Exploring flat worlds

Renato G. Bettiol

THE CITY UNIVERSITY OF NEW YORK

More important than any theorems in this talk ...

What many people told me when I was a PhD student:

More important than any theorems in this talk ...

What many people told me when I was a PhD student:

(and I sort of believed them

More important than any theorems in this talk ...

What many people told me when I was a PhD student:

(and I sort of believed them; turns out they were right)

More important than any theorems in this talk ...

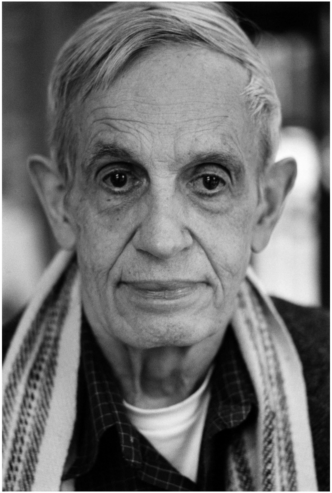
What many people told me when I was a PhD student:

(and I sort of believed them; turns out they were right)

- Find a problem you are **passionate** about, and study it

“I would not dare to say that there is a direct relation between mathematics and madness, but there is no doubt that great mathematicians suffer from maniacal characteristics, delirium and symptoms of schizophrenia.”

J. Nash



More important than any theorems in this talk ...

What many people told me when I was a PhD student:

(and I sort of believed them; turns out they were right)

- Find a problem you are **passionate** about, and study it
- Once in a while, **check** if you are still on Earth

More important than any theorems in this talk ...

What many people told me when I was a PhD student:

(and I sort of believed them; turns out they were right)

- Find a problem you are **passionate** about, and study it
- Once in a while, **check** if you are still on Earth
- **All** other things (papers, jobs, grants) sort themselves out

More important than any theorems in this talk ...

What many people told me when I was a PhD student:

(and I sort of believed them; turns out they were right)

- Find a problem you are **passionate** about, and study it
- Once in a while, **check** if you are still on Earth
- **All** other things (papers, jobs, grants) sort themselves out
- **Have fun** doing Mathematics!

Flat worlds

Definition

A metric space (X, d_X) is **locally isometric** to (Y, d_Y) if

Flat worlds

Definition

A metric space (X, d_X) is **locally isometric** to (Y, d_Y) if $\forall x \in X$,

Flat worlds

Definition

A metric space (X, d_X) is **locally isometric** to (Y, d_Y) if $\forall\, x \in X$, $\exists\, U \subset X$ neighborhood of x,

Flat worlds

Definition

A metric space (X, d_X) is **locally isometric** to (Y, d_Y) if $\forall\, x \in X$, $\exists\, U \subset X$ neighborhood of x, and $\phi\colon U \to Y$ so that

$$d_Y\big(\phi(p), \phi(q)\big) = d_X(p, q), \quad \forall\, p, q \in U$$

Flat worlds

Definition

A metric space (X, d_X) is **locally isometric** to (Y, d_Y) if $\forall\, x \in X$, $\exists\, U \subset X$ neighborhood of x, and $\phi\colon U \to Y$ so that

$$d_Y\big(\phi(p), \phi(q)\big) = d_X(p, q), \quad \forall\, p, q \in U$$

Definition

A **flat manifold** is a metric space locally isometric to $\mathbb{R}^n$.

Flat worlds

Definition

A metric space (X, d_X) is **locally isometric** to (Y, d_Y) if $\forall\, x \in X$, $\exists\, U \subset X$ neighborhood of x, and $\phi \colon U \to Y$ so that

$$d_Y\big(\phi(p), \phi(q)\big) = d_X(p, q), \quad \forall\, p, q \in U$$

Definition

A **flat manifold** is a metric space locally isometric to $\mathbb{R}^n$.
(always smooth!)

Flat worlds

Definition

A metric space (X, d_X) is **locally isometric** to (Y, d_Y) if $\forall\, x \in X$, $\exists\, U \subset X$ neighborhood of x, and $\phi\colon U \to Y$ so that

$$d_Y\big(\phi(p), \phi(q)\big) = d_X(p, q), \quad \forall\, p, q \in U$$

Definition

A **flat manifold** is a metric space locally isometric to $\mathbb{R}^n$.
(always smooth!)

A **flat orbifold** is a metric space locally isometric to $\mathbb{R}^n/\Gamma$, where $\Gamma < \mathrm{O}(n)$ is a finite group of (linear) isometries.

Flat worlds

Definition

A metric space (X, d_X) is **locally isometric** to (Y, d_Y) if $\forall\, x \in X$, $\exists\, U \subset X$ neighborhood of x, and $\phi\colon U \to Y$ so that

$$d_Y\big(\phi(p), \phi(q)\big) = d_X(p, q), \quad \forall\, p, q \in U$$

Definition

A **flat manifold** is a metric space locally isometric to $\mathbb{R}^n$.
(always smooth!)

A **flat orbifold** is a metric space locally isometric to $\mathbb{R}^n/\Gamma$, where $\Gamma < \mathrm{O}(n)$ is a finite group of (linear) isometries.
(only smooth where $\Gamma = \{1\}$!)

