TEMPLE UNIVERSITY – GSCAGT 2017

DEFORMING FLAT MANIFOLDS & FLAT ORBIFOLDS

6/4/2017

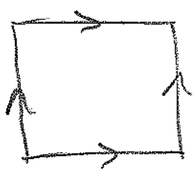
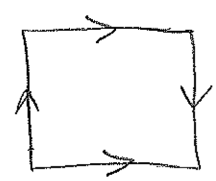
OUTLINE

1. FLAT MANIFOLDS & FLAT ORBIFOLDS ~ 30 min
2. DEFORMATIONS $(\mathcal{M}_{flat})$ ~ 10 min
3. LIMITS $(\partial \mathcal{M}_{flat})$ ~ 20 min

1. FLAT MANIFOLDS & FLAT ORBIFOLDS

<u>DEF:</u> A <u>FLAT MANIFOLD</u> M IS A METRIC SPACE LOCALLY ISOMETRIC TO $\mathbb{R}^n$.
A <u>FLAT ORBIFOLD</u> $\mathcal{O}$ IS A METRIC SPACE LOCALLY ISOMETRIC TO $\mathbb{R}^n/\Gamma$, WHERE $\Gamma < O(n)$ IS A FINITE GROUP OF (LINEAR) ISOMETRIES.

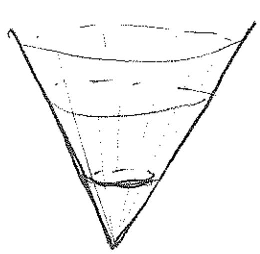
<u>EXAMPLES</u>: 2-DIMENSIONAL FLAT MANIFOLDS

T^2 (TORUS)  , K^2 (KLEIN BOTTLE) 

2-DIMENSIONAL FLAT ORBIFOLDS:

<u>LEMMA</u> ("DA VINCI THEOREM"): $\Gamma < O(2)$ FINITE $\Rightarrow$ $\underset{\text{(CYCLIC)}}{\Gamma \cong \mathbb{Z}_k}$ OR $\underset{\text{(DIHEDRAL)}}{\Gamma \cong D_k}$

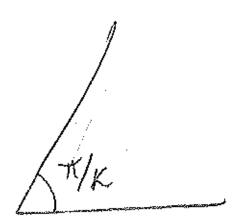
THUS, NEAR SINGULAR POINTS, ISOMETRIC TO:

$\mathbb{R}^2/\mathbb{Z}_k$ 

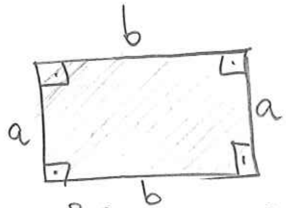
CONE W/ VERTEX ANGLE $2\pi/k$, $k \geq 2$

OR

$\mathbb{R}^2/D_k$

WEDGE W/ ANGLE π/k, $k \geq 1$ 

- STARTING WITH ANGLE $\pi/2$:

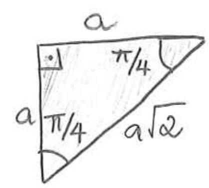


$D^2(;2,2,2,2)$

RECTANGLE

$\pi' = \left\langle \begin{matrix} (x,y)\mapsto(x+b,y), (x,y)\mapsto(x,y+a) \\ (x,y)\mapsto(-x,-y), (x,y)\mapsto(x,-y) \end{matrix} \right\rangle$

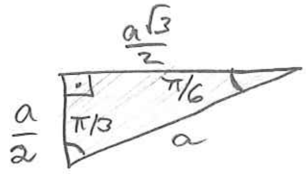
$H_\pi = D_2$ $d=2$



$D^2(;2,4,4)$

HALF SQUARE

$H_\pi = D_4$ $d=1$

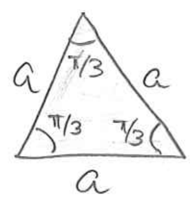


$D^2(;2,3,6)$

HALF EQUILATERAL TRIANGLE

$H_\pi = D_6$ $d=1$

- STARTING WITH ANGLE $\pi/3$:

