DEFORMING FLAT MANIFOLDS AND FLAT ORBIFOLDS 3/2017

L^2 (LAFAYETTE - LEHIGH) GEOMETRY - TOPOLOGY SEMINAR

OUTLINE:

1. FLAT MANIFOLDS, ORBIFOLDS AND GH-CONVERGENCE (w/ THM A AND BIEBERBACH THMS)
2. MODULI / TEICHMÜLLER SPACE (w/ THM B)
3. FLAT 2-ORBIFOLDS
4. LIMITS OF FLAT 3-MANIFOLDS (w/ THM D)

Why do we care?
- Apply to multiplicity results of Noncompact/singular Yamabe Problem.
- "Related" to question of density of sec≥0 mflds among Alex. spaces curv≥0

1. FLAT MANIFOLDS & ORBIFOLDS

- M FLAT MANIFOLD $\iff$ LOCALLY ISOMETRIC TO $\mathbb{R}^n$
- $\mathcal{O}$ FLAT ORBIFOLD $\iff$ LOCALLY ISOMETRIC TO $\mathbb{R}^n/\Gamma$ (Γ FINITE GROUP)
- π CRYSTALLOGRAPHIC GROUP IF $\pi \subset \mathrm{Iso}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ IS DISCRETE AND COCOMPACT
- π BIEBERBACH GROUP IF CRYSTALLOGRAPHIC AND TORSION-FREE, SO THAT $\pi \curvearrowright \mathbb{R}^n$ IS FREE.

THM. M FLAT MANIFOLD $\iff$ $M = \mathbb{R}^n/\pi$, π BIEBERBACH

$\mathcal{O}$ FLAT ORBIFOLD $\iff$ $\mathcal{O} = \mathbb{R}^n/\pi$, π CRYSTALLOGRAPHIC

IN PARTICULAR, FLAT ORBIFOLDS ARE "GOOD", I.E. GLOBAL QUOTIENT

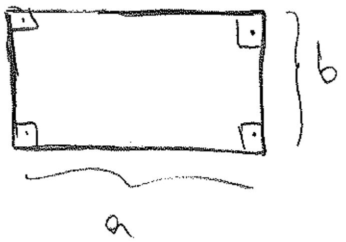
NOTE: $\pi_1(M) = \pi$, $\pi_1^{orb}(\mathcal{O}) = \pi$

FUNDAMENTAL GROUP, $\pi \curvearrowright \mathbb{R}^n$ DECK TRANSFORMATIONS.

EXAMPLE: $M = T^n = \mathbb{R}^n/\mathbb{Z}^n$ n-TORUS, $\pi = \mathbb{Z}^n \leftarrow$ translations in n lin. indep. directions

$\mathcal{O} = D^2(;2,2,2,2) = \mathbb{R}^2/\pi$, $\pi = \langle \underbrace{(x,y)\mapsto(x+a,y)}_{\text{translation } x}, \underbrace{(x,y)\mapsto(x,y+b)}_{\text{translation } y},$

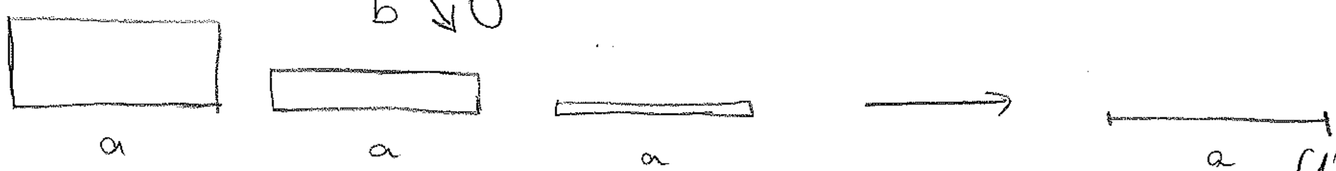
$\underbrace{(x,y)\mapsto(-x,-y)}_{\begin{pmatrix}-1&0\\0&-1\end{pmatrix}}, \underbrace{(x,y)\mapsto(x,-y)}_{\begin{pmatrix}1&0\\0&-1\end{pmatrix}} \rangle$

$\mathcal{O} =$  RECTANGLE.