Flat worlds

Definition

A metric space (X, d_X) is **locally isometric** to (Y, d_Y) if $\forall\, x \in X$, $\exists\, U \subset X$ neighborhood of x, and $\phi\colon U \to Y$ so that

$$d_Y\big(\phi(p), \phi(q)\big) = d_X(p, q), \quad \forall\, p, q \in U$$

Definition

A **flat manifold** is a metric space locally isometric to $\mathbb{R}^n$.
(always smooth!)

A **flat orbifold** is a metric space locally isometric to $\mathbb{R}^n/\Gamma$, where $\Gamma < \mathrm{O}(n)$ is a finite group of (linear) isometries.
(only smooth where $\Gamma = \{1\}$!)

- Recall: $\mathrm{O}(n) = \big\{A \in M_{n\times n}(\mathbb{R}) \,:\, A^t A = \mathrm{Id}\big\}$

Flat worlds

Definition

A metric space (X, d_X) is **locally isometric** to (Y, d_Y) if $\forall\, x \in X$, $\exists\, U \subset X$ neighborhood of x, and $\phi\colon U \to Y$ so that

$$d_Y\big(\phi(p), \phi(q)\big) = d_X(p, q), \quad \forall\, p, q \in U$$

Definition

A **flat manifold** is a metric space locally isometric to $\mathbb{R}^n$. (always smooth!)

A **flat orbifold** is a metric space locally isometric to $\mathbb{R}^n/\Gamma$, where $\Gamma < \mathrm{O}(n)$ is a finite group of (linear) isometries. (only smooth where $\Gamma = \{1\}$!)

- Recall: $\mathrm{O}(n) = \big\{A \in M_{n\times n}(\mathbb{R}) \,:\, A^t A = \mathrm{Id}\big\}$
- In this talk: only **compact** manifolds and orbifolds.

Two-dimensional flat manifolds

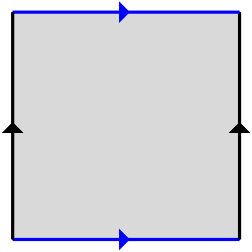
- Local model: $\mathbb{R}^2$
  All points have neighborhoods isometric to a subset of $\mathbb{R}^2$

Two-dimensional flat manifolds

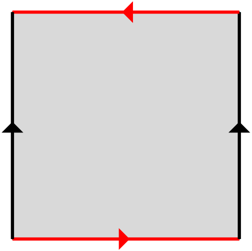
- Local model: $\mathbb{R}^2$
  All points have neighborhoods isometric to a subset of $\mathbb{R}^2$
- Compact: $\max_{x,y} d(x, y) < +\infty$

Two-dimensional flat manifolds

- Local model: $\mathbb{R}^2$
  All points have neighborhoods isometric to a subset of $\mathbb{R}^2$
- Compact: $\max_{x,y} d(x, y) < +\infty$
- Only possibilities are:



Torus T^2



Klein bottle K^2

Two-dimensional flat orbifolds

Local models: $\mathbb{R}^2/\Gamma$, where $\Gamma < O(2)$ is a finite subgroup

Two-dimensional flat orbifolds

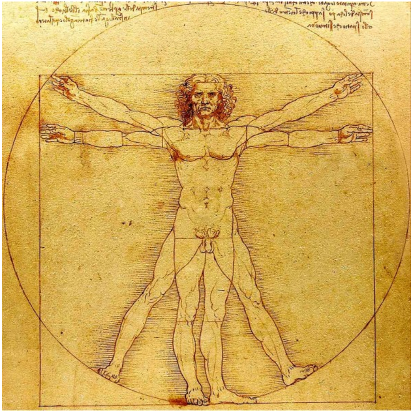
Local models: $\mathbb{R}^2/\Gamma$, where $\Gamma < O(2)$ is a finite subgroup

Theorem (Leonardo Da Vinci)

$\Gamma < \mathrm{O}(2)$ *is a finite subgroup*

Theorem (Leonardo Da Vinci)

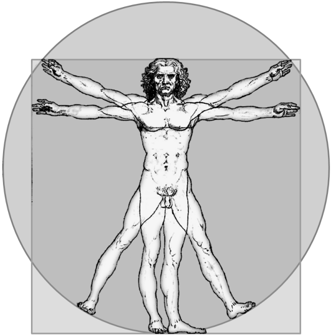
$\Gamma < O(2)$ is a finite subgroup $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



The Vitruvian Man

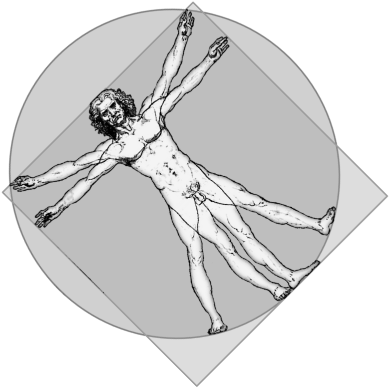
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\text{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\text{dihedral}}$



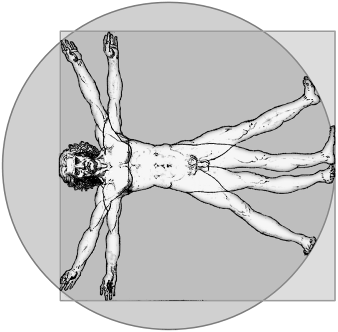
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



