DARTMOUTH GEOMETRY - TOPOLOGY SEMINAR 4/5/2017

NON-UNIQUENESS RESULTS FOR THE YAMABE AND Q-CURVATURE PROBLEMS

OUTLINE

1. YAMABE AND Q-CURVATURE PROBLEMS

2. BIFURCATION OF SOLUTIONS

3. APPLICATIONS ON NONCOMPACT MANIFOLDS

1. YAMABE AND Q-CURVATURE PROBLEMS

MOTIVATION:

> UNIFORMIZATION THM: FOR ALL (M^2, g_0), THERE EXISTS A COMPLETE CONFORMAL METRIC $g \in [g_0]$ SUCH THAT (M^2, g) HAS CONSTANT CURVATURE.

GENERALIZATIONS TO HIGHER DIMENSION:

(1)

> YAMABE PROBLEM: GIVEN (M^n, g_0), FIND A COMPLETE METRIC $g \in [g_0]$ WITH CONSTANT SCALAR CURVATURE: $\mathrm{scal}_g = \sum_{i,j} \sec(e_i, e_j)$

$\Updownarrow$

> FIND $u: M \to \mathbb{R}$, $u > 0$, AND $u \nearrow +\infty$ "FAST ENOUGH" SOLVING:
>
> $$4\frac{n-1}{n-2}\Delta_{g_0} u - \mathrm{scal}_{g_0} \cdot u = \mathrm{scal}_g \cdot u^{\frac{n+2}{n-2}} \quad \leadsto \quad g = u^{\frac{4}{n-2}} \cdot g_0$$

$\Updownarrow$ ⟵ IF M IS COMPACT!

> FIND CRITICAL POINT $g \in [g_0]_1$ OF THE FUNCTIONAL
>
> $$\mathcal{A} : [g_0]_1 \longrightarrow \mathbb{R}, \qquad \mathcal{A}(g) = \int_M \mathrm{scal}_g \cdot \mathrm{vol}_g$$

STATUS:

M COMPACT

- EXISTENCE: YES [YAMABE, TRUDINGER, AUBIN, SCHOEN]
- UNIQUENESS: NO* (BUT YES IF $\mathrm{scal}_{g_0} \leq 0$; OR IF (M, g_0) IS EINSTEIN AND NOT ROUND SPHERE. ALSO GENERICALLY)

M NONCOMPACT

- EXISTENCE: NO [JIN] (BUT YES IF $M(\infty)$ IS TAME ENOUGH... [MAZZEO–PACARD])
- UNIQUENESS: NO*

* = SUBJECT OF TODAY'S TALK!

☆ RELATED TO "BEST SOBOLEV CONSTANT" FOR $W^{1,2}(M) \hookrightarrow L^{\frac{2n}{n-2}}(M)$

HOW ABOUT OTHER SOBOLEV EMBEDDINGS, SAY $W^{2,2}(M) \hookrightarrow L^{\frac{2n}{n-4}}(M)$?

$W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ IF $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$

(2) Q-CURVATURE PROBLEM: GIVEN (M^n, g_0) FIND A COMPLETE METRIC $g \in [g_0]$ WITH CONSTANT Q-CURVATURE:

$$Q_g = -\frac{1}{2(n-1)} \Delta_g \mathrm{scal}_g - \frac{2}{(n-2)^2} |\mathrm{Ric}_g|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2 (n-2)^2} \mathrm{scal}_g^2 \quad \triangle!$$

$\Updownarrow$

FIND $u : M \to \mathbb{R}$, $u > 0$, AND $u \nearrow +\infty$ "FAST ENOUGH" SOLVING:

$$\underbrace{\frac{2}{n-4} \Delta_{g_0}^2 u + \frac{8}{(n-2)(n-4)} \mathrm{div}\left(\mathrm{Ric}_{g_0}(\nabla u, e_i) e_i\right) - \frac{n^2 - 4n + 8}{(n-1)(n-2)(n-4)} \mathrm{div}\left(\mathrm{scal}_{g_0} \nabla u\right)}_{\frac{2}{n-4} \cdot P_g(u)} + Q_{g_0} u = Q_g u^{\frac{n+4}{n-4}}$$

"PANEITZ OPERATOR" ← [INTRODUCED BEFORE Q-CURVATURE]

$\Updownarrow$ ← IF M IS COMPACT!