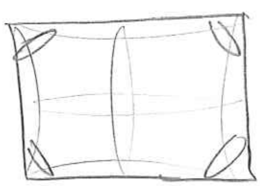


$D^2(;3,3,3)$

EQUILATERAL TRIANGLE

$H_\pi = D_3$ $d=1$

- DOUBLES: GLUE 2 COPIES OF ABOVE ALONG BOUNDARY

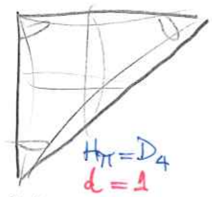


$S^2(2,2,2,2;)$

PILLOWCASE

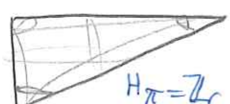
$H_\pi = D_2$ $d=3$

$\pi = \langle a,b,c,d \mid a^2=b^2=c^2=d^2=abcd=1\rangle$



$H_\pi = D_4$ $d=1$

$S^2(2,4,4;)$



$H_\pi = \mathbb{Z}_6$ $d=1$

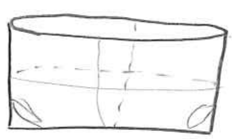
$S^2(2,3,6;)$

$H_\pi = \mathbb{Z}_3$ $d=1$

$S^2(3,3,3;)$

$\underbrace{S^2(2,4,4;),\ S^2(2,3,6;),\ S^2(3,3,3;)}_{\text{TURNOVERS}}$

- QUOTIENTS OF THE ABOVE



$D^2(2,2;)$

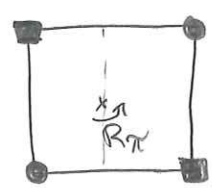
HALF PILLOWCASE

$H_\pi = D_2$ $d=2$

$\mathbb{RP}^2(2,2;)$

"PROJECTIVE PILLOWCASE"

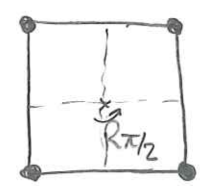
$H_\pi = D_2$ $d=2$



$/\mathbb{Z}_2$

$D^2(2;2,2)$

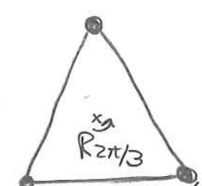
$H_\pi = D_2$ $d=2$



$/\mathbb{Z}_4$

$D^2(4;2)$

$H_\pi = D_4$ $d=1$

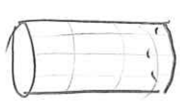


$/\mathbb{Z}_3$

$D^2(3;3)$

$H_\pi = D_3$ $d=1$

- "BORING" EXAMPLES:



$S^1 \times I$

CYLINDER

$H_\pi = \mathbb{Z}_2$ $d=2$

M^2

MÖBIUS STRIP

$H_\pi = \mathbb{Z}_2$ $d=2$

(AND T^2, K^2)

T^2: $H_\pi = 1$ $d=3$

K^2: $H_\pi = \mathbb{Z}_2$ $d=2$

17 TOTAL!

LOCAL TO GLOBAL:

$\pi \subset \mathrm{Iso}(\mathbb{R}^n) \cong O(n) \ltimes \mathbb{R}^n$ IS CRYSTALLOGRAPHIC IF IT IS DISCRETE / COCOMPACT

π IS BIEBERBACH IF IT IS CRYSTALLOGRAPHIC AND TORSION-FREE.

THM: M FLAT MANIFOLD $\iff M = \mathbb{R}^n/\pi$, π BIEBERBACH ← KILLING HOPF

$\mathcal{O}$ FLAT ORBIFOLD $\iff \mathcal{O} = \mathbb{R}^n/\pi$, π CRYSTALLOGRAPHIC ← THURSTON

REMARK: 17 FLAT 2-ORBIFOLDS $\leftrightsquigarrow$ 17 "WALLPAPER GROUPS" (CRYSTALLOGRAPHIC GROUPS IN DIM 3)

$r: \mathrm{Iso}(\mathbb{R}^n) \longrightarrow O(n)$ "ROTATIONAL PART"

$$0 \longrightarrow \underset{\text{(TRANSLATIONS)}}{L_\pi} \longrightarrow \pi \xrightarrow{r} \underset{\text{(ROTATIONS)}}{H_\pi} \longrightarrow 0 \quad \text{EXACT}$$

BIEBERBACH THEOREMS (1911-1912):