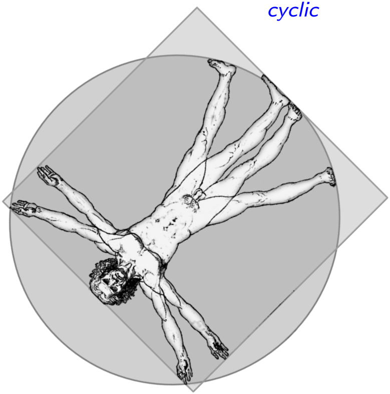
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



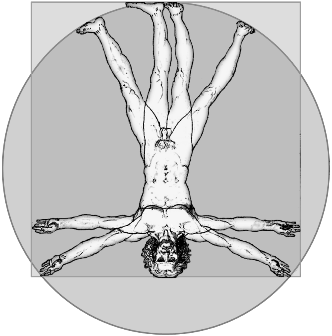
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\text{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\text{dihedral}}$



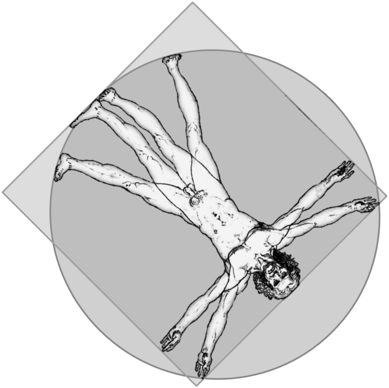
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ is a finite subgroup $\implies$ $\underbrace{\Gamma \cong \mathbb{Z}_k}_{\text{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\text{dihedral}}$



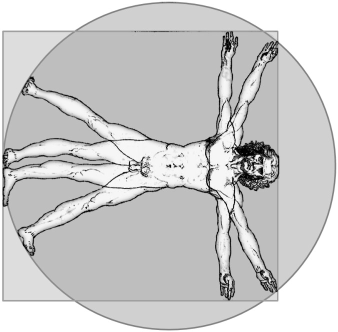
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\text{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\text{dihedral}}$



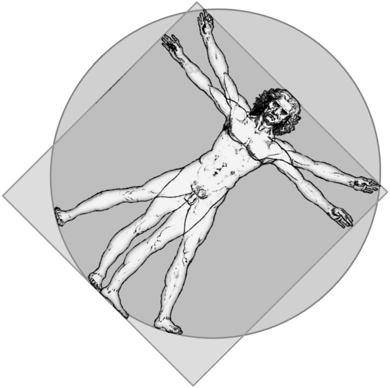
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



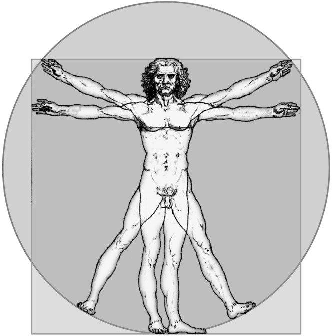
Theorem (Leonardo Da Vinci)

$\Gamma < \mathrm{O}(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



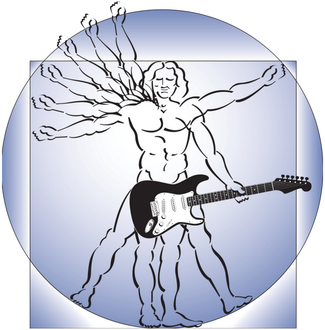
Theorem (Leonardo Da Vinci)

$\Gamma < \mathrm{O}(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



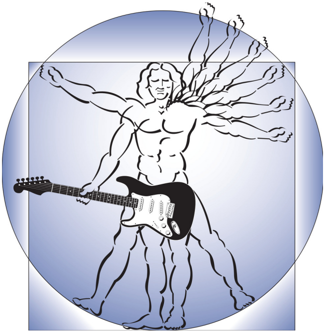
Theorem (Leonardo Da Vinci)

$\Gamma < \mathrm{O}(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



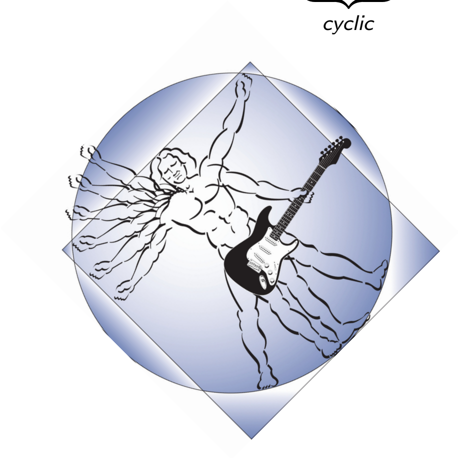
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{cyclic}$ *or* $\underbrace{\Gamma \cong D_k}_{dihedral}$



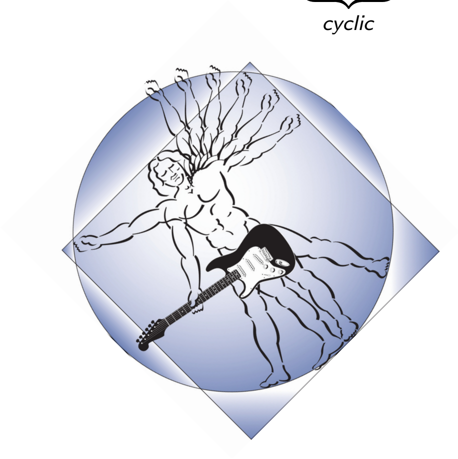
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