$0 \to \mathbb{Z}^2 \to \pi \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 0$

NOTE: CAN "DEFORM" T^n AND $D^2(;2,2,2,2)$ IN NONTRIVIAL WAYS:

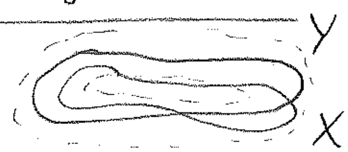
$b \searrow 0$



CLOSED INTERVAL (1-DIM FLAT ORBIFOLD)

Q: "DISTANCE" BETWEEN METRIC SPACES?

- **HAUSDORFF DISTANCE:** $X, Y \subset Z$



$$d_H^Z(X,Y) = \inf\{\varepsilon > 0 : B_\varepsilon(X) \supset Y,\ B_\varepsilon(Y) \supset X\}$$

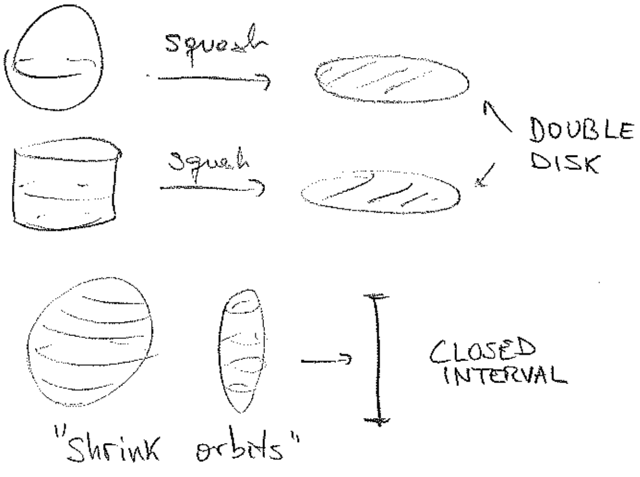
- **GROMOV-HAUSDORFF DISTANCE:** X, Y COMPACT METRIC SPACES

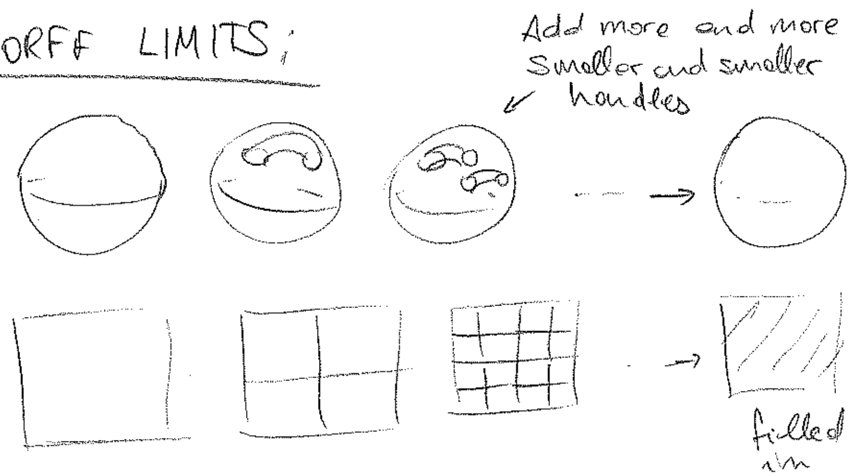
$$d_{GH}(X,Y) = \inf_{X,Y \hookrightarrow Z} d_H^Z(X,Y)$$

GROMOV: $\mathcal{M} = \{\text{COMPACT METRIC SPACES}\}/\text{ISOM.}$

$(\mathcal{M}, d_{GH})$ IS A METRIC SPACE.