FIND CRITICAL POINT $g \in [g_0]_1$ OF THE FUNCTIONAL

$$\mathcal{Q} : [g_0]_1 \longrightarrow \mathbb{R}, \quad \mathcal{Q}(g) = \int_M Q_g \cdot \mathrm{vol}_g$$

STATUS:

M COMPACT
- EXISTENCE: IN MOST CASES, YES [A. CHANG, M. GURSKY, F. HANG, P. YANG]
- UNIQUENESS: NO*

M NONCOMPACT
- EXISTENCE: ??? [PROBABLY NO]
- UNIQUENESS. NO*

* = SUBJECT OF TODAY'S TALK

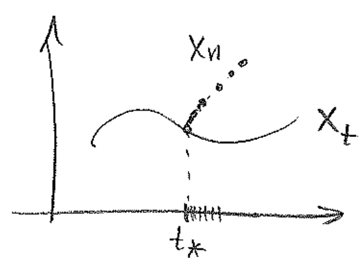
2. BIFURCATION OF SOLUTIONS

$f_t : X \longrightarrow \mathbb{R}$ 1-PARAMETER FAMILY OF FUNCTIONALS

$x_t \in X$ "TRIVIAL BRANCH" OF SOLUTIONS: $df_t(x_t) = 0$

DEF: BIFURCATION OCCURS AT t_* IF $\exists\, t_n \longrightarrow t_*$, $\exists\, x_n \longrightarrow x_{t_*}$ S.T. $df_{t_n}(x_n) = 0$ AND $x_n \neq x_{t_n}$

I.E., THE IMPLICIT FUNCTION THEOREM "FAILS" AT x_{t_*}.



• DEGENERACY AT x_{t_*} IS NECESSARY BUT NOT SUFFICIENT!

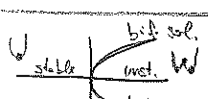
THM (KRASNOSEL'SKII). ASSUME x_t IS SUCH THAT $df_t(x_t) = 0$ AND:

(i) $\exists\, a < b$ S.T. x_a AND x_b ARE NONDEGENERATE CRITICAL POINTS WITH $i_{Morse}(x_a) \neq i_{Morse}(x_b)$

(ii) $d^2 f_t$ IS A FREDHOLM OPERATOR OF INDEX ZERO.

THEN $\exists\, t_* \in (a,b)$ A BIFURCATION INSTANT FOR x_t.

SILLY EXAMPLE: $f_t : \mathbb{R}^2 \to \mathbb{R}^2$, $f_t(x,y) = x^2 + y^4 - ty^2$, $df_t(x,y) = (2x, 4y^3 - 2ty)$, $d^2 f_t(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2t \end{pmatrix}$, BIFURCATING BRANCH $(x,y) = (0, \pm\sqrt{t/2})$. $i_{Morse}(0,0) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$



Q: WHAT IS $i_{Morse}(g)$ FOR A CRITICAL POINT $g \in [g_0]_1$?

A: FOR THE YAMABE PROBLEM, IF $dA(g) = 0$, THEN:

$$d^2 A(g)(\psi,\psi) = \int_M \left(\Delta_g \psi - \frac{scal_g}{n-1}\psi\right)\psi$$

$$\Longrightarrow \boxed{i^{Y}_{Morse}(g) = \# \, Spec(\Delta_g) \cap \left(-\infty, \frac{scal_g}{n-1}\right)}$$

FOR THE Q-CURVATURE PROBLEM, IF $dQ(g) = 0$, THEN:

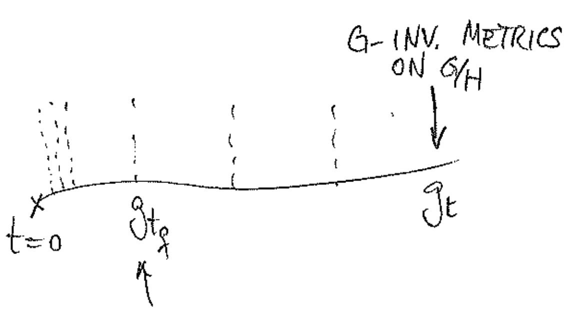
$$d^2 Q(g)(\psi,\psi) = \int_M \left(\frac{1}{2} P_g \psi - \frac{n+4}{4} Q_g \psi\right)\psi$$

$$\Longrightarrow \boxed{i^{Q}_{Morse}(g) = \# \, Spec(P_g) \cap \left(-\infty, \frac{n+4}{2} Q_g\right)}$$

