NON-UNIQUENESS OF CONFORMAL METRICS W/ CONSTANT Q-CURVATURE

JOINT W/ P. PICCIONE AND Y. SIRE

CHICAGO, OCT 2017

OUTLINE

1. Q-CURVATURE AND EXISTENCE PROBLEM
2. NON-UNIQUENESS VIA COVERINGS
3. NON-UNIQUENESS VIA BIFURCATION

1. Q-CURVATURE AND EXISTENCE PROBLEM

(M^n, g) RIEM. MANIFOLD, $n \geq 5$

$$Q_g = \frac{1}{2(n-1)} \Delta_g \mathrm{scal}_g - \frac{2}{(n-2)^2} \|\mathrm{Ric}_g\|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} \mathrm{scal}_g^2$$

MOTIVATIONS:
- FUNCTIONAL DETERMINANTS / SCATTERING THEORY FOR CONFORMALLY COMPACT EINSTEIN MANIFOLDS [FEFFERMAN-GRAHAM] ← CHARLES, NOT ROBERT...
- BEST SOBOLEV CONSTANT

- $H^1(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ CRITICAL SOBOLEV EMBEDDING:

$$\|\varphi\|^2_{L^{\frac{2n}{n-2}}} \leq \sigma_{1,n} \cdot \|\nabla \varphi\|^2_{L^2}, \quad \sigma_{1,n} = \frac{4\,\frac{n-1}{n-2}}{Y(S^n)} = \frac{4\,\frac{n-1}{n-2}}{n(n-1)\,\mathrm{Vol}(S^n)^{\frac{2}{n}}}$$

- $H^2(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-4}}(\mathbb{R}^n)$ CRITICAL SOBOLEV EMBEDDING:

$$\|\varphi\|^2_{L^{\frac{2n}{n-4}}} \leq \sigma_{2,n} \|\Delta \varphi\|^2_{L^2},$$

$\sigma_{2,n}$ IS KNOWN [LIONS, EDMUNDS-FORTUNATO-JANNELLI]

- $\sigma_{2,n}$ IS ATTAINED AT RADIAL FUNCTIONS $u: \mathbb{R}^n \to \mathbb{R}$ S.T.

$$(*)_{\mathbb{R}^n} \quad \Delta^2 u = u^{\frac{n+4}{n-4}} \longleftarrow \boxed{\text{CONFORMALLY INVARIANT}}$$

INVARIANT UNDER:
- TRANSLATIONS $u(x) \mapsto u(x+v)$
- DILATIONS $u(x) \mapsto t^{\frac{n-4}{2}} u(tx)$
- INVERSIONS $u(x) \mapsto |x|^{4-n} u\left(\frac{x}{|x|^2}\right)$

- MINIMIZERS ARE POSITIVE [VAN DER VORST]

Q: ANALOGOUS TO $(*)_{\mathbb{R}^n}$ FOR RIEMANNIAN MANIFOLDS?

THM (PANEITZ, 1983). GIVEN (M^n, g), $n \geq 5$, THERE EXISTS A DIFF. OP. $P_g = \Delta_g^2$ + LOWER ORDER TERMS SUCH THAT:

$$\tilde{g} = u^{\frac{4}{n-4}} g \implies P_{\tilde{g}} \varphi = u^{-\frac{n+4}{n-4}} P_g(u\varphi)$$

EXAMPLES:
$P_{\mathbb{R}^n} = \Delta_{\mathbb{R}^n}^2$
$P_{S^n} = \Delta_{S^n}(\Delta_{S^n} - c)$

$$P_g \varphi = \Delta_g^2 \varphi + \frac{4}{n-2} \operatorname{div}_g\left(\operatorname{Ric}_g(\nabla\varphi, e_i) e_i\right) - \frac{n^2 - 4n + 8}{2(n-1)(n-2)} \operatorname{div}_g(\operatorname{Scal}_g \nabla\varphi) + \frac{n-4}{2} Q_g \cdot \varphi$$

Q_g IS THE ZEROTH ORDER TERM: $\quad Q_g = \frac{2}{n-4} P_g(1)$

ANALOGOUS TO THE YAMABE PROBLEM:

CONSTANT Q-CURVATURE PROBLEM: GIVEN (M^n, g_0), $n \geq 5$, FIND A COMPLETE CONFORMAL METRIC $g \in [g_0]$ SUCH THAT $Q_g = \text{const}$.

