal{T}_{flat}(\mathcal{O}) = \prod_{i=1}^{\ell} \frac{GL(m_i, K_i)}{O(m_i, K_i)}$$

WHERE $O(m, K) = O(m), U(m), Sp(m)$ ACCORDINGLY

EXAMPLE: TORUS $T^n = \mathbb{R}^n / \mathbb{Z}^n$, $H_\pi = 1$. $\mathcal{T}_{flat}(T^n) = O(n) \backslash GL(n,\mathbb{R})$

↑ SPACE OF $n \times n$ POSITIVE DEF. MATRICES (INNER PRODUCTS ON $\mathbb{R}^n$)

$$\mathcal{M}_{flat}(T^n) = O(n) \backslash GL(n,\mathbb{R}) / GL(n,\mathbb{Z})$$

3. LIMITS

Q2: "LIMITS" OF FLAT MANIFOLDS / FLAT ORBIFOLDS?

$\rightsquigarrow \partial \mathcal{M}_{flat} = ?$, $\partial \mathcal{T}_{flat} = ?$

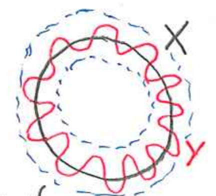
$\boxed{\text{THM (B.-DERDZINSKI-PICCIONE). THE (GROMOV-HAUSDORFF) LIMIT OF FLAT MANIFOLDS IS A FLAT ORBIFOLD. CONVERSELY, EVERY FLAT ORBIFOLD IS THE (GROMOV-HAUSDORFF) LIMIT OF FLAT MANIFOLDS.}}$

DIAGONAL ARGUMENT

COR: {FLAT ORBIFOLDS}/ISOM IS CLOSED IN $(\mathcal{M}, d_{GH})$.

Q: WHAT ARE GROMOV-HAUSDORFF LIMITS?

- HAUSDORFF DISTANCE: $X, Y \subset Z$



$$d_H^Z(X,Y) := \inf \{ \varepsilon > 0 : B_\varepsilon(X) \supset Y,\ B_\varepsilon(X) \supset Y \}$$

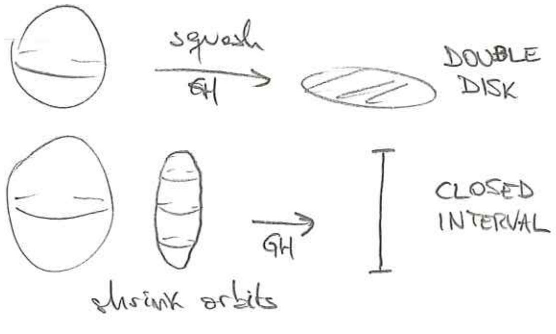
- GROMOV-HAUSDORFF DISTANCE: X, Y COMPACT METRIC SPACES.

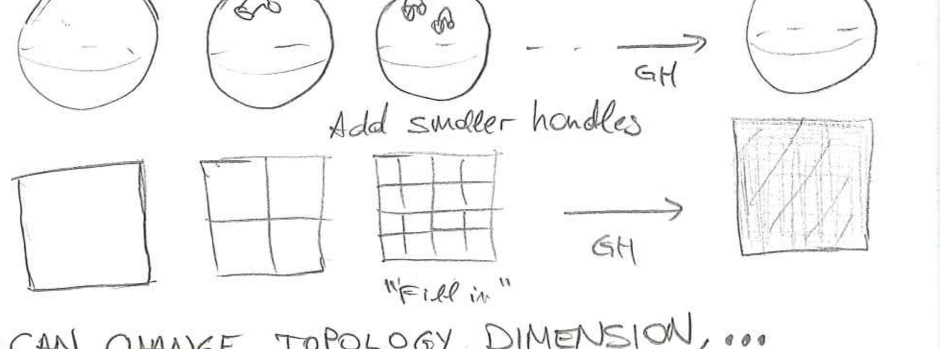
$$d_{GH}(X,Y) = \inf_{\substack{X \hookrightarrow Z \\ Y \hookrightarrow Z}} d_H^Z(X,Y)$$

GROMOV: $\mathcal{M} = \{\text{COMPACT METRIC SPACES}\} / \text{ISOMETRIES}$

$(\mathcal{M}, d_{GH})$ IS A METRIC SPACE.

