How to find non-trivial solutions out of trivial ones

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Of course, I have no clue ...

... but try deforming trivial solutions
until they become *very unstable*
and that might give you new solutions!

Buckling under compressive stress

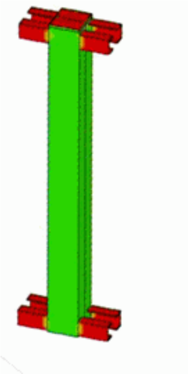

Image credit: Paul Rumbach (University of Notre Dame)
https://www3.nd.edu/~prumbach/AME20217/B4/index.html

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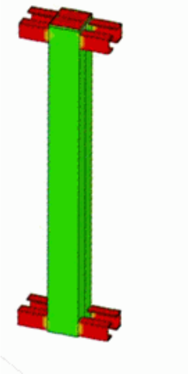

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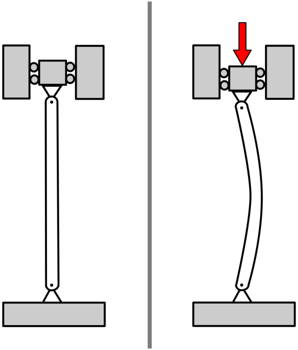
Buckling under compressive stress

Euler (1757)



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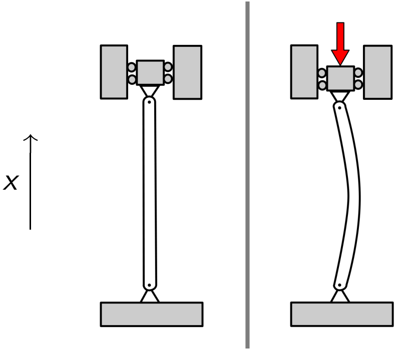

$x \in [0, L]$

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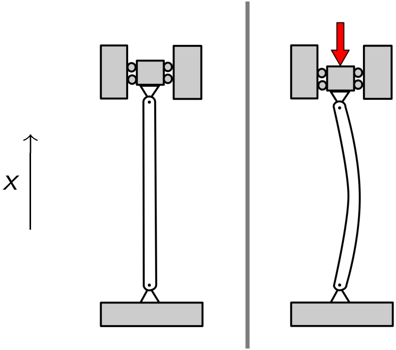

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$w(x) =$ lateral deflection at x

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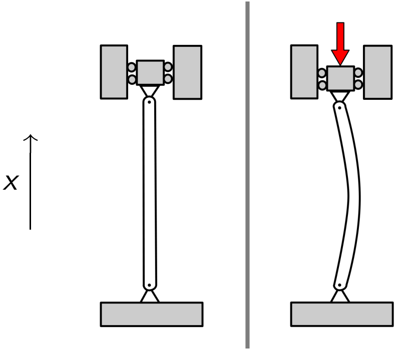

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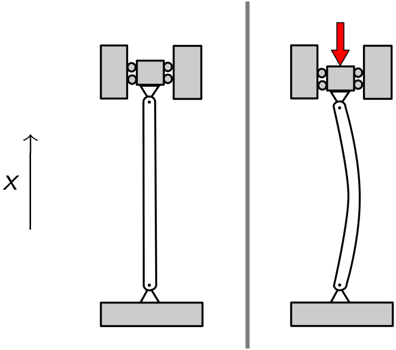

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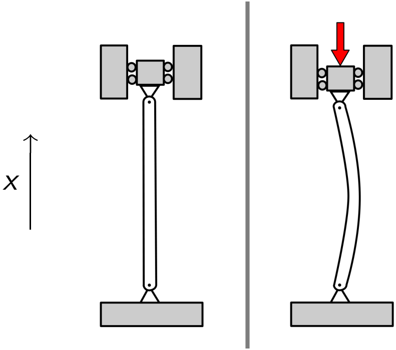

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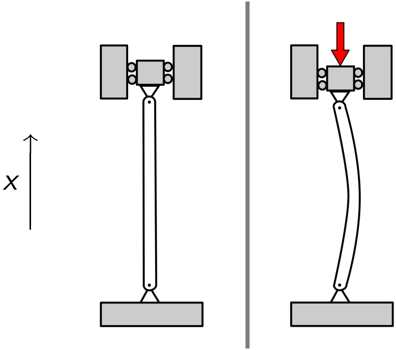

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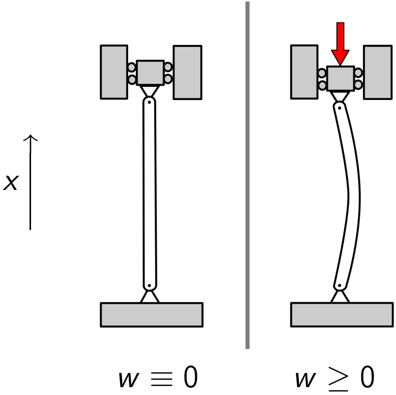

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  Only trivial solution exists; no buckling!

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- $P \geq \frac{n^2\pi^2 E}{L^2} \implies \boxed{w_j(x) = A\sin(\lambda_j x),\ \lambda_j = \frac{j\pi}{L}, 0 \leq j \leq n}$

  Nontrivial solutions appear; at least n buckling modes!

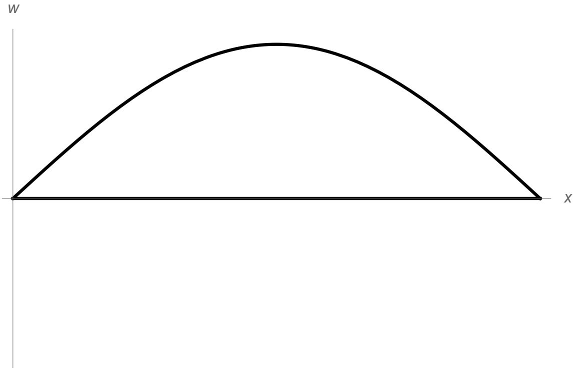
Increasing the load P

Solutions to $E\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + Pw = 0$ with $0 \leq P < \frac{\pi^2 E}{L^2}$

w_0

Increasing the load P

Solutions to $E\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + Pw = 0$ with $\frac{\pi^2 E}{L^2} \leq P < \frac{2^2\pi^2 E}{L^2}$



w_0, w_1

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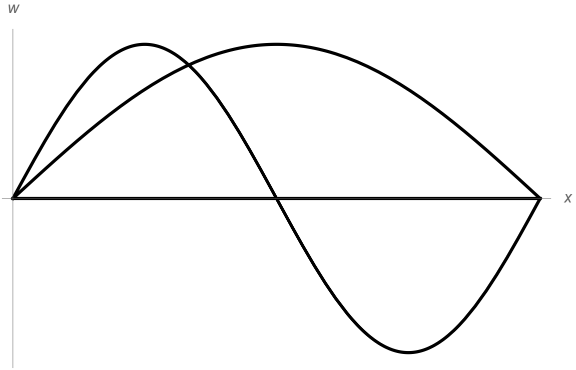

w_0, w_1, w_2

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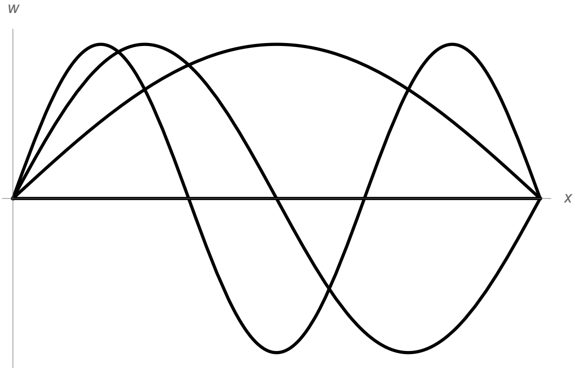

w_0, w_1, w_2, w_3

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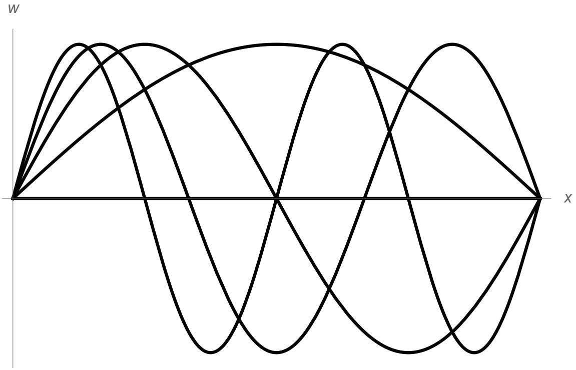

w_0, w_1, w_2, w_3, w_4

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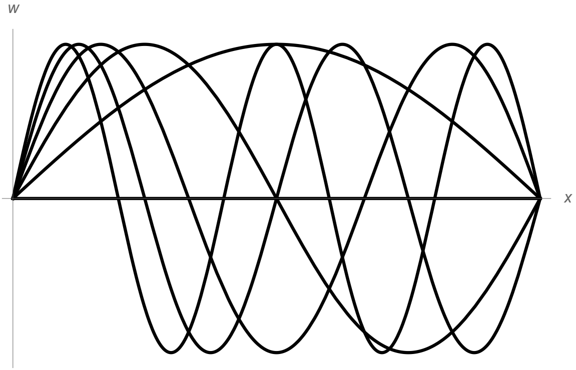