EXAMPLES OF GROMOV-HAUSDORFF LIMITS:





CAN CHANGE TOPOLOGY, DIMENSION, ...

THM (B.-DERDZINSKI-PICCIONE). THE GH-LIMIT OF CLOSED FLAT MANIFOLDS IS A FLAT ORBIFOLD. CONVERSELY, EVERY FLAT ORBIFOLD IS THE GH-LIMIT OF CLOSED FLAT MANIFOLDS.

COR: $\{$FLAT ORBIFOLDS$\} \subseteq \mathcal{M}$ IS CLOSED (UNDER GH-LIMITS)

COR: FLAT ORBIFOLDS ADMIT "FLAT DESINGULARIZATIONS"

THROUGH HIGHER DIMENSIONAL OBJECTS...

- STARTING POINT FOR PROOF:

$\pi \subset \mathrm{Iso}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ CRYSTALLOGRAPHIC GROUP

$r: O(n) \ltimes \mathbb{R}^n \to O(n)$ ROTATIONAL PART

$$0 \longrightarrow \underset{\substack{\| \\ \mathrm{Ker}(r|_\pi)}}{L_\pi} \longrightarrow \pi \longrightarrow \underset{\substack{\| \\ r(\pi)}}{H_\pi} \longrightarrow 0$$ SHORT EXACT SEQUENCE

BIEBERBACH THM (ALGEBRAIC VERSION):

I. L_π IS A LATTICE THAT SPANS $\mathbb{R}^n$, AND $H_\pi \subset O(n)$ IS FINITE

II. $\pi, \pi' \subset \mathrm{Iso}(\mathbb{R}^n)$ CRYSTALLOGRAPHIC, $\phi: \pi \to \pi'$ ISOMORPHISM. THEN ϕ IS A CONJUGATION IN $\mathrm{Aff}(\mathbb{R}^n) = GL(n) \ltimes \mathbb{R}^n$

III. $\forall n$, $\exists$ ONLY FINITELY MANY ISOM. TYPES OF $\pi \subset \mathrm{Iso}(\mathbb{R}^n)$ CRYST.

BIEBERBACH THM (GEOMETRIC VERSION):

I. $\mathcal{O}^n$ CLOSED FLAT ORBIFOLD IS COVERED BY A FLAT TORUS $T^n = \mathbb{R}^n / L_\pi$

II. $\mathcal{O}, \mathcal{O}'$ FLAT ORBIFOLDS, $\dim \mathcal{O} = \dim \mathcal{O}'$. THEN $\mathcal{O} \underset{\substack{\text{AFF.} \\ \text{EQUIV.}}}{\cong} \mathcal{O}' \iff \pi_1^{orb}(\mathcal{O}) \cong \pi_1^{orb}(\mathcal{O}')$

III. $\forall n$, $\exists$ ONLY FINITELY MANY AFFINE TYPES OF CLOSED FLAT ORBIFOLDS.

NOTE. III SOLVED HILBERT'S 18^{th} PROBLEM (BIEBERBACH, 1911-1912)

MORE THAN 100 YEARS AGO!

H_π = HOLONOMY GROUP OF $\mathbb{R}^n/\pi$.

SKETCH OF PROOF THAT GH-LIMITS ARE FLAT ORBIFOLDS:

$$M = \mathbb{R}^n/\pi \text{ COLLAPSES } \underset{\text{DIAM. ESTIMATES}}{\Longleftrightarrow} T^n = \mathbb{R}^n/L_\pi \text{ COLLAPSES}$$

USING PREVIOUS WORK OF FUKAYA-YAMAGUCHI

LEMMA: GH-LIMIT OF FLAT TORI ARE FLAT TORI

CAN TAKE THE LIMIT OF ACTION

$$\begin{tikzcd} \pi \curvearrowright \mathbb{R}^n \arrow[d] & \Rightarrow & H_\pi \curvearrowright T^n = \mathbb{R}^n/L_\pi \arrow[d] \arrow[r, "\text{COLLAPSE}"] & H_\pi \curvearrowright T^k \arrow[d] \\ M & & M & \lim M \end{tikzcd}$$

THIS ACTION MAY BE NO LONGER FREE, SO $\lim M$ IS A FLAT ORBIFOLD

$\square$

2. MODULI / TEICHMÜLLER SPACES

Important in applications to the Yamabe problem and others...