I. L_π IS AN n-DIMENSIONAL LATTICE (SO $\mathbb{R}^n/L_\pi$ IS A TORUS)
$H_\pi \subset O(n)$ IS FINITE (HOLONOMY GROUP OF $\mathcal{O} = \mathbb{R}^n/\pi$)

II. $\pi, \pi' \subset \mathrm{Iso}(\mathbb{R}^n)$ CRYSTALLOGRAPHIC, $\pi \underset{\text{isom}}{\cong} \pi' \iff \pi, \pi'$ CONJUGATE IN $\mathrm{Aff}(\mathbb{R}^n)$ $= GL(n) \ltimes \mathbb{R}^n$
($\mathcal{O}, \mathcal{O}'$ FLAT n-ORBIFOLDS, $\mathcal{O} \underset{\text{affine equiv}}{\cong} \mathcal{O}' \iff \pi_1^{orb}(\mathcal{O}) \cong \pi_1^{orb}(\mathcal{O}')$)

III. $\forall n$, $\exists$ ONLY FINITELY MANY ISOM. TYPES OF $\pi \subset \mathrm{Iso}(\mathbb{R}^n)$ CRYSTALLOGRAPHIC GPS
($\forall n$, $\exists$ ONLY FINITELY MANY AFFINE TYPES OF FLAT n-ORBIFOLDS)

← SOLVED HILBERTS 18th PROBLEM.

n	# BIEBERBACH GROUPS	# CRYSTALLOGRAPHIC GROUPS
2	2	17
3	10	219
4	74	4,783
5	1,060	222,018
6	38,746	28,927,922

↓ COMPUTER ASSISTED.

2. DEFORMATIONS ($\mathcal{M}_{flat}$)

$$\boxed{\mathcal{M}_{flat}(\mathcal{O}) = \{\text{FLAT METRICS ON } \mathcal{O}\} \big/ \text{ISOMETRIES} \qquad \text{"MODULI SPACE"}}$$

- FITS THURSTON'S ABSTRACT FRAMEWORK OF DEFORMATIONS OF (X, G)-STRUCTURES WITH $X = \mathbb{R}^n$, $G = \mathrm{Iso}(\mathbb{R}^n)$...

  X: LOCAL MODEL; G: TRANSITION MAPS

- PREVIOUSLY STUDIED IN [WOLF '73], [GOLDMAN '88], [BAUES 2000], ... ?

THM (B.-DERDZINSKI-PICCIONE, 2017). THERE EXISTS A "TEICHMÜLLER SPACE" $\mathcal{T}_{flat}(\mathcal{O}) \cong \mathbb{R}^d$ SUCH THAT $\mathcal{M}_{flat}(\mathcal{O}) = \mathcal{T}_{flat}(\mathcal{O}) \big/ N_\pi$, WHERE N_π IS DISCRETE. MORE PRECISELY, IF

$$H_\pi \curvearrowright \mathbb{R}^n = \bigoplus_{i=1}^{\ell} W_i$$

ISOTYPIC COMPONENTS OF TYPE $\mathbb{K}_i = \mathbb{R}, \mathbb{C}, \mathbb{H}$ AND MULTIPLICITY $m_i \geq 1$ (# OF COPIES OF THE SAME IRREDUCIBLE)

THEN $$\mathcal{T}_{flat}(\mathcal{O}) = \prod_{i=1}^{\ell} \frac{GL(m_i, \mathbb{K}_i)}{O(m_i, \mathbb{K}_i)}$$ WHERE $O(m, \mathbb{K}) = O(m), U(m), Sp(m)$ ACCORDINGLY.

HISS-SZCZEPAŃSKI: π BIEBERBACH $\Rightarrow H_\pi \curvearrowright \mathbb{R}^n$ REDUCIBLE

COROLLARY: FLAT MANIFOLDS ALWAYS ADMIT (NONHOMOTHETIC) DEFORMATIONS!
FLAT ORBIFOLDS MAY BE RIGID

EXAMPLE: TORUS $T^n = \mathbb{R}^n / \mathbb{Z}^n$

$H_\pi = \{1\} \curvearrowright \mathbb{R}^n$ TRIVIAL

$$\mathcal{T}_{flat}(T^n) = O(n) \backslash GL(n, \mathbb{R}) \cong \mathbb{R}^{\frac{n(n+1)}{2}}$$ SPACE OF INNER PRODUCTS IN $\mathbb{R}^n$ ($n \times n$ POS-DEF SYMM. MATRICES)