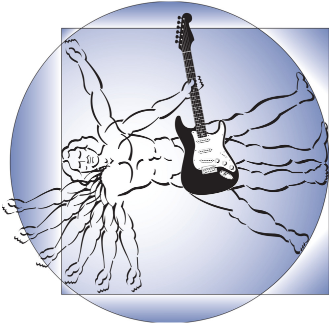
Theorem (Leonardo Da Vinci)

$\Gamma < \mathrm{O}(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



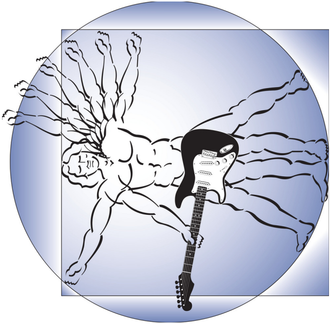
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\Longrightarrow \underbrace{\Gamma \cong \mathbb{Z}_k}_{cyclic}$ *or* $\underbrace{\Gamma \cong D_k}_{dihedral}$



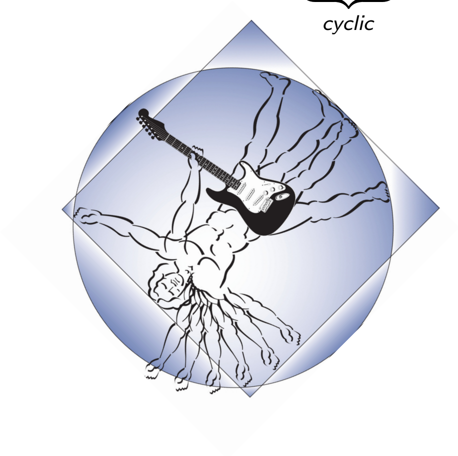
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



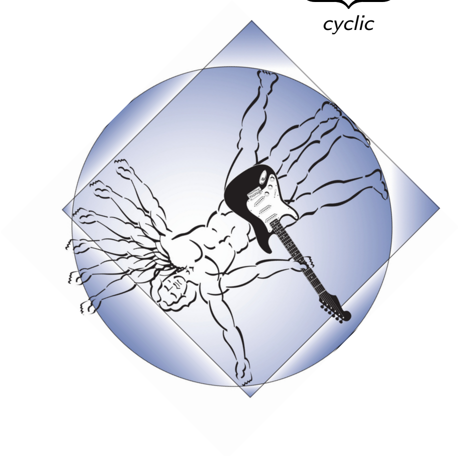
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



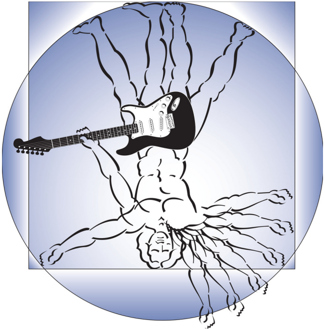
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\Longrightarrow \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



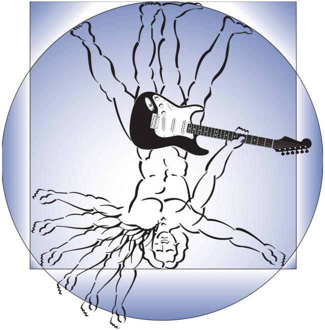
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\text{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\text{dihedral}}$



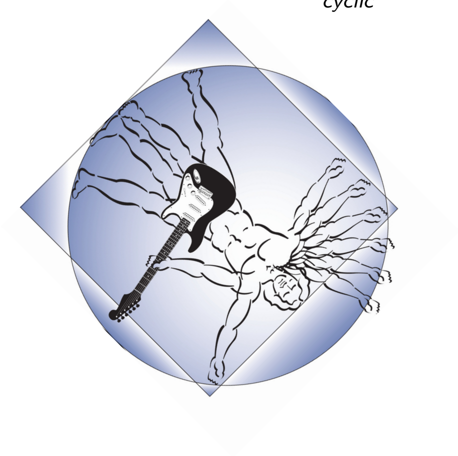
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



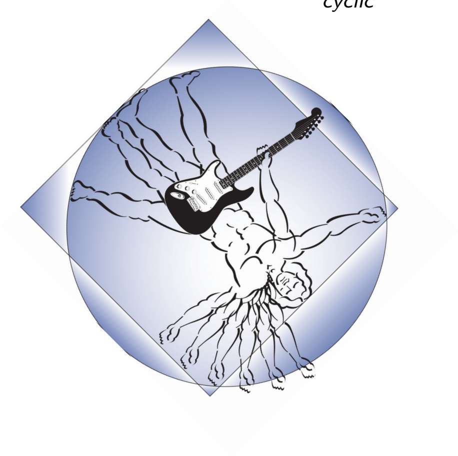
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\Longrightarrow \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



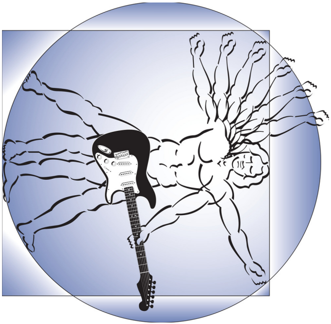
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