$\boxed{\text{Q: CAN ALL FLAT MANIFOLDS/ORBIFOLDS COLLAPSE NONTRIVIALLY?}}$

FIX $\mathcal{O} = \mathbb{R}^n/\pi$ FLAT ORBIFOLD (COULD BE A FLAT MANIFOLD)

LET $\mathcal{M}_{flat}(\mathcal{O}) = \{\text{FLAT METRICS ON } \mathcal{O}\}/\text{ISOM.}$ MODULI SPACE

$H_\pi \curvearrowright \mathbb{R}^n$ ORTHOGONALLY.

$$\mathbb{R}^n = \bigoplus_{i=1}^{\ell} \underbrace{\left[\bigoplus_{j=1}^{m_i} V_i\right]}_{W_i}$$

ISOTYPIC COMPONENT W_i HAS m_i COPIES OF IRREDUCIBLE V_i, OF REAL, COMPLEX OR QUATERNIONIC TYPE

$$\mathbb{K}_i = \mathbb{R}, \mathbb{C}, \mathbb{H}$$

ACCORDINGLY TO $\mathrm{Aut}^{H_\pi}(V_i) \cong \mathbb{K}_i$

THM (B.-DERDZINSKI-PICCIONE). THERE EXISTS A TEICHMÜLLER SPACE

$$\mathcal{T}_{flat}(\mathcal{O}) = \prod_{i=1}^{\ell} \frac{GL(m_i, \mathbb{K}_i)}{O(m_i, \mathbb{K}_i)} \cong \mathbb{R}^d \quad \text{SUCH THAT} \quad \mathcal{M}_{flat}(\mathcal{O}) = \mathcal{T}_{flat}(\mathcal{O})/N_\pi$$

WHERE N_π IS A DISCRETE "MAPPING GROUP" AND $O(m, \mathbb{K})$ IS $O(m)$, $U(m)$ OR $Sp(m)$ ACCORDING TO $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

THE ABOVE BUILDS ON PREVIOUS WORK OF J. WOLF.

EXAMPLES

1) $T^n = \mathbb{R}^n/\mathbb{Z}^n$, $\quad \pi = \mathbb{Z}^n = L_\pi$, $\quad H_\pi = \{1\}$

$$\Longrightarrow \quad \mathcal{T}_{flat}(T^n) = \frac{GL(n,\mathbb{R})}{O(n)} \cong \mathbb{R}^{n(n+1)/2}$$

$\mathcal{T}_{flat}(S^1) \cong \mathbb{R}$

$\mathcal{T}_{flat}(T^2) \cong \mathbb{R}^3$

[Identified w/ pos.-def. matrices, via polar decomposition]

$$\& \quad \mathcal{M}_{flat}(T^n) = O(n) \backslash GL(n,\mathbb{R}) / GL(n,\mathbb{Z})$$

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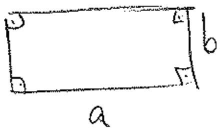
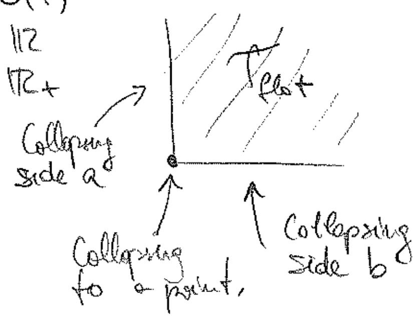
2) $D(;2,2,2,2)$ $\quad \mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ $\quad$ 2 ISOTYPIC COMPONENTS



$$\mathcal{T}_{flat}(D(;2,2,2,2)) = \underbrace{\frac{GL(1,\mathbb{R})}{O(1)}}_{\mathbb{R}_+} \times \underbrace{\frac{GL(1,\mathbb{R})}{O(1)}}_{\mathbb{R}_+} \cong \mathbb{R}^2$$