$w_0, w_1, w_2, w_3, w_4, w_5$

Bifurcation

H. Poincaré. "L'Équilibre d'une masse fluide animée d'un mouvement de rotation". Acta Math., vol. 7, pp. 259-380, 1885.

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Parameter: P (load)

Bifurcation values: $\frac{n^2\pi^2 E}{L^2}$

Bifurcation

270 H. Poincaré.

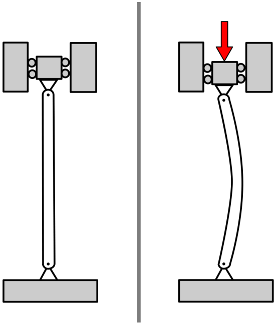
Il pourra d'ailleurs arriver qu'une même forme d'équilibre appartienne à la fois à deux ou plusieurs séries linéaires. Nous dirons alors que c'est une *forme de bifurcation*. On peut en effet, pour une valeur de y infiniment voisine de celle qui correspond à cette forme, trouver *deux* formes d'équilibre qui diffèrent infiniment peu de la forme de bifurcation.

Il peut arriver également que deux séries linéaires de formes d'équi-

Bifurcation phenomena

In Physics, Engineering, Finance, and other Applied Sciences:

- Buckling under compressive stress;



Euler beam equation

$$E\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + {\color{red}P} w = 0$$

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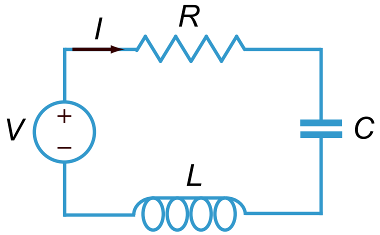

Kirchhoff voltage law

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{LC} I = 0$$

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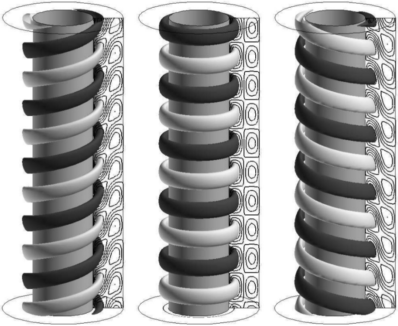

Navier-Stokes equation

$$\rho \frac{\partial u}{\partial t} = -\nabla p + \nabla \cdot \tau + \rho g$$

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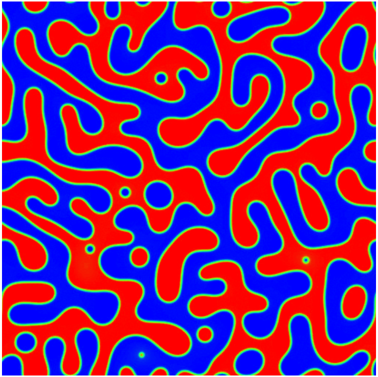

$$\frac{\partial c}{\partial t} = d\nabla^2(c^3 - c - \gamma\nabla^2 c)$$

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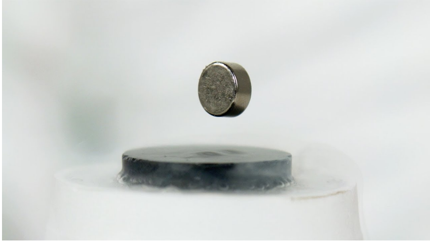
- Ginzburg–Landau equation for superconductors;



$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m}(-i\hbar\nabla - 2eA)^2\psi = 0$$

$$j = \frac{2e}{m}\mathrm{Re}(\psi^*(-i\hbar\nabla - 2eA)\psi)$$

$$\nabla \times B = \mu_0 j$$

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- Competitive equilibria in iterative auctions

...

Walras Law

$$\sum_j p_j(D_j - S_j) = 0$$

Bifurcation phenomena

In Mathematics:

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A way to use instability of trivial solutions to produce nontrivial solutions to differential equations.

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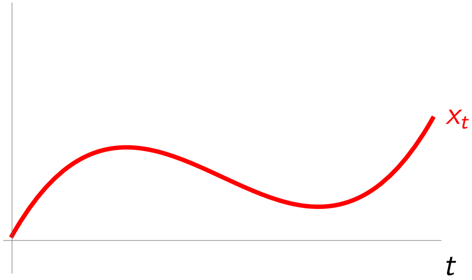
- $x_t =$ "ground state", typically minimizes $f_t(x)$

  Principle of Least Action: x_t is state observed in nature

Bifurcation

x_t trivial branch

$$\mathrm{d}f_t(x_t) = 0$$



Bifurcation

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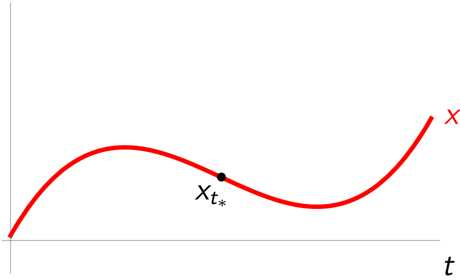
Definition

Bifurcation occurs at x_{t_*} if:



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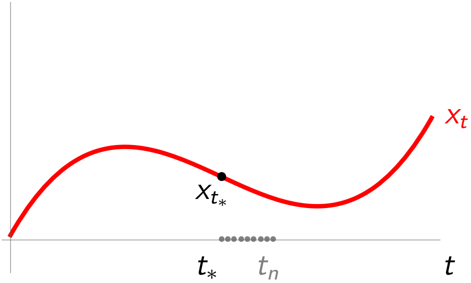

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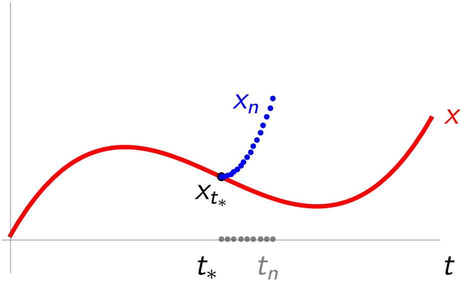

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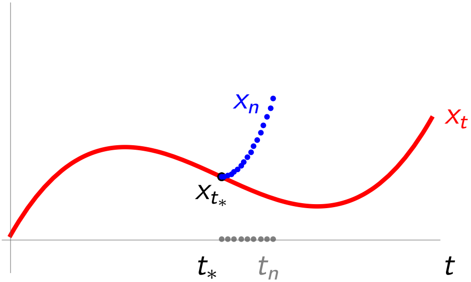

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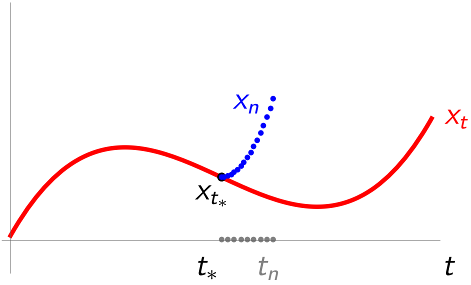

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Thus, $\ker \mathrm{d}^2 f_{t_*}(x_{t_*}) \neq \{0\}$ is a **necessary condition**

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Bifurcation occurs at x_{t_*} if:

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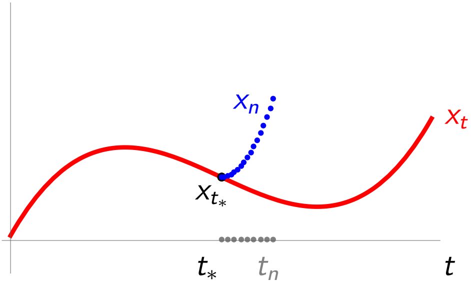
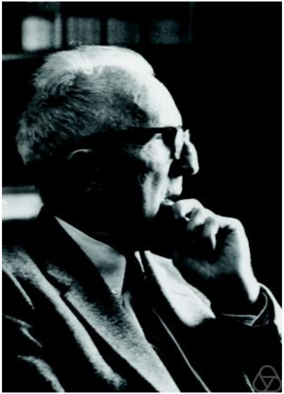
- $\exists\, x_n \to x_{t_*}, \mathrm{d}f_{t_n}(x_n) = 0$, $x_n \neq x_{t_n}$



Equivalently, the Implicit Function Theorem **fails** at x_{t_*}!

Thus, $\ker \mathrm{d}^2 f_{t_*}(x_{t_*}) \neq \{0\}$ is a **necessary condition** but it is **not sufficient**...