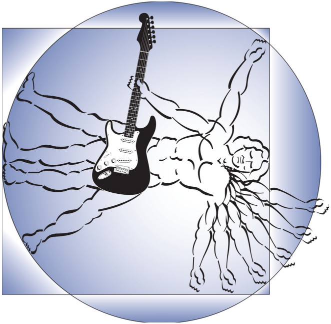
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\text{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\text{dihedral}}$



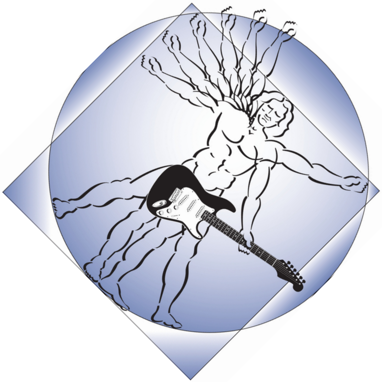
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



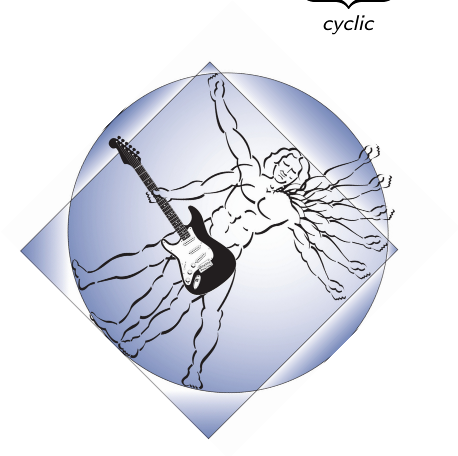
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\Longrightarrow \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



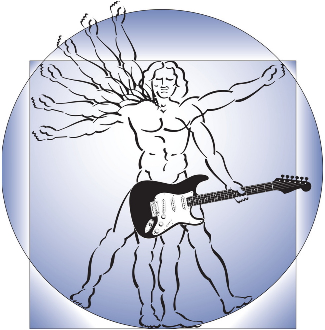
Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\Longrightarrow \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$



Theorem (Leonardo Da Vinci)

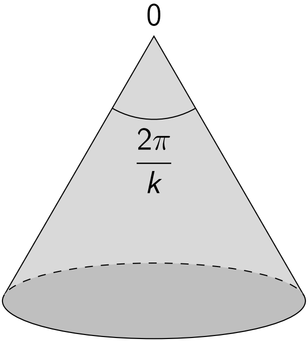
$\Gamma < \mathrm{O}(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\text{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\text{dihedral}}$

$\implies$ Models are $\mathbb{R}^2/\mathbb{Z}_k$, $\mathbb{R}^2/D_k$

Theorem (Leonardo Da Vinci)

$\Gamma < O(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\text{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\text{dihedral}}$

$\implies$ Models are $\mathbb{R}^2/\mathbb{Z}_k$, $\mathbb{R}^2/D_k$



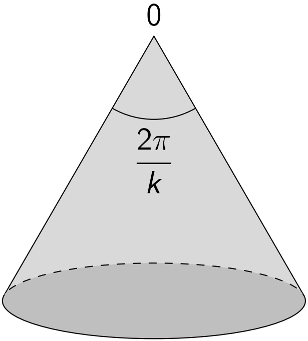
$\mathbb{R}^2/\mathbb{Z}_k$

Cone with angle $\frac{2\pi}{k}$

Theorem (Leonardo Da Vinci)

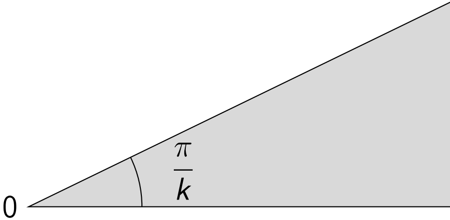
$\Gamma < \mathrm{O}(2)$ *is a finite subgroup* $\implies \underbrace{\Gamma \cong \mathbb{Z}_k}_{\textit{cyclic}}$ *or* $\underbrace{\Gamma \cong D_k}_{\textit{dihedral}}$

$\implies$ Models are $\mathbb{R}^2/\mathbb{Z}_k$, $\mathbb{R}^2/D_k$



$\mathbb{R}^2/\mathbb{Z}_k$

Cone with angle $\dfrac{2\pi}{k}$



$\mathbb{R}^2/D_k$

Wedge with angle $\dfrac{\pi}{k}$

Two-dimensional flat orbifolds

Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$

Two-dimensional flat orbifolds

Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$

Two-dimensional flat orbifolds

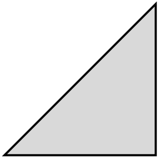
Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$

(1) "Rectangle"
$D^2(;2,2,2,2)$

Two-dimensional flat orbifolds

Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$

(1) "Rectangle"
$D^2(;2,2,2,2)$

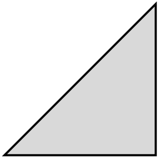


(2) "Half square"
$D^2(;2,4,4)$

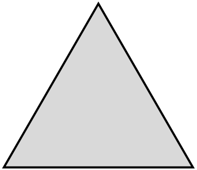
Two-dimensional flat orbifolds

Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$

(1) "Rectangle"
$D^2(;2,2,2,2)$



(2) "Half square"
$D^2(;2,4,4)$



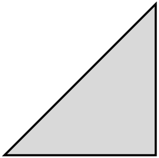
(3) "Equilateral triangle"
$D^2(;3,3,3)$