> **FACT:** IF $\pi < Iso(\mathbb{R}^n)$ IS A BIEBERBACH GROUP, THEN $H_\pi \curvearrowright \mathbb{R}^n$ IS REDUCIBLE. NOT NEC. TRUE FOR π CRYSTALLOGRAPHIC.

[HISS-SZCZEPAŃSKI '91]

$\Downarrow$

Note: This still doesn't imply they can collapse nontrivially...

> **COR:** FLAT MANIFOLDS CAN ALWAYS BE DEFORMED; WHILE FLAT ORBIFOLDS MAY BE RIGID!

3. FLAT 2-ORBIFOLDS: 17 AFFINE TYPES ⟷ 17 "WALLPAPER" GROUPS

#	$\mathcal{O}$	$\lvert\mathcal{O}\rvert$	H_π	dim $\mathcal{T}_{flat}(\mathcal{O})$	GH-LIMIT OF FLAT 3-MANIFOLD
1)	T^2 torus	T^2	$\{1\}$	3	✓
2)	K^2 Klein bottle	K^2	$\mathbb{Z}_2$	2	✓
3)	$S^1 \times I$ cylinder	$S^1 \times I$	$\mathbb{Z}_2$	2	✓
4)	M^2 Möbius band	M^2	$\mathbb{Z}_2$	2	✓

#	$\mathcal{O}$	$\lvert\mathcal{O}\rvert$	H_π	$\dim T_{flat}(\mathcal{O})$	
5)	$D^2(;3,3,3)$ equil. triangle	D^2	D_3	1	
6)	$D^2(;2,3,6)$ $\pi/3$ $\pi/6$	D^2	D_6	1	
7)	$D^2(;2,4,4)$ $\pi/4$ $\pi/4$	D^2	D_4	1	
8)	$D^2(;2,2,2,2)$ rectangle	D^2	D_2	2	
9)	$D^2(2;2,2)$ square $/\mathbb{Z}_2=\langle R_\pi\rangle$	D^2	D_2	2	
10)	$D^2(4;2)$ square $/\mathbb{Z}_4=\langle R_{\pi/2}\rangle$	D^2	D_4	1	✓
11)	$D^2(3;3)$ $/\mathbb{Z}_3=\langle R_{\pi/3}\rangle$	D^2	D_3	1	✓
12)	$D^2(2,2;)$ half pillowcase	D^2	D_2	2	✓
13)	$S^2(2,2,2,2;)$ pillowcase (double rectangle)	S^2	D_2	3	✓
14)	$S^2(3,3,3;)$ turnover (double triangle)	S^2	$\mathbb{Z}_3$	1	✓
15)	$S^2(2,3,6;)$ turnover	S^2	$\mathbb{Z}_6$	1	
16)	$S^2(2,4,4;)$ turnover	S^2	D_4	1	
17)	$\mathbb{RP}^2(2,2;)$ pillowcase$/\mathbb{Z}_2$	$\mathbb{RP}^2$	D_2	2	✓

4. LIMITS OF FLAT 3-MANIFOLDS

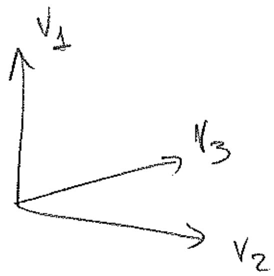
- THERE ARE <u>10 AFFINE TYPES OF</u> <u>FLAT 3-MANIFOLDS</u>
  (219 AFFINE TYPES OF FLAT 3-ORBIFOLDS!)