Sufficient condition for bifurcation



M. Morse
(1965)

Definition (Morse index)

$i_{\mathrm{Morse}}(x) = \#\,\mathsf{Spec}\left(\mathrm{d}^2 f_t(x)\right) \cap (-\infty, 0),$

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⚠ Technical warnings (for experts):

- $\mathrm{d}^2 f_t$ must be Fredholm (of index 0)
- x_a and x_b must be nondegenerate.

Toy example from Calculus

- $X = \mathbb{R}^2$

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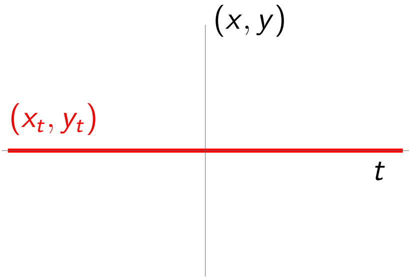

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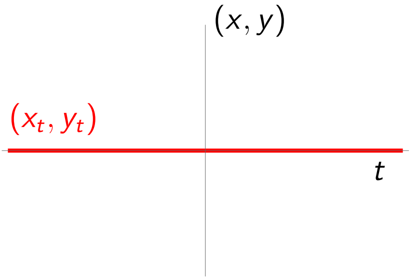

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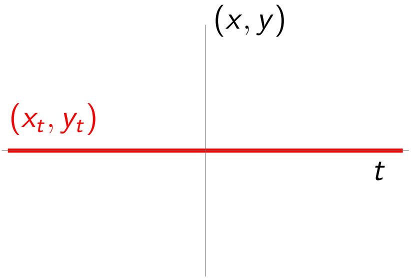

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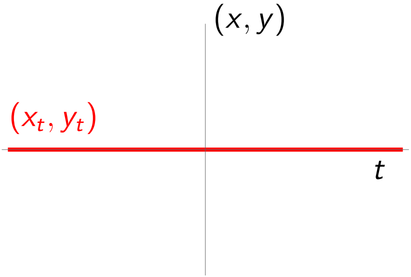

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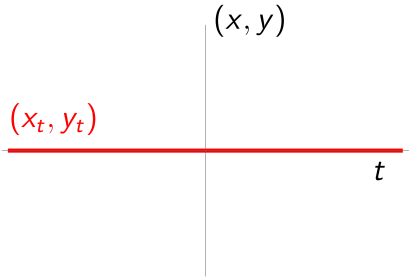

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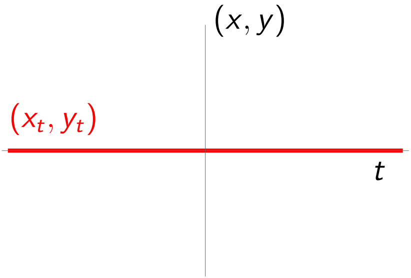

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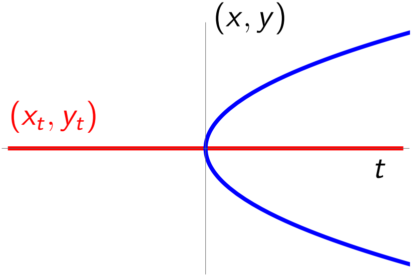
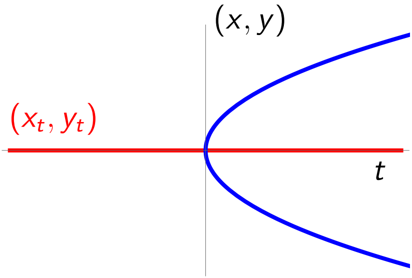


Two bifurcating branches issue from $x_{t_*} = (0,0)$

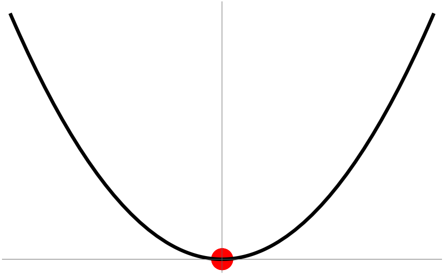
$(x_t, y_t) = \left(0, \pm\sqrt{t/2}\right)$, $t > 0$

“Pitchfork bifurcation”



(x_t, y_t) trivial solutions

(x_t, y_t) bifurcating solutions



$t < 0$

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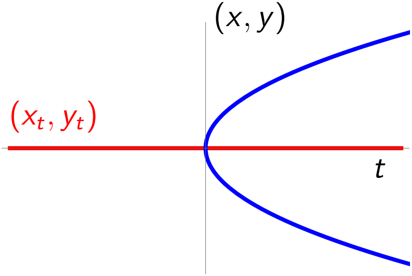

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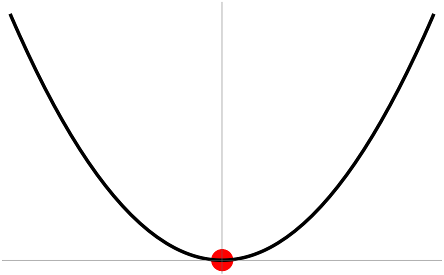

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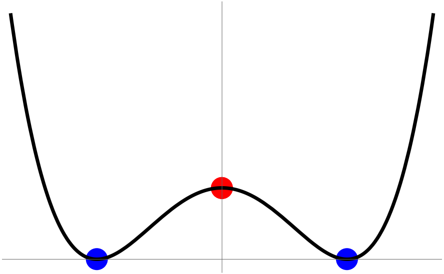
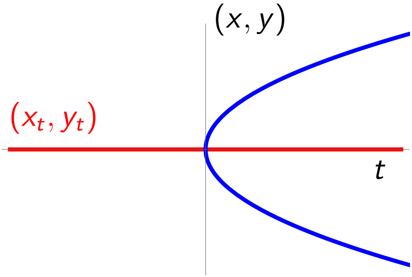


$t > 0$

“Pitchfork bifurcation”



(x_t, y_t) trivial solutions

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(x_t, y_t) bifurcating solutions

H. Poincaré. "L'Équilibre d'une masse fluide animée d'un mouvement de rotation". Acta Math., vol. 7, pp. 259-380, 1885.

Avant de démontrer ce résultat général, donnons quelques exemples. Soit:

$$F = Ax_1^2 + \frac{1}{3}x_2^3 - y^2 x_2 - \alpha y x_2.$$

Il vient pour les équations d'équilibre:

$$x_1 = 0, \qquad x_2 = \pm\sqrt{y^2 + \alpha y}$$

d'où

$$\Delta = 4Ax_2 = \pm 4A\sqrt{y^2 + \alpha y}.$$

PDE Application: Semilinear elliptic equations

$$\text{(PDE)} \quad \begin{cases} \Delta u(x) + t\, f(u(x)) = 0 & \text{in } B^n \\ \alpha u(x) - t\, \beta \dfrac{\partial u}{\partial \nu}(x) = 0 & \text{on } \partial B^n \end{cases}$$

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Look for *radial solutions*, i.e., $u = u(r)$,
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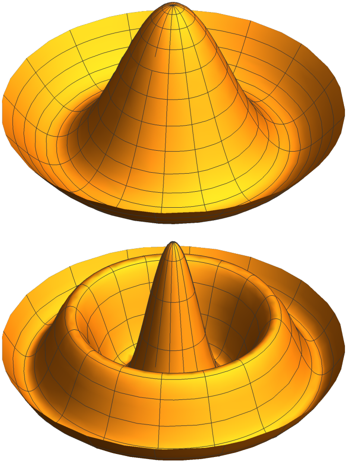
Theorem (Smoller-Wasserman, 1990)

There are infinitely many bifurcations from u_t *as* $t \nearrow +\infty$ *by nonradial solutions to* (PDE).

J. Smoller

A. Wasserman

Symmetry-breaking



Radial solutions u_t

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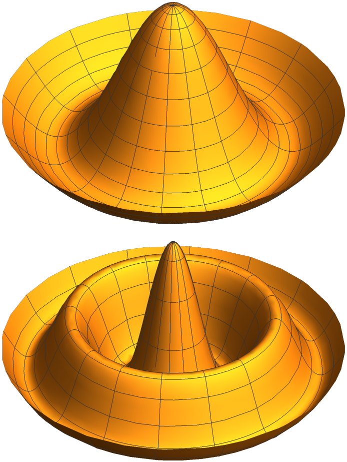

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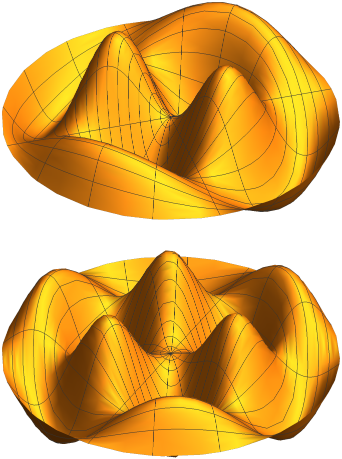
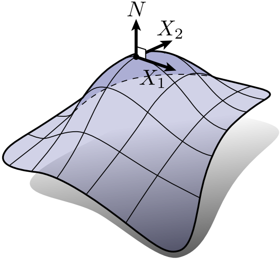