Two-dimensional flat orbifolds

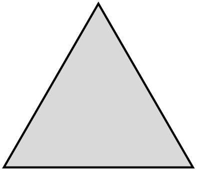
Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$

$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$

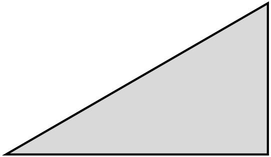
(1) “Rectangle”
$D^2(;2,2,2,2)$



(2) “Half square”
$D^2(;2,4,4)$



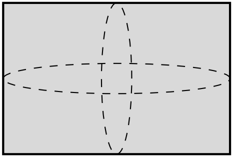
(3) “Equilateral triangle”
$D^2(;3,3,3)$



(4) “Half equilateral triangle”
$D^2(;2,3,6)$

Two-dimensional flat orbifolds

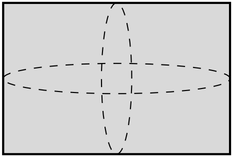
Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$



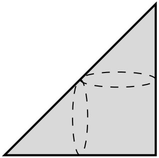
(5) "Pillowcase"
$S^2(2,2,2,2;\,)$

Two-dimensional flat orbifolds

Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$



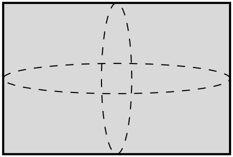
(5) "Pillowcase"
$S^2(2, 2, 2, 2;)$



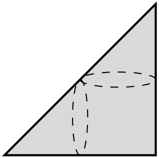
(6) "Turnover"
$S^2(2, 4, 4;)$

Two-dimensional flat orbifolds

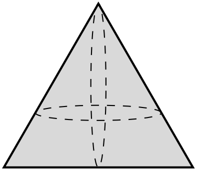
Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$



(5) “Pillowcase”
$S^2(2,2,2,2;)$



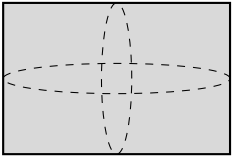
(6) “Turnover”
$S^2(2,4,4;)$



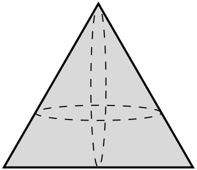
(7) “Turnover”
$S^2(3,3,3;)$

Two-dimensional flat orbifolds

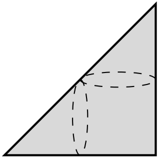
Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$



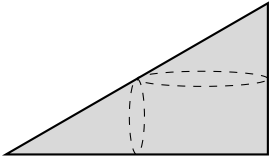
(5) "Pillowcase"
$S^2(2,2,2,2;\,)$



(7) "Turnover"
$S^2(3,3,3;\,)$



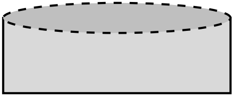
(6) "Turnover"
$S^2(2,4,4;\,)$



(8) "Turnover"
$S^2(2,3,6;\,)$

Two-dimensional flat orbifolds

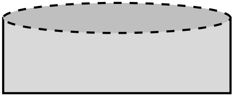
Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$



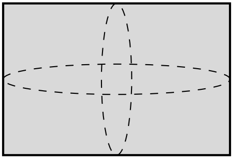
(9) "Half pillowcase"
$D^2(2,2;)$

Two-dimensional flat orbifolds

Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$



(9) "Half pillowcase"
$D^2(2, 2;)$

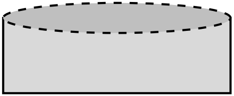


$/\mathbb{Z}_2$

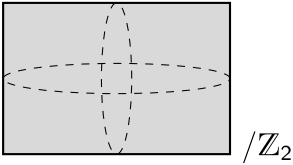
(10) "Projective pillowcase"
$\mathbb{R}P^2(2, 2;)$

Two-dimensional flat orbifolds

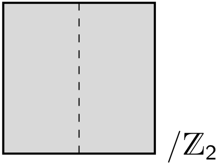
Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$



(9) "Half pillowcase"
$D^2(2,2;)$



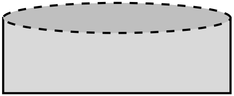
(10) "Projective pillowcase"
$\mathbb{R}P^2(2,2;)$



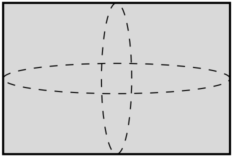
(11) $D^2(2;2,2)$

Two-dimensional flat orbifolds

Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$

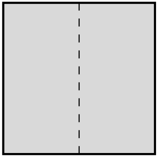


(9) "Half pillowcase"
$D^2(2,2;\,)$



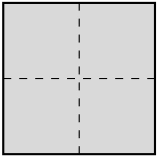
$/\mathbb{Z}_2$

(10) "Projective pillowcase"
$\mathbb{R}P^2(2,2;\,)$



$/\mathbb{Z}_2$

(11) $D^2(2;2,2)$

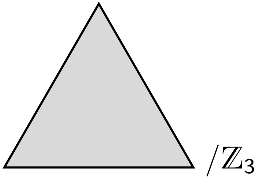


$/\mathbb{Z}_4$

(12) $D^2(4;2)$

Two-dimensional flat orbifolds

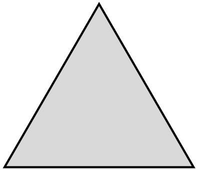
Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$