<u>THM</u> (B.-DEROZINSKI-PICCIONE). THE GROMOV-HAUSDORFF LIMIT OF A SEQUENCE OF CLOSED FLAT 3-MANIFOLDS IS EITHER A:

- CLOSED FLAT 3-MANIFOLD; OR A COLLAPSING CASE:
- $\{p\}$ FLAT 0-MANIFOLD
- I, S^1 FLAT 1-ORBIFOLD
- T^2, K^2, $S^1\times I$, M^2, $D^2(4;2)$, $D^2(3;3)$, $D^2(2,2;)$, $S^2(3,3,3;)$, $S^2(2,2,2,2;)$, $\mathbb{R}P^2(2,2;)$ FLAT 2-ORBIFOLD.

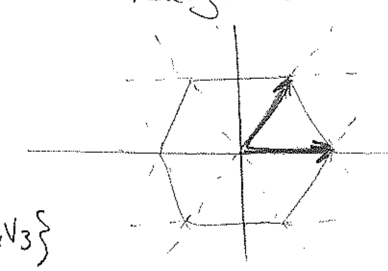
<u>EXAMPLE:</u> $L_\pi = \mathrm{span}_{\mathbb{Z}}\{v_1, v_2, v_3\}$, $v_1 \perp v_2, v_3$, $\|v_2\| = \|v_3\|$.

span an hexagonal lattice!



$$\pi = \langle L_\pi, (A, \tfrac{1}{2}v_1)\rangle$$

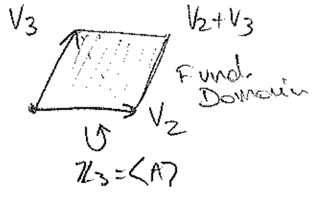
$$A = \begin{pmatrix} 1 & \\ & \left[R_{\frac{2\pi}{3}}\right] \end{pmatrix} \quad \{v_1, v_2, v_3\}$$



$$\mathbb{R}^3 = \underset{\substack{\| \\ \mathrm{span}\{v_1\}}}{W_1} \oplus \underset{\substack{\wr\| \\ \mathrm{span}\{v_2, v_3\}}}{W_2}$$

COLLAPSE $W_2 \rightsquigarrow$ GH-LIMIT IS S^1

COLLAPSE $W_1 \rightsquigarrow$ GH-LIMIT IS $T^2/\mathbb{Z}_3$, $T^2 = \mathbb{R}^2/L_\pi \circlearrowleft \mathbb{Z}_3 = \langle A\rangle$



$S^2(3,3,3;)$

APPENDIX:

$$\boxed{\text{THM (AUSLANDER-KURANISHI). ANY FINITE GROUP } H \text{ IS THE HOLONOMY GROUP } H_\pi \text{ OF A CLOSED FLAT MANIFOLD.}}$$

PF (CONVERSE STATEMENT IN THM A)

$$\mathcal{O} = \mathbb{R}^n/\pi_{\mathcal{O}}, \quad H := H_{\pi_{\mathcal{O}}} \overset{[AK]}{\Longrightarrow} \exists\, \pi_M \subset \mathrm{Iso}(\mathbb{R}^m) \text{ BIEBERBACH}$$

$$H_{\pi_M} = H.$$

$\Delta H \curvearrowright T^n \times T^m$ DIAGONAL ACTION (FREE), $T^n = \mathbb{R}^n/L_{\pi_{\mathcal{O}}}$, $T^m = \mathbb{R}^m/L_{\pi_M}$

$N = \dfrac{T^n \times T^m}{\Delta H}$ IS A FLAT MANIFOLD, $N \xrightarrow[\mathcal{G}H]{\text{Collapse } T^m} \mathcal{O}$. $\square$

$\left(g_\lambda = g_{T^n} \oplus \lambda\, g_{T^m} \text{ IS } H\text{-INVARIANT}\right)$