Bifurcating solutions

Geometric Application I: Constant Mean Curvature

$\Sigma^n \subset \mathbb{R}^{n+1}$

hypersurface



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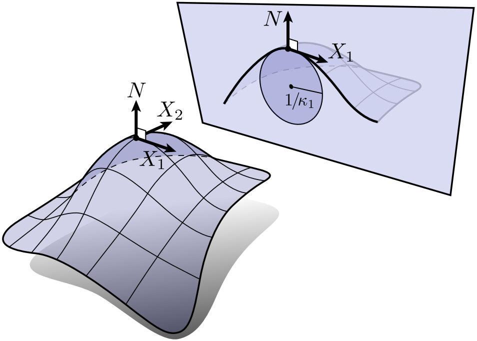
Principal curvatures:

κ_1



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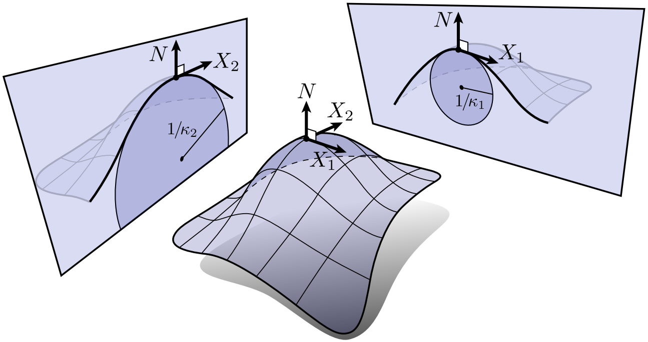
Principal curvatures:
κ_1, κ_2



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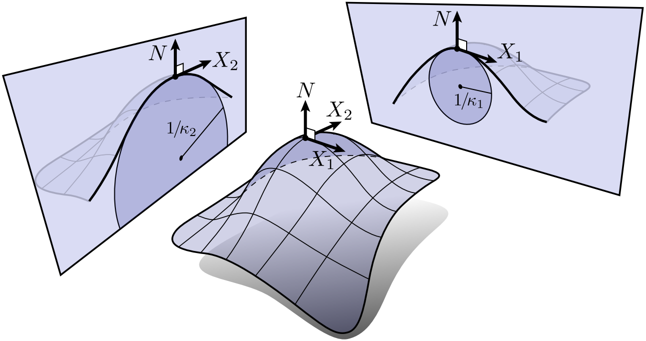

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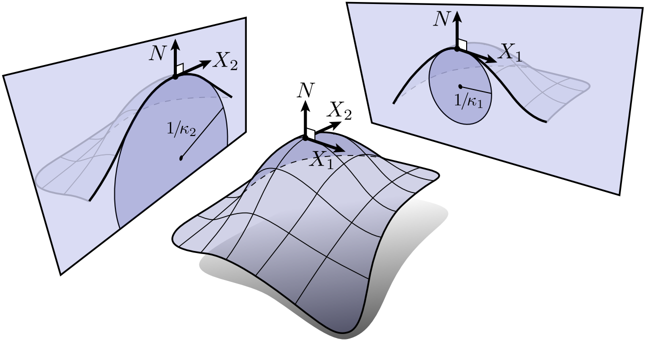

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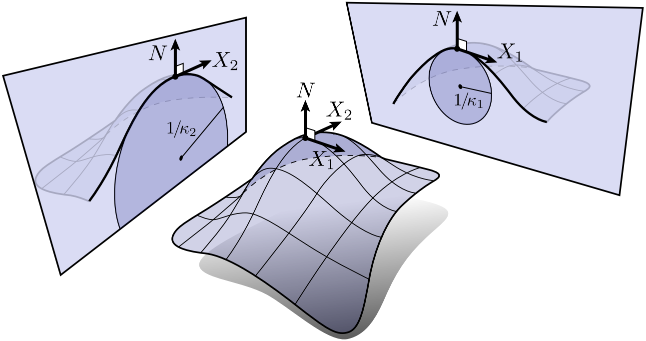

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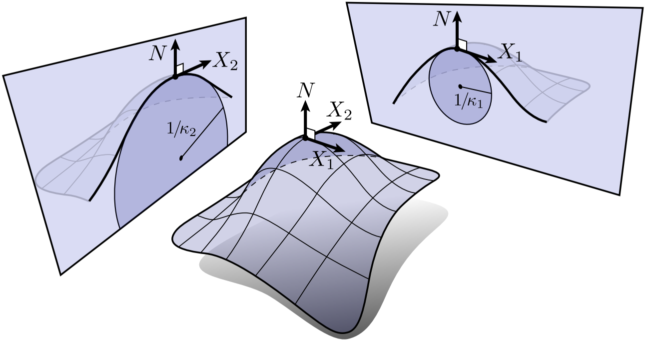

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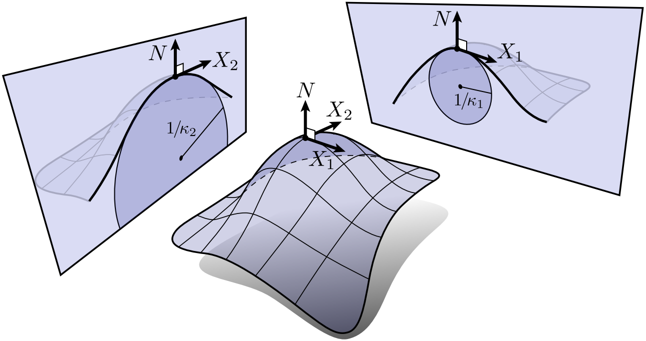

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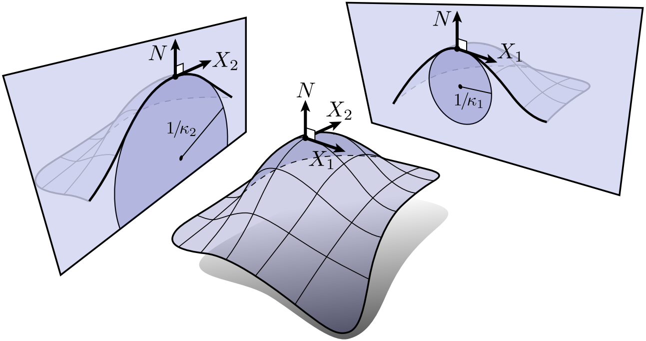

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- Center of Mass in General Relativity

Roulette of a conic section

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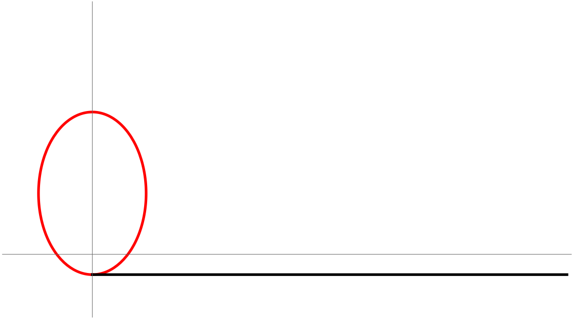

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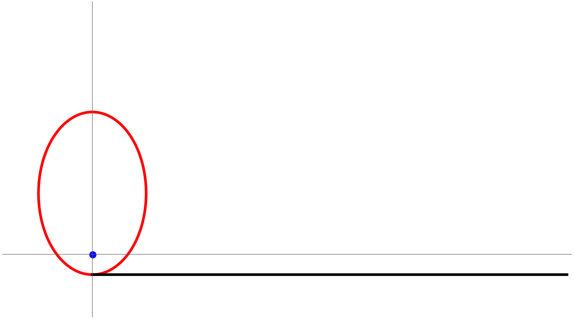

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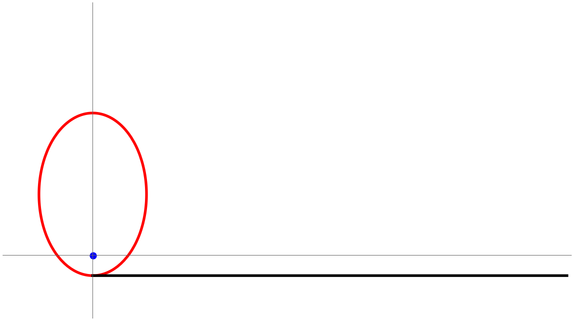

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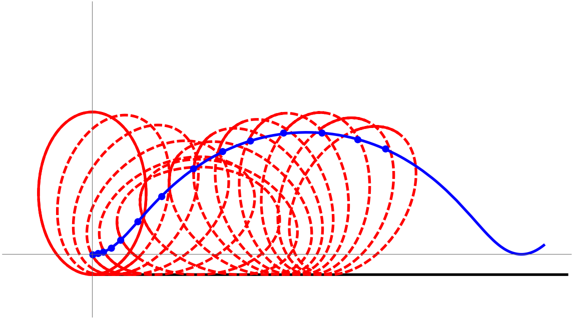