(13) $D^2(3;3)$

Two-dimensional flat orbifolds

Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$



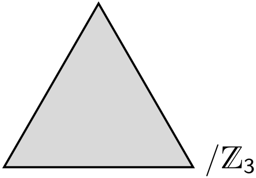
$/\mathbb{Z}_3$

(13) $D^2(3;3)$

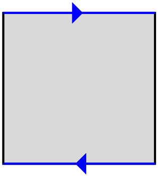
(14) "Cylinder" $S^1 \times I$

Two-dimensional flat orbifolds

Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$



(13) $D^2(3;3)$



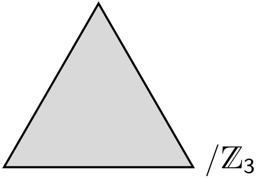
(15) "Möbius strip" M^2

(14) "Cylinder" $S^1 \times I$

Two-dimensional flat orbifolds

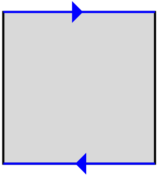
Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$

$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$



(13) $D^2(3;3)$

(14) “Cylinder” $S^1 \times I$

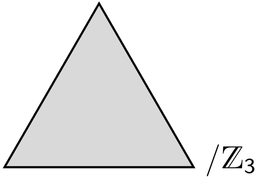


(15) “Möbius strip” M^2

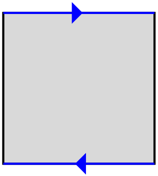
(16) “Torus” T^2

Two-dimensional flat orbifolds

Local models: $\mathbb{R}^2/\mathbb{Z}_k$, cone with angle $\frac{2\pi}{k}$
$\mathbb{R}^2/D_k$, wedge with angle $\frac{\pi}{k}$



(13) $D^2(3;3)$



(15) "Möbius strip" M^2

(14) "Cylinder" $S^1 \times I$

(16) "Torus" T^2

(17) "Klein bottle" K^2

Local to global

Q: How to classify flat orbifolds and flat manifolds?

Local to global

Q: How to classify flat orbifolds and flat manifolds?

Definition

A subgroup $\pi < \mathrm{Iso}(\mathbb{R}^n) \cong \mathrm{O}(n) \ltimes \mathbb{R}^n$ is

- **crystallographic** if π is discrete and cocompact;

Local to global

Q: How to classify flat orbifolds and flat manifolds?

Definition

A subgroup $\pi < \mathrm{Iso}(\mathbb{R}^n) \cong \mathrm{O}(n) \ltimes \mathbb{R}^n$ is

- **crystallographic** if π is discrete and cocompact;
  (i.e., orbit space of $\pi \curvearrowright \mathbb{R}^n$ is Hausdorff and compact)

Local to global

Q: How to classify flat orbifolds and flat manifolds?

Definition

A subgroup $\pi < \mathrm{Iso}(\mathbb{R}^n) \cong \mathrm{O}(n) \ltimes \mathbb{R}^n$ is

- **crystallographic** if π is discrete and cocompact;
  (i.e., orbit space of $\pi \curvearrowright \mathbb{R}^n$ is Hausdorff and compact)
- **Bieberbach** if π is crystallographic and torsion-free.

Local to global

Q: How to classify flat orbifolds and flat manifolds?

Definition

A subgroup $\pi < \mathrm{Iso}(\mathbb{R}^n) \cong \mathrm{O}(n) \ltimes \mathbb{R}^n$ is

- **crystallographic** if π is discrete and cocompact;
  (i.e., orbit space of $\pi \curvearrowright \mathbb{R}^n$ is Hausdorff and compact)
- **Bieberbach** if π is crystallographic and torsion-free.
  (i.e., $\pi \curvearrowright \mathbb{R}^n$ is free; orbit space is smooth)

Local to global

Q: How to classify flat orbifolds and flat manifolds?

Definition

A subgroup $\pi < \mathrm{Iso}(\mathbb{R}^n) \cong \mathrm{O}(n) \ltimes \mathbb{R}^n$ is

- **crystallographic** if π is discrete and cocompact;
  (i.e., orbit space of $\pi \curvearrowright \mathbb{R}^n$ is Hausdorff and compact)
- **Bieberbach** if π is crystallographic and torsion-free.
  (i.e., $\pi \curvearrowright \mathbb{R}^n$ is free; orbit space is smooth)

Theorem

- *M is a flat manifold $\iff M = \mathbb{R}^n/\pi$, π Bieberbach*

Local to global

Q: How to classify flat orbifolds and flat manifolds?

Definition

A subgroup $\pi < \mathrm{Iso}(\mathbb{R}^n) \cong \mathrm{O}(n) \ltimes \mathbb{R}^n$ is

- **crystallographic** if π is discrete and cocompact;
  (i.e., orbit space of $\pi \curvearrowright \mathbb{R}^n$ is Hausdorff and compact)
- **Bieberbach** if π is crystallographic and torsion-free.
  (i.e., $\pi \curvearrowright \mathbb{R}^n$ is free; orbit space is smooth)

Theorem

- *M is a flat manifold $\iff M = \mathbb{R}^n/\pi$, π Bieberbach*
- *$\mathcal{O}$ is a flat orbifold $\iff \mathcal{O} = \mathbb{R}^n/\pi$, π crystallographic*

Local to global

Q: How to classify flat orbifolds and flat manifolds?