THM (B.-PICCIONE). LET $H < K < G$ BE COMPACT LIE GROUPS WITH EITHER $H \triangleleft K$ OR $K \triangleleft G$, AND $scal_{K/H} > 0$. LET g_t BE THE HOMOGENEOUS METRICS ON G/H OBTAINED BY RESCALING THE FIBERS OF

$$K/H \longrightarrow G/H \longrightarrow G/K$$

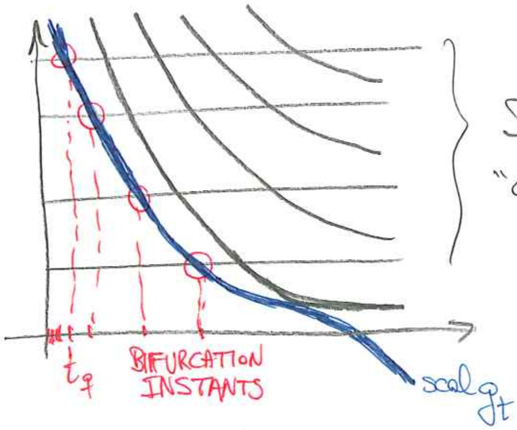
THEN $\exists \, t_q \searrow 0$ SEQUENCE OF BIFURCATION INSTANTS FOR YAMABE PROBLEM



EACH g_{t_q} IS A LIMIT OF A SEQUENCE OF OTHER METRICS W/ CONST. SCAL. (K-INVARIANT)

EXAMPLE:
$$tS^3 \longrightarrow S^{4n+3}_t \longrightarrow \mathbb{H}P^n$$
$$tS^7 \longrightarrow S^{15}_t \longrightarrow S^8(1/2)$$

SKETCH OF PROOF:



$\mathrm{Spec}(\Delta_{G/K})$
"CONSTANT EIGENVALUES"

$K/H \to (G/H, g_t) \longrightarrow G/K$ HAS TOTALLY GEODESIC FIBERS, HENCE

$$\underbrace{\mathrm{Spec}(\Delta_{G/K})}_{\text{INDEPENDENT OF } t} \subset \mathrm{Spec}(\Delta_{g_t})$$

AND $\mathrm{scal}_{g_t} \nearrow +\infty$ AS $t \searrow 0$.

THUS, $i^{Y}_{\mathrm{Morse}}(g_t) \nearrow +\infty$ AS $t \searrow 0$, AND THUS GET BIFURCATION INSTANTS. $\square$

⚠ COMPENSATION ISSUE: USE EQUIVARIANT BIFURCATION AND $H \trianglelefteq K$ OR $K \trianglelefteq G$ (OR MAYBE THE IDEAS OF [OTOBA - PETEAN]?)

THM. (B. - PICCIONE - SIRE) IN THE SAME SETUP, UNDER SOME EXTRA CONDITIONS INVOLVING $\mathrm{scal}_{K/H}$, $\|\mathrm{Ric}_{K/H}\|$, A, THERE EXIST SEQUENCES $t_a \searrow 0$ AND $t_b \nearrow +\infty$ OF BIFURCATION INSTANTS FOR Q-CURVATURE PROBLEM.

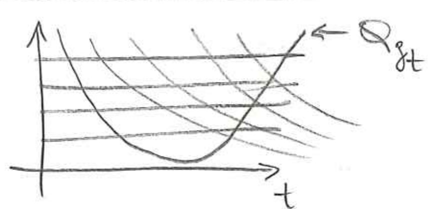
SKETCH OF PROOF: $P_{g_t} = \alpha \Delta^2_{g_t} + \beta \Delta_{g_t} + \gamma =: p(\Delta_{g_t})$ IS POLYNOMIAL ON Δ_{g_t}

$$\Rightarrow \mathrm{Spec}(P_{g_t}) = \{p(\lambda) : \lambda \in \mathrm{Spec}(\Delta_{g_t})\} \supset \underbrace{\{p(\lambda) : \lambda \in \mathrm{Spec}(\Delta_{G/K})\}}_{\text{"CONSTANT EIGENVALUES"}}$$

AND $Q_{g_t} \nearrow +\infty$ AS $t \searrow 0$ AND AS $t \nearrow +\infty$.