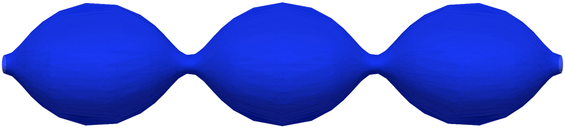

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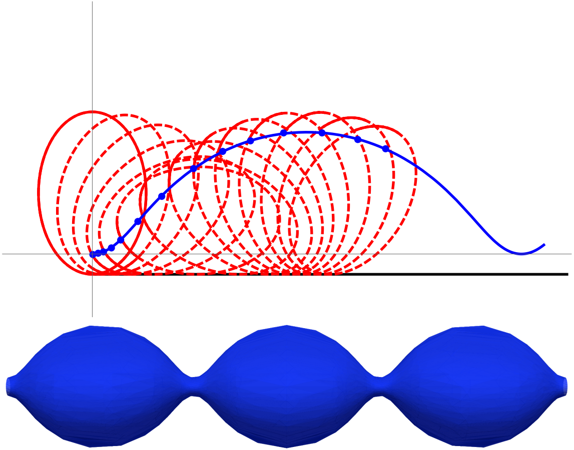
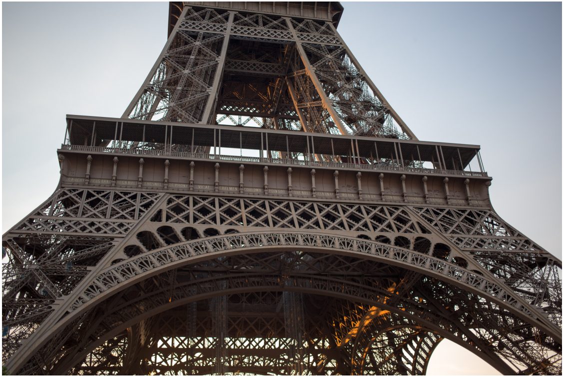


Theorem (Delaunay, 1841)

Surface of revolution $\Sigma \subset \mathbb{R}^3$ has CMC	$\Longleftrightarrow$	Profile curve of Σ is the roulette of a conic section.

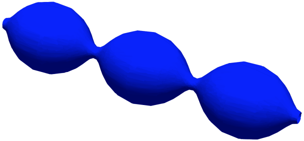
Delaunay



C.-E. Delaunay

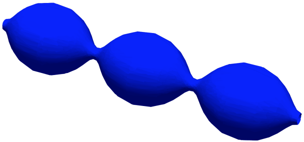
Southeast side of the Eiffel tower:

Delaunay surfaces



Unduloid
(ellipse)

Delaunay surfaces



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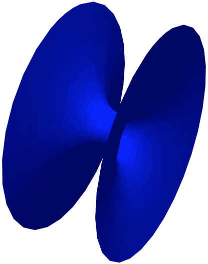

Catenoid
(parabola)

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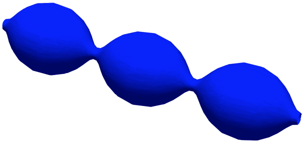

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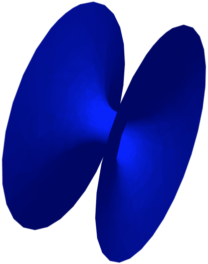

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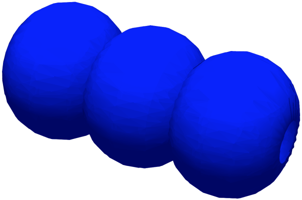

Nodoid
(hyperbola)

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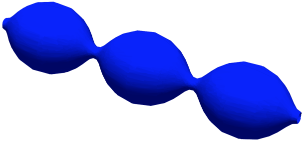

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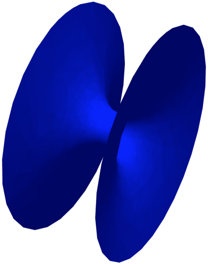

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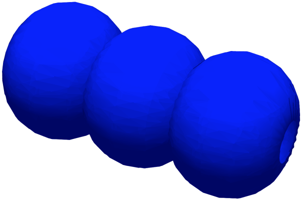

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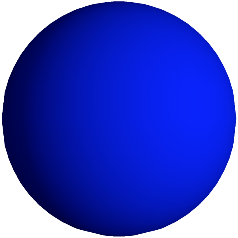
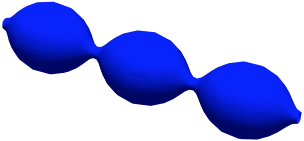


Sphere

Delaunay surfaces



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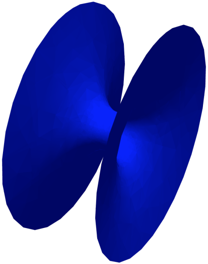

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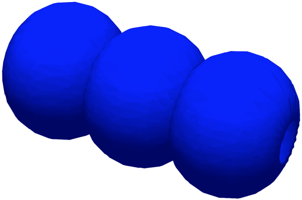

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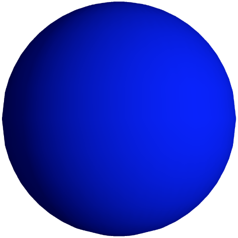

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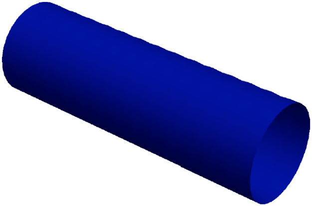
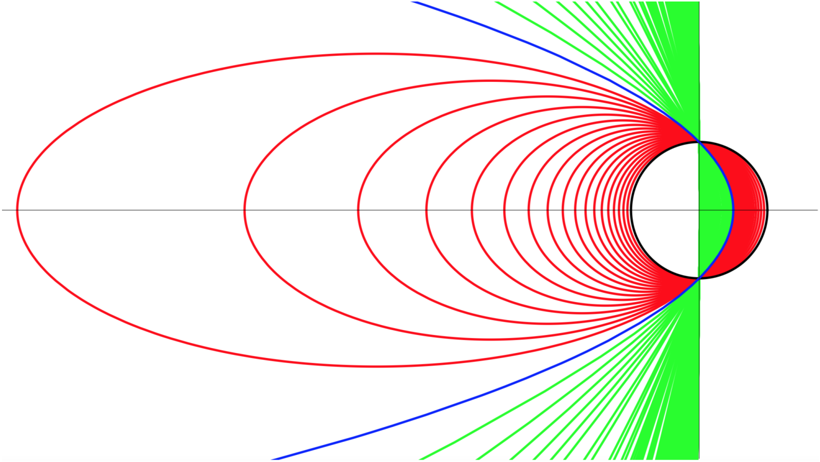


Cylinder

Conics of varying eccentricity



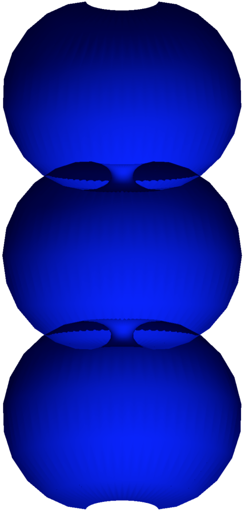
ellipses $0 < e < 1$

parabola $e = 1$

hyperbolae $1 < e < +\infty$

Video credit: GeometrieWerkstatt Gallery
http://service.ifam.uni-hannover.de/~geometriewerkstatt/

Bifurcating Nodoids



Theorem (Mazzeo-Pacard, 2002)

There are infinitely many families of CMC surfaces in $\mathbb{R}^3$ that bifurcate from nodoids as their eccentricity goes to $+\infty$.