EXAMPLES:





- CAN CHANGE TOPOLOGY, DIMENSION, ...
- BEST BEHAVED UNDER CURVATURE BOUNDS
  ↳ (Δ COMPARISON)
- USED IN PERELMAN'S SOLUTION OF POINCARÉ CONJECTURE!

SKETCH OF THE PROOF:

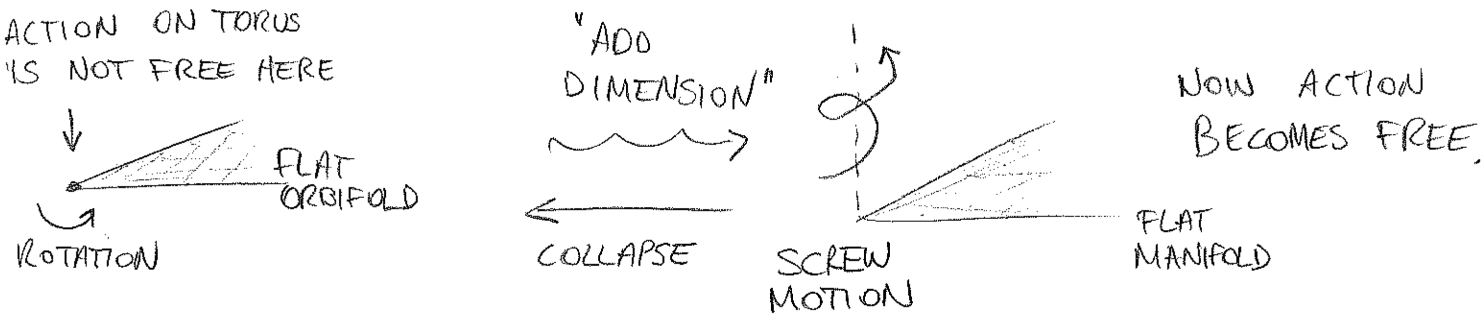
$$M = \mathbb{R}^n/\pi \text{ COLLAPSES} \overset{\text{DIAMETER ESTIMATES}}{\Longleftrightarrow} T^n = \mathbb{R}^n/L_\pi \text{ COLLAPSES.}$$

LEMMA. GH-LIMIT OF FLAT TORI ARE FLAT TORI ← USES [FUKAYA-YAMAGUCHI]

$$\begin{array}{ccccc} \pi \curvearrowright \mathbb{R}^n & & H_\pi \curvearrowright T^n = \mathbb{R}^n/L_\pi & & H_\pi \curvearrowright T^K \\ \downarrow & \Longrightarrow & \downarrow & \xrightarrow[\text{COLLAPSE}]{} & \downarrow \\ M & & M & & \lim M \end{array}$$

LIMIT ACTION MAY NO LONGER BE FREE; SO QUOTIENT IS FLAT ORBIFOLD.

CONVERSE:



□

REMARK: 10 OUT OF THE 17 FLAT 2-ORBIFOLDS ARE LIMITS OF FLAT 3-MANIFOLDS:

> **THM** (B.-DERDZINSKI-PICCIONE), THE GROMOV-HAUSDORFF LIMIT OF A SEQUENCE OF FLAT 3-MANIFOLDS IS EITHER A
>
> - FLAT 3-MANIFOLD; OR A COLLAPSING CASE:
> - $\{p\}$ FLAT 0-MANIFOLD;
> - I, S^1 FLAT 1-ORBIFOLD;
> - T^2, K^2, $S^2 \times I$, M^2, $D^2(4;2)$, $D^2(3;3)$, $D^2(2,2;)$, $S^2(3,3,3;)$, $S^2(2,2,2,2;)$, $\mathbb{RP}^2(2,2;)$ FLAT 2-ORBIFOLD.