$\Longrightarrow i^{Q}_{\mathrm{Morse}}(g_t) \nearrow +\infty$ AS $t \searrow 0$ AND AS $t \nearrow +\infty$, THUS GET BIFURCATION. $\square$

REMARK: THE ABOVE APPLIES TO $tS^3 \to S^{4n+3}_t \to \mathbb{H}P^n$, AND PRODUCES MULTIPLE METRICS WITH CONSTANT Q-CURVATURE WHEN $\mathrm{scal} < 0$, THUS WHEN SOLUTION TO YAMABE PROBLEM IS UNIQUE.
IN PARTICULAR, THIS PRODUCES g WITH $Q_g = \mathrm{const.}$ BUT $\mathrm{scal}_g \neq \mathrm{const.}$



3. APPLICATIONS ON NON COMPACT MANIFOLDS

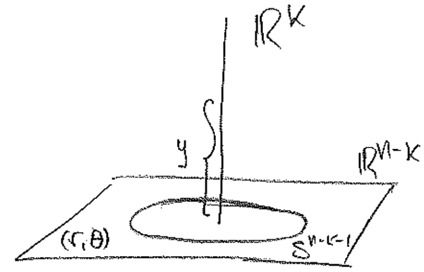
THM (B.–PICCIONE–SANTORO 2015, B.–PICCIONE 2016). THERE EXIST INFINITELY MANY PAIRWISE NONHOMOTHETIC COMPLETE CONSTANT SCALAR CURVATURE METRICS ON $S^n \setminus S^k$, $0 \leq k < \frac{n-2}{2}$ THAT ARE CONFORMAL TO THE ROUND (INCOMPLETE) METRIC.

I.E., SOLUTIONS TO THE "SINGULAR YAMABE PROBLEM"

SKETCH OF PROOF:

$$S^n \setminus S^k \longrightarrow \mathbb{R}^n \setminus \mathbb{R}^k \longrightarrow S^{n-k-1} \times \mathbb{H}^{k+1}$$

$$g_{round} \xrightarrow[\text{projection}]{\text{stereographic}} \underbrace{dr^2 + r^2 d\theta^2 + dy^2}_{g_{flat}} \xrightarrow{\cdot \frac{1}{r^2}} \underbrace{d\theta^2 + \frac{dr^2 + dy^2}{r^2}}_{g_{prod}}$$



PULLING BACK BY ABOVE CONFORMAL EQUIVALENCE, GET "TRIVIAL SOLUTION" (ONLY ONE PREVIOUSLY KNOWN) WITH $\mathrm{scal} = (n-2k-2)(n-1)$.

SUPPOSE $k=1$: $\quad S^n \setminus S^1 \cong S^{n-2} \times \mathbb{H}^2$

$$\downarrow$$

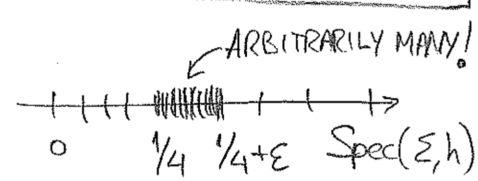
$\left(\Sigma_t^2 = \mathbb{H}^2 / \Gamma_t \text{ HYP. SURFACES} \right) \quad S^{n-2} \times \Sigma_t^2$

IDEA: BIFURCATE BY DEFORMING HYPERBOLIC STRUCTURE ON A COMPACT QUOTIENT!

$$i^Y_{Morse}(g_{round} \oplus h_t) = \#\Big\{ \lambda_i(S^{n-2}, g_{round}) + \underbrace{\lambda_j(\Sigma, h_t)}_{\text{Only part that varies with } t!} < \underbrace{n-4}_{= \frac{\mathrm{scal}}{n-1}} \Big\}$$

THM (BUSER '77). $\forall \varepsilon > 0$, $j \in \mathbb{N}$, $\exists h \in \mathcal{H}(\Sigma)$ WITH $\lambda_j(\Sigma, h) < \frac{1}{4} + \varepsilon$.

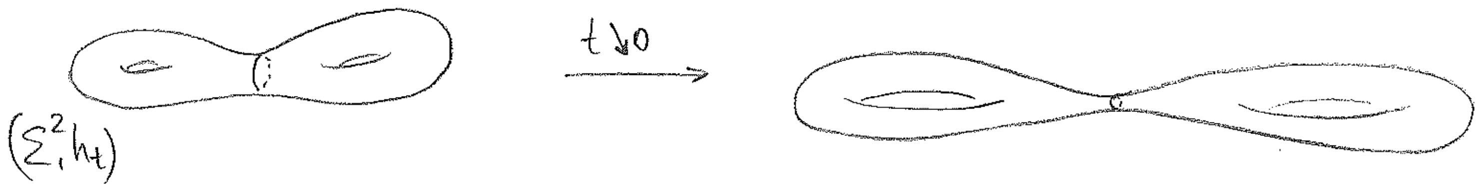
SO CAN MAKE $i^Y_{Morse}(g_t) \nearrow \infty$ AND GET INFINITELY MANY SOLUTIONS!