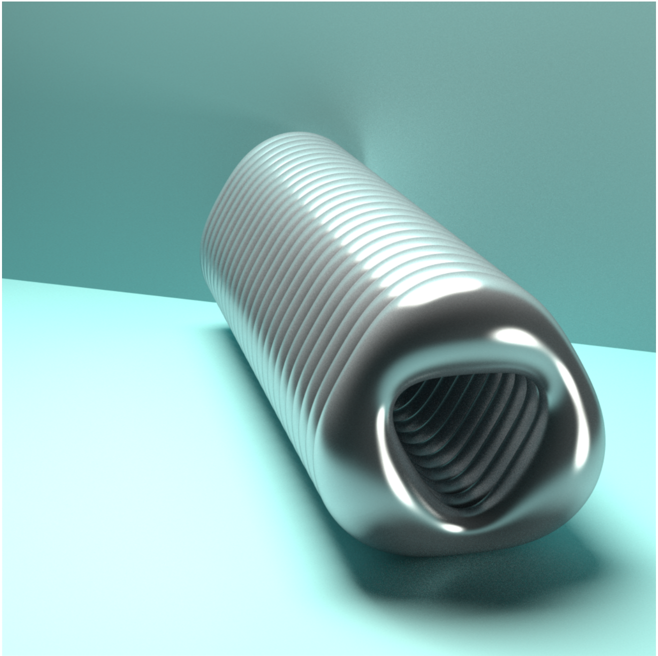
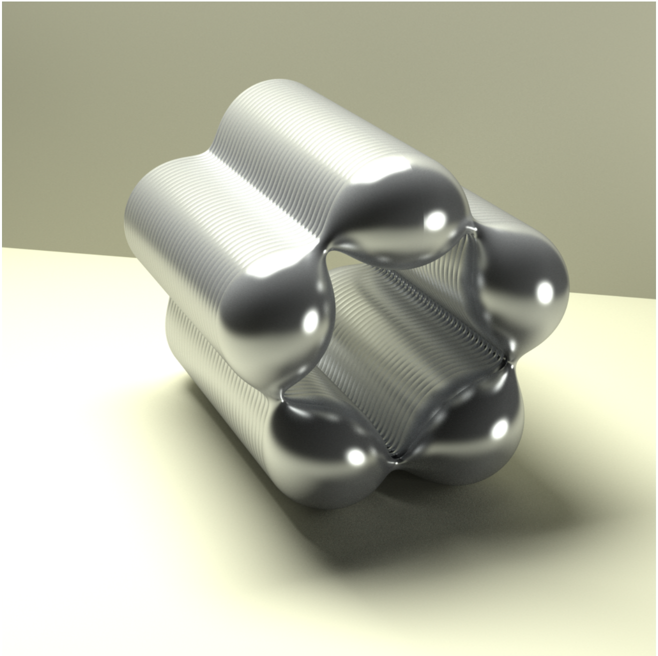


Image credit: GeometrieWerkstatt Gallery
http://service.ifam.uni-hannover.de/~geometriewerkstatt/gallery/0003.html



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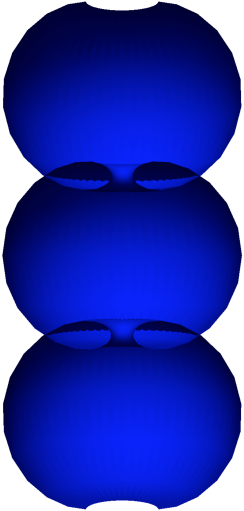

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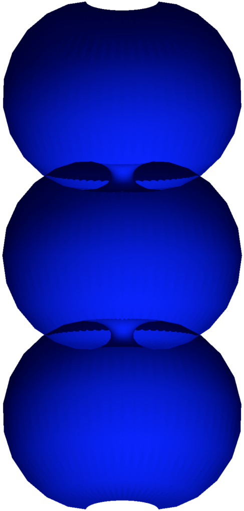

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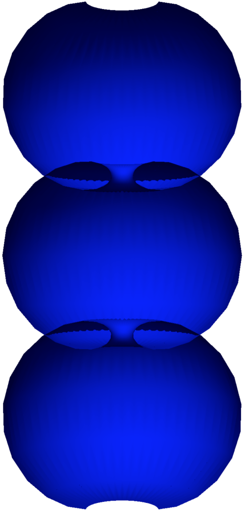

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Theorem (Koiso-Palmer-Piccione, 2015)

There are infinitely many families of CMC surfaces in $\mathbb{R}^3$ with boundary on two fixed coaxial circles that bifurcate from portions of nodoids as their conormal angle varies.

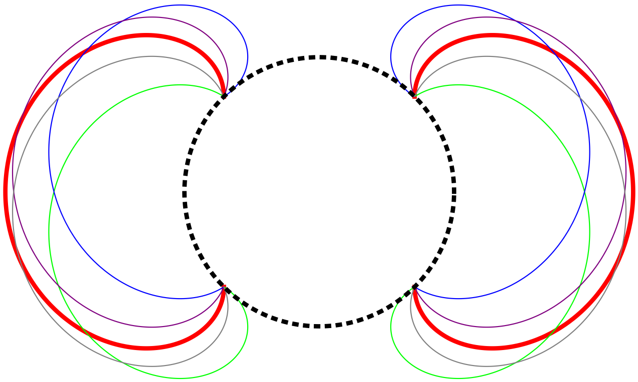

Image credit: Koiso-Palmer-Piccione, 2015

https://www.ime.usp.br/~piccione/Downloads/NodoidBifurcation_revisionACV.pdf

New Delaunay-type hypersurfaces

Plot twist:

- Bifurcate Delaunay surfaces into new CMC surfaces

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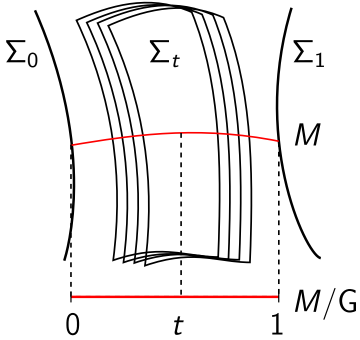

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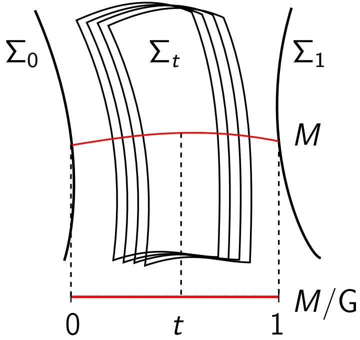

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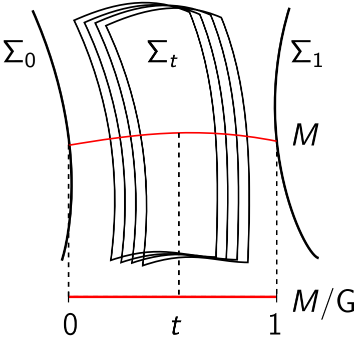

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Significance: These are Delaunay-type hypersurfaces in these spaces!

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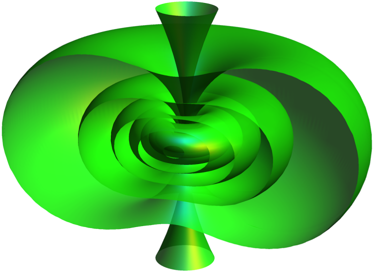

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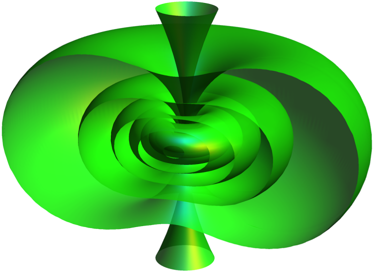

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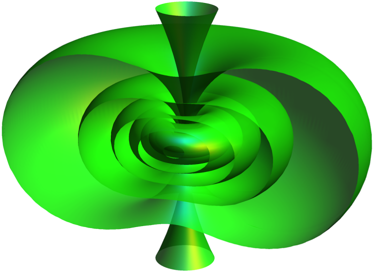

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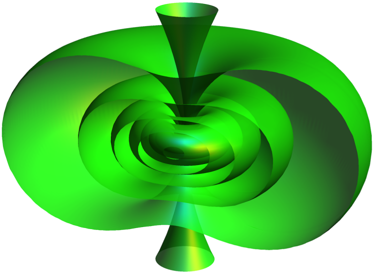

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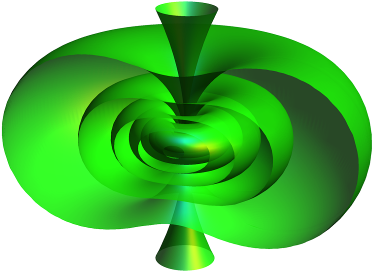