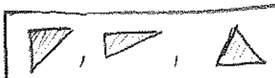
$$\mathcal{M}_{flat}(T^n) = O(n) \backslash GL(n, \mathbb{R}) \big/ GL(n, \mathbb{Z})$$ HAS SINGULARITIES

KLEIN BOTTLE $K^2 = \mathbb{R}^2 / \pi$

$H_\pi = \mathbb{Z}_2 \curvearrowright \mathbb{R}^2 = \underset{\text{(TRIVIAL)}}{\mathbb{R}} \oplus \underset{\text{(NONTRIVIAL)}}{\mathbb{R}}$

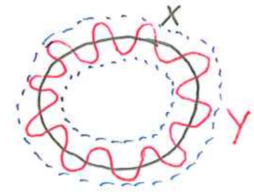
$$\text{"} O(1) \backslash GL(1, \mathbb{R}) \text{"} \cong \mathbb{Z}_2 \backslash (\mathbb{R} \setminus \{0\}) \cong \mathbb{R}$$

$$\mathcal{T}_{flat}(K^2) \cong \mathbb{R}^2$$



& DOUBLES ARE RIGID! $H_\pi \curvearrowright \mathbb{R}^2$ IRRED.

3. LIMITS ($\partial \mathcal{M}_{flat}$)



- HAUSDORFF DISTANCE: $X, Y \subset Z$

$$d_H^Z(X,Y) := \inf \{ \varepsilon > 0 : B_\varepsilon(X) \supset Y, \ B_\varepsilon(Y) \supset X \}$$

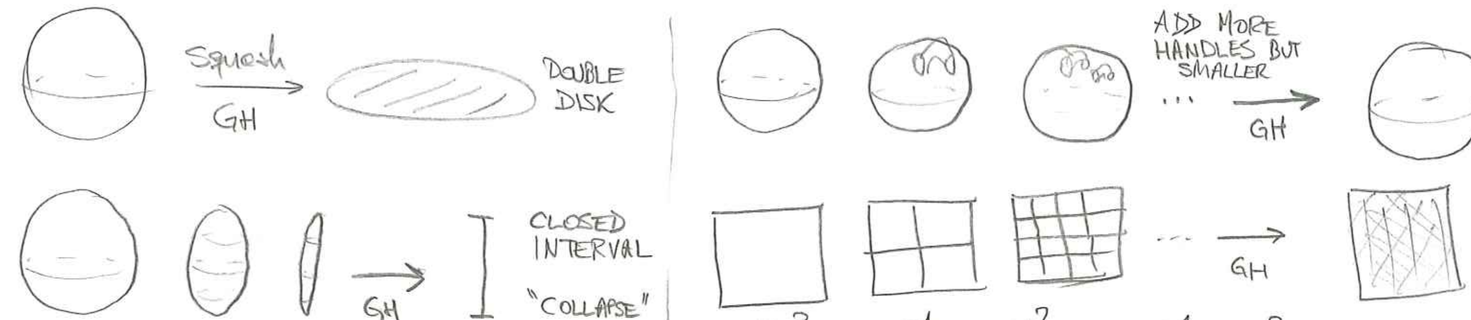
- GROMOV-HAUSDORFF DISTANCE: X, Y COMPACT METRIC SPACES

$$d_{GH}(X,Y) := \inf_{\substack{X \hookrightarrow Z \\ Y \hookrightarrow Z}} d_H^Z(X,Y)$$

GROMOV: $\mathcal{M} = \{\text{COMPACT METRIC SPACES}\} / \text{ISOMETRIES}$

$(\mathcal{M}, d_{GH})$ IS A PATH-CONNECTED COMPLETE METRIC SPACE!