$\Updownarrow$

FIND $u: M \to \mathbb{R}$, $u > 0$, $u \nearrow +\infty$ FAST ENOUGH (IF M NONCOMPACT) SUCH THAT

$$P_{g_0} u = \lambda \cdot u^{\frac{n+4}{n-4}}, \quad \lambda = \frac{n-4}{2} Q_g$$

$\Updownarrow \leftarrow \boxed{\text{M COMPACT}}$

FIND $u: M \to \mathbb{R}$, $u > 0$, CRITICAL POINT OF $E: \{u \in H^2(M) : \|u\|_{L^{\frac{2n}{n-4}}} = \text{const}\} \to \mathbb{R}$,

$$E(u) = \int_M u\, P_{g_0} u \ \mathrm{vol}_M$$

STATUS:

EXISTENCE: KNOWN IN SOME CASES [A. CHANG, M. GURSKY, F. HANG, P. YANG]

⚠
- MINIMIZING SEQUENCE MAY CONVERGE (WEAKLY) TO 0 (NONCPCT EMBEDDING)
- MINIMIZERS MAY NOT BE POSITIVE (FOURTH ORDER/NO MAX. PRINCIPLE)

UNIQUENESS (* TODAY'S TALK!): FAILS IN FUNDAMENTAL WAYS...

TO DISCUSS EXISTENCE, DEFINE:

NOT KNOWN IN GENERAL WHEN $Y_4 > 0$

$$Y_4(M, g_0) = \inf_{u \in H^2(M) \setminus \{0\}} \frac{E(u)}{\|u\|^2_{L^{\frac{2n}{n-4}}}}$$

HANG-YANG' 2016: IF $Y(M, g_0) > 0$ AND $Q_{g_0} \geq 0$, $Q_{g_0}(p) > 0$, THEN $P_{g_0} > 0$ (AND $\operatorname{Ker} P_{g_0} = \{0\}$).

- GREEN'S FUNCTION $G_p > 0$ ON $M \times M$

$P(G_p(x,y)) = \delta_x(y)$

- INVERSE OF P_{g_0}:

$$(G_p f)(p) = \int_M G_p(p,q) f(q) \, dq$$

- NEW INVARIANT $\left(\simeq (Y_4^+)^{-1}\right)$:

HARDY-LITTLEWOOD-SOBOLEV INEQUALITY

$$\Theta_4(M, g_0) = \sup_{f \in L^{\frac{2n}{n+4}}} \frac{\int_M G_p f \cdot f}{\|f\|^2_{L^{\frac{2n}{n+4}}}} < \infty \quad \left(f = u^{\frac{n+4}{n-4}}\right)$$

- ABOVE SUPREMUM IS ACHIEVED AT SOME $f \in L^{\frac{2n}{n+4}}$, $f > 0$ (B/C $G_p > 0$) AND $g = f^{\frac{4}{n-4}} g_0$ HAS $Q_g = \text{const}$.

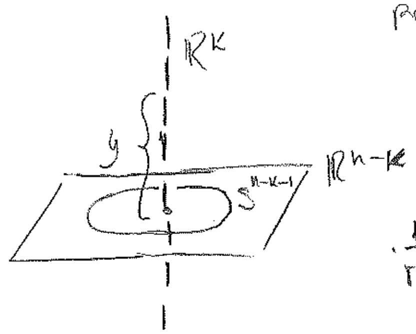
- AUBIN-TYPE INEQUALITY: $\Theta_4(M^n, g_0) \geq \Theta_4(S^n, g_{\text{round}})$

2. NON-UNIQUENESS VIA COVERINGS

THM (B.-PICCIONE-SIRE). THERE ARE INFINITELY MANY NON-HOMOTHETIC COMPLETE CONFORMAL METRICS ON $(S^n \setminus S^k, g_{\text{round}})$, $0 \leq k < \frac{n-4}{2}$ WITH CONSTANT Q-CURVATURE.