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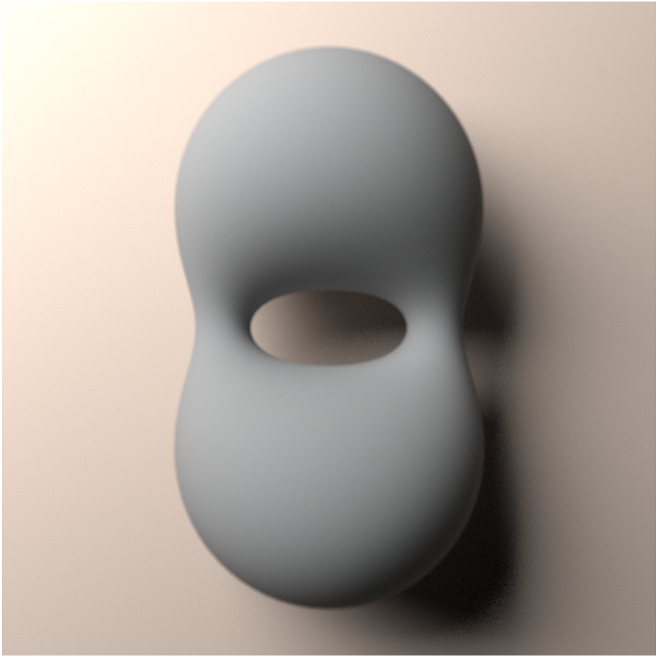
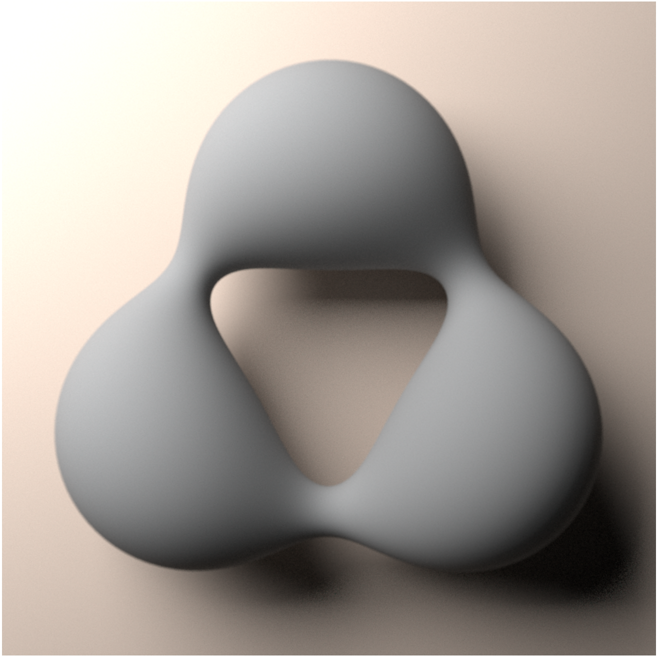


Image credit: GeometrieWerkstatt Gallery

http://service.ifam.uni-hannover.de/~geometriewerkstatt/gallery/0600.html



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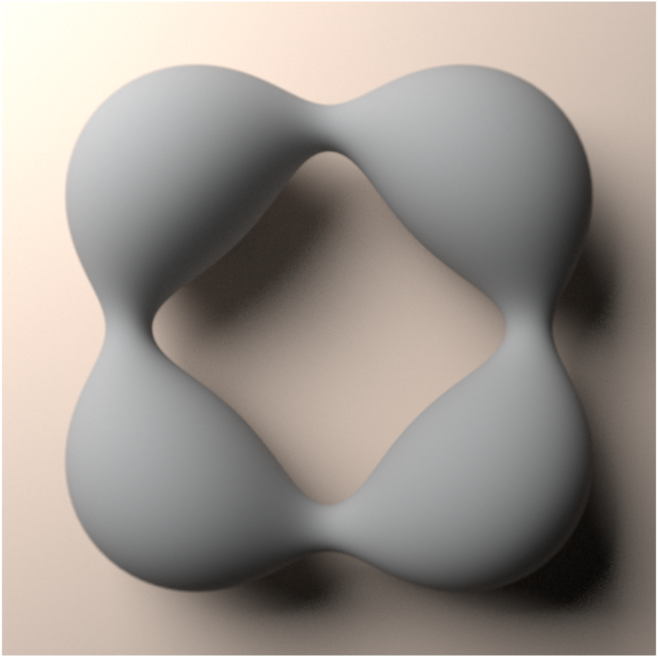

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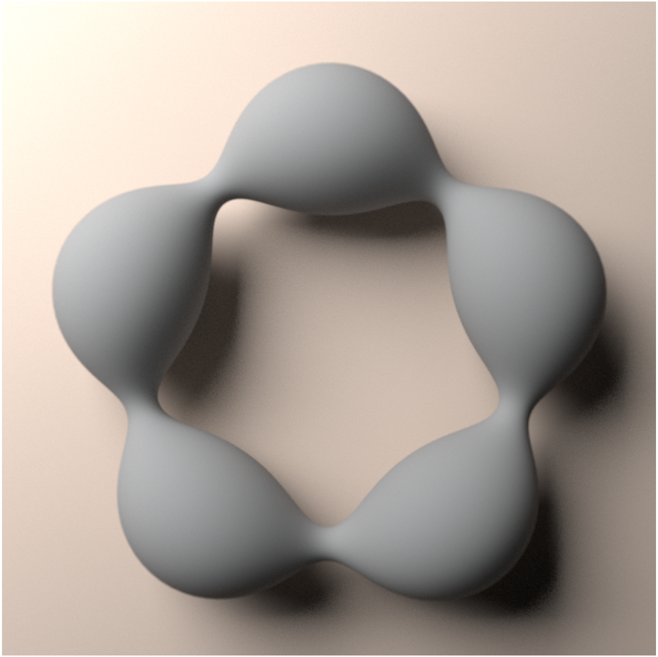

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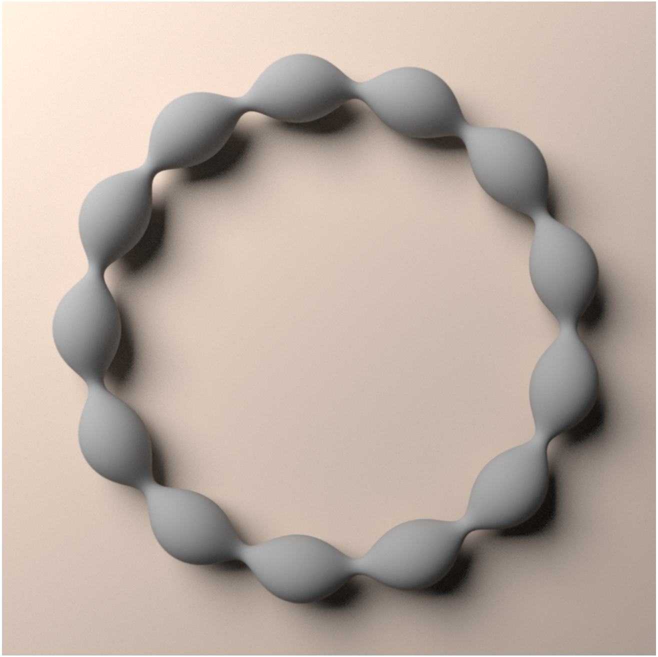

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- Krasnosel'skij's Theorem $\Longrightarrow$ Infinitely many bifurcations

$\square$

Geometric Application II: Conformal Deformations

Uniformization Theorem (Poincaré 1882; Klein 1883)

Every closed surface

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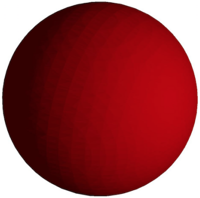

Genus 0

Image credit: Thomas Krämer

`https://www2.mathematik.hu-berlin.de/~kraemeth/riemann-surfaces/index.html`

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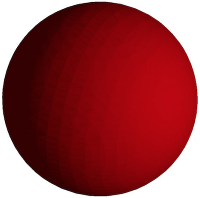

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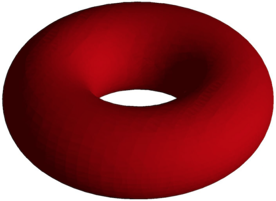

Genus 1

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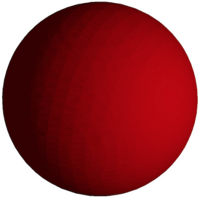
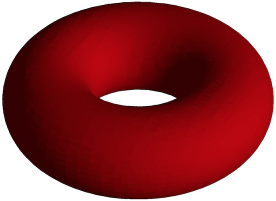
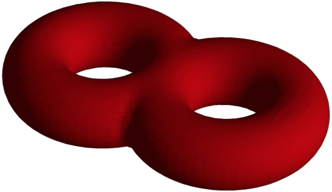


Genus 0



Genus 1



Genus ≥ 2

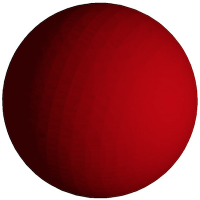
Image credit: Thomas Krämer

`https://www2.mathematik.hu-berlin.de/~kraemeth/riemann-surfaces/index.html`

Geometric Application II: Conformal Deformations

Uniformization Theorem (Poincaré 1882; Klein 1883)

Every closed surface admits a metric of constant curvature



Genus 0
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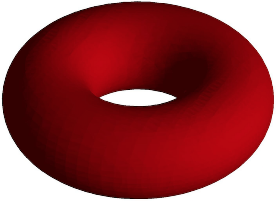

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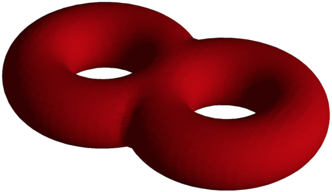

Genus ≥ 2
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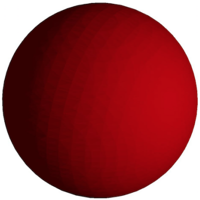
Geometric Application II: Conformal Deformations

Uniformization Theorem (Poincaré 1882; Klein 1883)

Every closed surface admits a metric of constant curvature which can be achieved with a conformal deformation.