EXAMPLES:



$T^2 \xrightarrow[GH]{} S^1, \quad K^2 \xrightarrow[GH]{} S^1, \quad K^2 \xrightarrow[GH]{} I$

- TOPOLOGY & DIMENSION MAY CHANGE IN THE LIMIT!
- BEST BEHAVED UNDER CURVATURE BOUNDS (Δ-COMPARISON SENSE)
- USED IN PERELMAN'S SOLUTION OF THE POINCARÉ CONJECTURE

THM (B.-DERDZINSKI-PICCIONE '2017). THE GROMOV-HAUSDORFF LIMIT OF FLAT MANIFOLDS IS A FLAT ORBIFOLD. CONVERSELY, EVERY FLAT ORBIFOLD IS THE GROMOV-HAUSDORFF LIMIT OF FLAT MANIFOLDS.

COR: $\{\text{FLAT ORBIFOLDS}\}/\text{ISOM.}$ IS CLOSED IN $(\mathcal{M}, d_{GH})$.

SO, MORALLY, $\partial \mathcal{M}_{flat}(M) \cong \left\{ \begin{array}{l} \text{FLAT ORBIFOLDS TO} \\ \text{WHICH } M \text{ COLLAPSES} \end{array} \right\}$ ⚠ BUT d_{GH} DOES NOT EXTEND CONTINUOUSLY FROM $\mathcal{M}_{flat}$ TO $\partial \mathcal{M}_{flat}$!

SKETCH OF PROOF:

$$M = \mathbb{R}^n/\pi \text{ COLLAPSES } \overset{\text{DIAMETER ESTIMATES}}{\Longleftrightarrow} T^n = \mathbb{R}^n/L_\pi \text{ COLLAPSES}$$

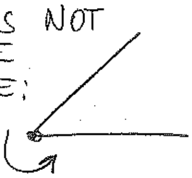
LEMMA: GH-LIMIT OF FLAT TORI IS A FLAT TORUS $\longleftarrow$ USING, E.G., [FUKAYA-YAMAGUCHI]

$$\begin{array}{ccccc} \pi \curvearrowright \mathbb{R}^n & & H_\pi \curvearrowright T^n & & H_\pi \curvearrowright T^k \\ \downarrow & \Longrightarrow & \downarrow & \xrightarrow[\text{COLLAPSE}]{} & \downarrow \\ M & & M & & \mathcal{O} = \lim M \end{array}$$

(MAY BE NOT FREE!)

CONVERSE:

ACTION ON TORUS NOT FREE HERE:

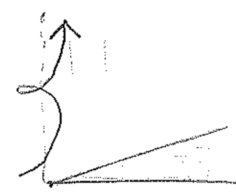


FLAT ORBIFOLD

ROTATION

"ADD DIMENSION & SCREW MOTION" $\rightsquigarrow$

$\longleftarrow$ COLLAPSE



FREE!

FLAT MANIFOLD

ACTUAL PROOF USES AUSLANDER-KURANISHI THM $\longleftarrow$ ANY FINITE GROUP IS THE HOLONOMY GROUP OF A CLOSED FLAT MANIFOLD.

$\square$

> THM (B.-DERDZINSKI-PICCIONE '2017). THE GROMOV-HAUSDORFF LIMIT OF A SEQUENCE OF FLAT 3-MANIFOLDS IS EITHER A:
>
> - FLAT 3-MANIFOLD; OR A COLLAPSING CASE:
> - $\{p\}$ FLAT 0-ORBIFOLD
> - I, S^1 FLAT 1-ORBIFOLD
> - T^2, K^2, $S^2 \times I$, M^2, $D^2(4;2)$, $D^2(3;3)$, $D^2(2,2;)$, $S^2(3,3,3;)$, $S^2(2,2,2,2;)$, $\mathbb{R}P^2(2,2;)$ FLAT 2-ORBIFOLD.
>
> 10 OUT OF 17 FLAT 2-ORBIFOLDS

Q: WHAT IS THE LOWEST DIMENSION OF A COLLAPSING SEQUENCE TO $\mathcal{O} = \mathbb{R}^n/\pi$?

ESTIMATES KNOWN IF H_π IS CYCLIC, DIHEDRAL, SEMI-DIHEDRAL, ELEMENTARY ABELIAN p-GROUP, GENERALIZED QUATERNION GROUP, SIMPLE $PSL(2,p)$, p PRIME

DIFFICULTY: COMPUTING $H^2(G)$ WITH SPECIAL COEFF.

EXAMPLE: $D^2(;2,2,2,2)$ IS THE GH-LIMIT OF $K^2 \times K^2$ (BUT NOT OF FLAT 3-MANIFOLDS