NOTE: DIFFERENT CONFORMAL CLASSES IN THE QUOTIENT, BUT PULLBACK TO SAME ONE ON $S^n \setminus S^1$

⚠ NON DEGENERACY ISSUE: WOLPERT

GEOMETRICALLY: PINCH OFF CLOSED GEODESIC (SYSTOLE $\searrow 0$)



FOR $K \geq 2$: MOSTOW RIGIDITY $\Rightarrow$ CANNOT APPLY SAME IDEA...

$\rightsquigarrow$ USE ANOTHER ARGUMENT WITH FINITE COVERINGS

$\Sigma = \mathbb{H}^{K+1}/\Gamma \Rightarrow \pi_1(\Sigma)$ IS RESIDUALLY FINITE

$\Rightarrow$ LOTS OF FINITE COVERINGS. $\square$

> THM (B.-PICCIONE-SIRE). SAME HOLDS FOR CONSTANT Q-CURVATURE.

COR: THERE ARE INFINITELY MANY CONFORMAL (NON-HOMOTHETIC) METRICS ON $\mathbb{H}^d \times S^n$ WITH CONSTANT SCALAR CURVATURE/CONSTANT Q-CURVATURE

Q: HOW ABOUT $\mathbb{R}^d \times S^n$?

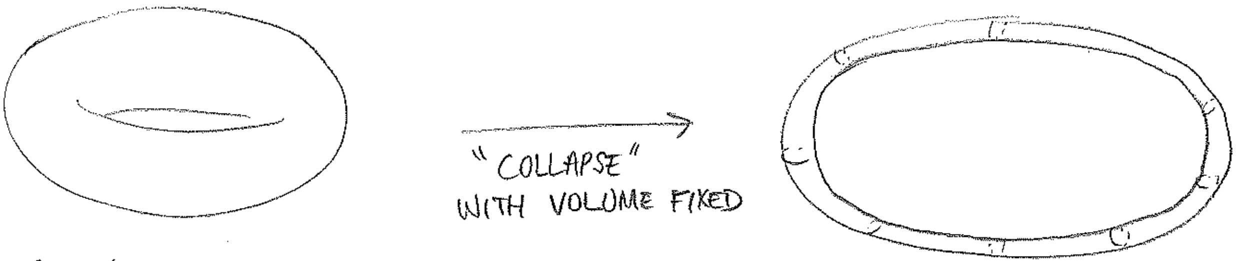
> THM (B.-PICCIONE). LET (M^n, g) BE A CLOSED MANIFOLD WITH $\mathrm{scal}_g = \mathrm{const} > 0$ AND $\pi \subset \mathrm{Iso}(\mathbb{R}^d)$ A BIEBERBACH GROUP, $d \geq 2$. THEN THERE EXIST INFINITELY MANY BRANCHES OF π-PERIODIC SOLUTIONS TO THE YAMABE PROBLEM ON $(M \times \mathbb{R}^d, g \oplus g_{flat})$.

> THM (B.-PICCIONE-SIRE). SAME HOLDS FOR CONSTANT Q-CURVATURE.

SKETCH OF PROOFS: AS BEFORE, TO GET $i_{Morse}(g_t) \nearrow +\infty$, KEY IS TO HAVE ARBITRARILY SMALL EIGENVALUES OF Δ_{g_t} ON THE COMPACT QUOTIENT $F = \mathbb{R}^d/\pi$

$$i_{Morse}(g \oplus h_t) = \#\left\{ \lambda_i(M,g) + \lambda_j(F,h_t) < \frac{\mathrm{scal}_g}{n+d} \right\} \nearrow \infty \text{ IF } \lambda_j(F,h_t) \searrow 0.$$

$\rightsquigarrow$ COLLAPSE OF FLAT MANIFOLDS (COLLOQUIUM TOPIC)



(T^d, h_t) CAN BE COLLAPSED NONTRIVIALLY, SAME FOR OTHER $F = \mathbb{R}^d/\pi$!

REPLACEMENT FOR BUSER'S THM:

$\boxed{\text{LEMMA: } \forall \varepsilon > 0,\ j \in \mathbb{N},\ \exists h \in \mathcal{M}_{flat}(F) \text{ WITH } Vol(F,h) = 1 \text{ AND } \lambda_j(F,h) < \varepsilon.}$

SO BIFURCATE SOLUTIONS ON $M \times F$ AND HENCE ON $M \times \mathbb{R}^d$

DIFFERENT CONFORMAL CLASSES

ALL PULL BACK TO THE CONFORMAL CLASS OF g_{prod}

NOTE: ABOVE WORKS FOR YAMABE & CONSTANT Q-CURVATURE.

$\square$