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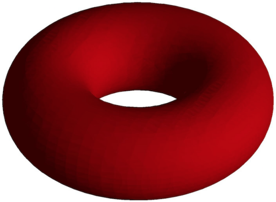

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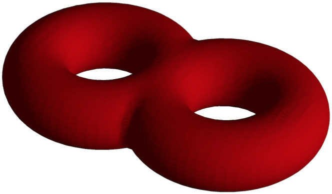

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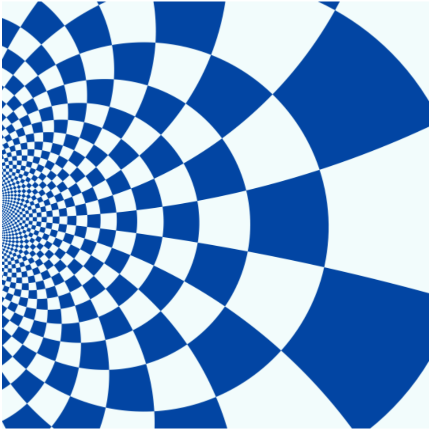

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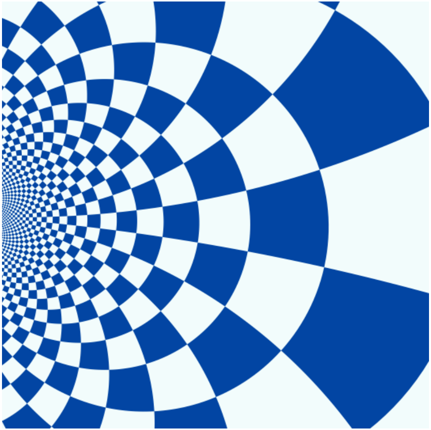

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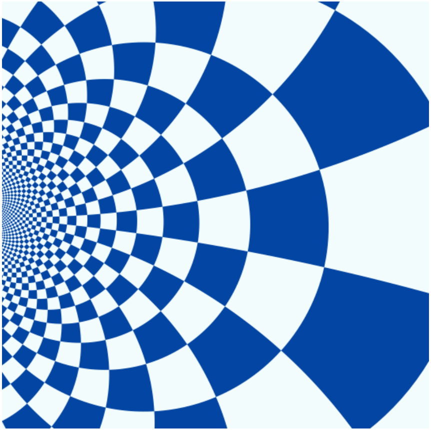

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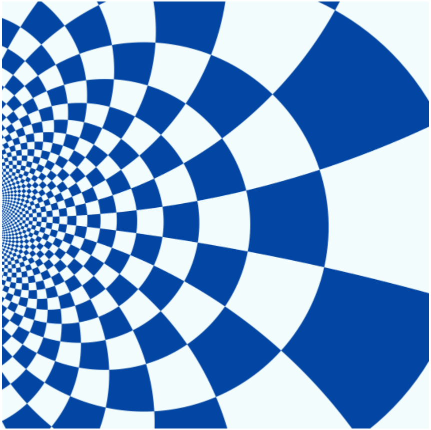

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(M, g)
n-manifold

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Is it unique?

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Yamabe problem: non-uniqueness results

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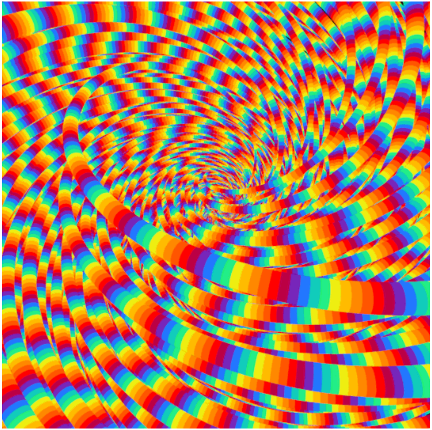
Then $\big(M \times N, \mathrm{g}_t\big)$ *bifurcates infinitely many times as $t \searrow 0$.*

"Twisted products": Hopf bundles



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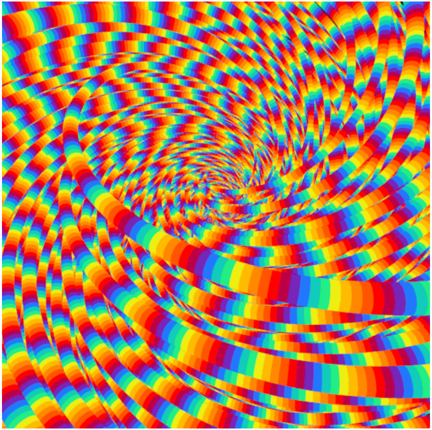

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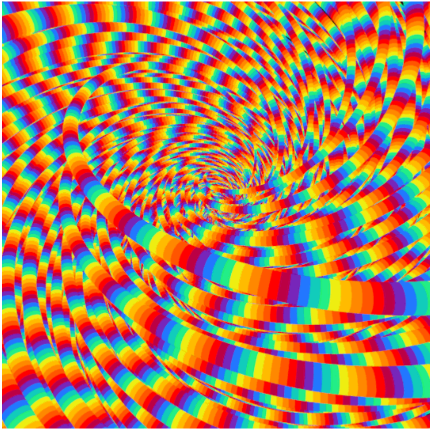

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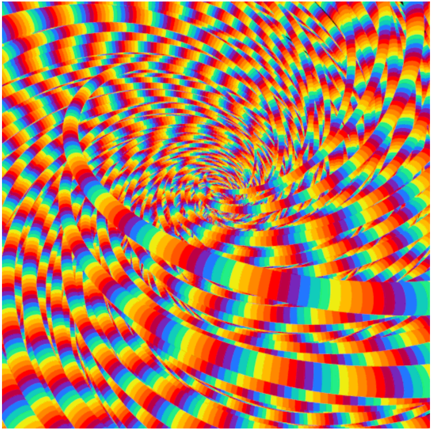

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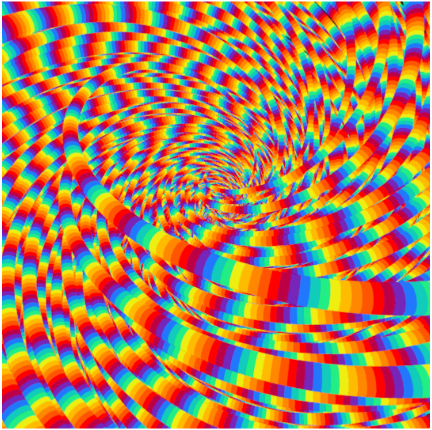

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Rescale vertical space by $t > 0$

Yamabe problem: non-uniqueness results

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The Berger spheres (S^{4n+3}, g_t) *and* (S^{15}, g_t) *bifurcate infinitely many times as* $t \searrow 0$.

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with $\mathrm{scal}_{\mathsf{K}/\mathsf{H}} > 0$ *bifurcates infinitely many times as* $t \searrow 0$.

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- $$i_{\mathrm{Morse}}(\mathrm{g}) = \#\Big(\underbrace{\mathrm{Spec}(\Delta_\mathrm{g})}_{\text{Laplace eigenvalues}} \cap \Big(-\infty, \underbrace{\mathrm{scal}_\mathrm{g}/(n-1)}_{\text{``geometric potential''}} \Big)\Big)$$

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- Krasnosel'skij's Theorem $\Longrightarrow$ Infinitely many bifurcations

□

A singular cousin

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- If $1 < k < \frac{n-2}{2}$, other methods are needed [B.-Piccione, 2018]

A fourth-order cousin

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Find complete conformal metrics on (M^n, g) *such that*

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$$Q_{\mathrm{g}} = \frac{1}{2(n-1)}\Delta_{\mathrm{g}}\,\mathrm{scal}_{\mathrm{g}} - \frac{2}{(n-2)^2}\left\|\mathrm{Ric}_{\mathrm{g}}\right\|^2 + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}\,\mathrm{scal}_{\mathrm{g}}^2$$

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$$Q_{\mathrm{g}} = \frac{1}{2(n-1)}\Delta_{\mathrm{g}}\,\mathrm{scal}_{\mathrm{g}} - \frac{2}{(n-2)^2}\left\|\mathrm{Ric}_{\mathrm{g}}\right\|^2 + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}\,\mathrm{scal}_{\mathrm{g}}^2 \equiv \mathit{const}.$$

A fourth-order cousin

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Theorem (B.-Piccione-Sire, 2018)

Hopf bundle	Infinitely many bifurcations as $t \searrow 0$	Infinitely many bifurcations as $t \nearrow +\infty$
$S^1 \to S^{2q+1} \to \mathbb{C}P^q$	no	if $q \geq 6$
$S^3 \to S^{4q+3} \to \mathbb{H}P^q$	if $q \geq 1$	if $q \geq 2$
$\mathbb{C}P^1 \to \mathbb{C}P^{2q+1} \to \mathbb{H}P^q$	if $q \geq 2$	if $q \geq 3$
$S^7 \to S^{15} \to S^8(1/2)$	yes	yes

Thank you for
your attention!