SKETCH OF PROOF:

$$(S^n \setminus S^k, g_{round}) \xrightarrow[\text{stereog. proj}]{\cong} (\mathbb{R}^n \setminus \mathbb{R}^k, g_{flat}) \xrightarrow[\cdot \frac{1}{r^2}]{\cong} (S^{n-k-1} \times \mathbb{H}^{k+1}, g_{prod})$$



$$g_{flat} = dr^2 + r^2 d\theta^2 + dy^2$$

$$\Big\downarrow \cdot \frac{1}{r^2}$$

$$d\theta^2 + \frac{dr^2 + dy^2}{r^2} = g_{S^{n-k-1}} \oplus g_{\mathbb{H}^{k+1}} =: g_{prod}.$$

NOTE: $Q_{g_{prod}} > 0 \iff \underbrace{0 \leq k < \frac{n-4}{2}}$ OR $k > \frac{n}{2}$

$\mathrm{scal}_{g_{prod}} > 0 \iff 0 \leq k < \frac{n-2}{2} \leftarrow$ NEEDED TO QUOTE [HANG-YANG] EXISTENCE THM

LET $\Sigma^{k+1} = \mathbb{H}^{k+1}/\Gamma$ COMPACT HYPERBOLIC MANIFOLD.

$\pi_1(\Sigma) \cong \Gamma$ IS RESIDUALLY FINITE $\Rightarrow$ INFINITE CHAIN OF COVERINGS:

$$\Sigma = \Sigma_1 \leftarrow \Sigma_2 \leftarrow \cdots \leftarrow \Sigma_j \leftarrow \cdots \leftarrow \mathbb{H}^{k+1}$$

$$\Theta_4(S^{n-k-1} \times \Sigma_j^{k+1}, \tilde{g}) = c \cdot \mathrm{Vol}(S^{n-k-1} \times \Sigma_j^{k+1})^{-\frac{4}{n}} \searrow 0 \text{ AS } j \nearrow +\infty$$

$j \gg 1 \Rightarrow$ VIOLATES AUBIN INEQUALITY $\Rightarrow$ NEW SOLUTION (Θ_4- MAXIMIZER)

ITERATE, PULL BACK METRICS TO $S^n \setminus S^k$. $\square$

MORE GENERALLY: (M, g) CLOSED MANIFOLD, (N, h) SYMMETRIC SPACE OF NONCOMPACT OR EUCLIDEAN TYPE, S.T. $(M \times N, g \oplus h)$ HAS $\mathrm{scal} \geq 0$ AND $Q \geq 0$ NOT IDENTICALLY ZERO. THEN THERE ARE INFINITELY MANY NONHOMOTHETIC SOLUTIONS ON $(M \times N, g \oplus h)$.

3. NON-UNIQUENESS VIA BIFURCATION

- $F \to M \xrightarrow{\pi} B$ RIEM. SUBMERSION W/ TOTALLY GEODESIC FIBERS
- $Ric_F = \Lambda_F g_F$, $Ric_B = \Lambda_B g_B$, (F & B ARE EINSTEIN)
- A - TENSOR SATISFIES $(AU, AV) = \sum_i \langle A_{X_i} U, A_{X_i} V \rangle = \eta \cdot g(U,V)$

$$(A_X, A_Y) = \sum_j \langle A_X U_j, A_Y U_j \rangle = \rho \cdot g(X,Y)$$

THEN: (M, g_t), $g_t = t g_F \oplus g_B$ HAS CONSTANT Q-CURVATURE

(AND CONSTANT SCALAR CURVATURE)

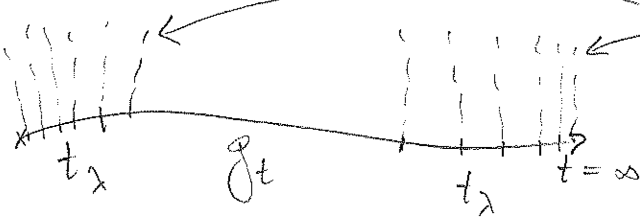
THM (B.-PICCIONE-SIRE). IN THE ABOVE SETUP:

(1) IF $\Lambda_F > 0$, THEN $\exists t_\lambda \searrow 0$ SEQ. OF BIFURCATION INSTANTS FOR g_t

(2) IF $\eta/\rho \approx 1$, THEN $\exists t_\lambda \nearrow +\infty$ SEQ. OF BIFURCATION INSTANTS FOR g_t

MOREOVER, BIFURCATING SOLUTIONS SUFFICIENTLY CLOSE TO g_t DO NOT HAVE CONSTANT SCALAR CURVATURE.

FIRST EXAMPLES OF CLOSED MFLD W/ $Q = const$ BUT $scal \neq const$



SEQUENCES OF OTHER METRICS W/ $Q = const$.

IN MANY CASES, $t \nearrow +\infty \Rightarrow \underbrace{Q_t \searrow -\infty \text{ AND } scal_t \searrow -\infty}$

VERY DIFFERENT FROM YAMABE PROBLEM

⚠ UNIQUENESS FAILS EVEN IN THIS REGIME; EPITOMIZES LACK OF MAXIMUM PRINCIPLE

EXAMPLES: HOPF BUNDLES:

$S^1 \to S^{2n+1} \to \mathbb{C}P^n$ (2) HOLDS IF $n \geq 6$ $Q_t \downarrow -\infty$ AS $t \nearrow +\infty$ IF $n \leq 9$	$S^3 \to S^{4n+3} \to \mathbb{H}P^n$ (1) & (2) HOLD $\forall n \geq 2$ $Q_t \downarrow -\infty$ AS $t \nearrow +\infty$ IF $n=2$	$S^7 \to S^{15} \to S^8(\frac{1}{2})$ (1) & (2) HOLD

$\mathbb{C}P^1 \to \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$ (1) HOLDS $\forall n \geq 2$ (2) HOLDS $\forall n \geq 3$ $Q_t \downarrow -\infty$ AS $t \nearrow +\infty$ IF $n=3$

SKETCH OF PROOF:

- $C^\infty(B) \xhookrightarrow[\text{HORIZ. LIFT}]{} C^\infty(M)$ NATURAL CONSTRAINT:

$$u \in C^\infty(B): \quad dE(u) = 0 \iff dE|_{C^\infty(B)}(u) = 0.$$

Q_t IS CONSTANT ALONG FIBERS AND $C^\infty(M) = C^\infty(B) \oplus \left\{ f \in C^\infty(M) : \forall b \in B, \int_{\pi^{-1}(b)} f = 0 \right\}$

- EULER-LAGRANGE EQN FOR $dE|_{C^\infty(B)}(u)=0$ IS SUBCRITICAL

- HYPOTHESES ON Ric $\Rightarrow P_t|_{C^\infty(B)} = p(\Delta)$ $\quad p(x) = ax^2+bx+c$ IS A (QUADRATIC) POLYNOMIAL IN THE LAPLACIAN.

- ASYMPTOTIC EXPANSION OF $\operatorname{Spec}(P_t) = \{ p(\lambda) : \lambda \in \operatorname{Spec}(\Delta_t) \}$ AND Q_t AS $t \downarrow 0$ AND $t \nearrow +\infty$

$\Rightarrow$ $\boxed{\text{MORSE INDEX OF } u=1 \text{ AS CRITICAL POINT OF } E_t : C^\infty(M) \to \mathbb{R} \text{ EXPLODES } i_{Morse}(1) \nearrow +\infty \text{ AS } t \downarrow 0 \text{ OR } t \nearrow +\infty}$ $\xRightarrow{\text{KRASNOSEL'SKII THEOREM}}$ $\boxed{\exists\, t_\lambda \downarrow 0 \text{ SEQUENCE OF BIFURCATION INSTANTS FOR } g_t}$

$\hookrightarrow$ I.E., IMPLICIT FUNCTION THM FAILS FOR g_t, $t \approx t_\lambda$.

$\square$