Definition

A subgroup $\pi < \mathrm{Iso}(\mathbb{R}^n) \cong \mathrm{O}(n) \ltimes \mathbb{R}^n$ is

- **crystallographic** if π is discrete and cocompact;
  (i.e., orbit space of $\pi \curvearrowright \mathbb{R}^n$ is Hausdorff and compact)
- **Bieberbach** if π is crystallographic and torsion-free.
  (i.e., $\pi \curvearrowright \mathbb{R}^n$ is free; orbit space is smooth)

Theorem

- *M is a flat manifold $\iff M = \mathbb{R}^n/\pi$, π Bieberbach*
- *$\mathcal{O}$ is a flat orbifold $\iff \mathcal{O} = \mathbb{R}^n/\pi$, π crystallographic*

A: Classify crystallographic and Bieberbach groups!

Structure of crystallographic groups

$$\rho\colon \; \mathsf{Iso}(\mathbb{R}^n) \cong \mathsf{O}(n) \ltimes \mathbb{R}^n \longrightarrow \mathsf{O}(n)$$

Structure of crystallographic groups

$$\rho\colon \mathrm{Iso}(\mathbb{R}^n) \cong \mathrm{O}(n) \ltimes \mathbb{R}^n \longrightarrow \mathrm{O}(n)$$

$$0 \longrightarrow L_\pi \longrightarrow \pi \xrightarrow{\rho} H_\pi \longrightarrow 0$$

Structure of crystallographic groups

$$\rho\colon \operatorname{Iso}(\mathbb{R}^n) \cong \mathrm{O}(n) \ltimes \mathbb{R}^n \longrightarrow \mathrm{O}(n)$$
$$0 \longrightarrow L_\pi \longrightarrow \pi \xrightarrow{\rho} H_\pi \longrightarrow 0$$

$\ker \rho = L_\pi \quad \leftrightsquigarrow \quad$ translations

Structure of crystallographic groups

$$\rho\colon \operatorname{Iso}(\mathbb{R}^n) \cong \mathrm{O}(n) \ltimes \mathbb{R}^n \longrightarrow \mathrm{O}(n)$$

$$0 \longrightarrow L_\pi \longrightarrow \pi \xrightarrow{\rho} H_\pi \longrightarrow 0$$

$\ker \rho = L_\pi$ $\quad\leftrightsquigarrow\quad$ translations

$\operatorname{im} \rho = H_\pi$ $\quad\leftrightsquigarrow\quad$ rotations and reflections

Structure of crystallographic groups

$$\rho\colon \operatorname{Iso}(\mathbb{R}^n) \cong \mathrm{O}(n) \ltimes \mathbb{R}^n \longrightarrow \mathrm{O}(n)$$

$$0 \longrightarrow L_\pi \longrightarrow \pi \xrightarrow{\rho} H_\pi \longrightarrow 0$$

$\ker\rho = L_\pi$	$\leftrightsquigarrow$	translations
$\operatorname{im}\rho = H_\pi$	$\leftrightsquigarrow$	rotations and reflections

Theorem (Bieberbach, 1911)

1. *$L_\pi \cong \mathbb{Z}^n$ is a lattice and $H_\pi < \mathrm{O}(n)$ is finite;*

Structure of crystallographic groups

$$\rho\colon \ \mathsf{Iso}(\mathbb{R}^n) \cong \mathsf{O}(n) \ltimes \mathbb{R}^n \longrightarrow \mathsf{O}(n)$$
$$0 \longrightarrow L_\pi \longrightarrow \pi \xrightarrow{\rho} H_\pi \longrightarrow 0$$

$\ker \rho = L_\pi$ $\quad\leftrightsquigarrow\quad$ translations
$\operatorname{im} \rho = H_\pi$ $\quad\leftrightsquigarrow\quad$ rotations and reflections

Theorem (Bieberbach, 1911)

1. *$L_\pi \cong \mathbb{Z}^n$ is a lattice and $H_\pi < \mathsf{O}(n)$ is finite;*
2. *$\pi, \pi' < \mathsf{Iso}(\mathbb{R}^n)$ crystallographic groups are isomorphic if and only if π, π' are conjugate in $\mathrm{Aff}(\mathbb{R}^n) = \mathsf{GL}(n) \ltimes \mathbb{R}^n$;*

Structure of crystallographic groups

$$\rho\colon \mathsf{Iso}(\mathbb{R}^n) \cong \mathsf{O}(n) \ltimes \mathbb{R}^n \longrightarrow \mathsf{O}(n)$$
$$0 \longrightarrow L_\pi \longrightarrow \pi \xrightarrow{\rho} H_\pi \longrightarrow 0$$

$\ker \rho = L_\pi$ $\quad\leftrightsquigarrow\quad$ translations
$\operatorname{im} \rho = H_\pi$ $\quad\leftrightsquigarrow\quad$ rotations and reflections

Theorem (Bieberbach, 1911)

1. *$L_\pi \cong \mathbb{Z}^n$ is a lattice and $H_\pi < \mathsf{O}(n)$ is finite;*
2. *$\pi, \pi' < \mathsf{Iso}(\mathbb{R}^n)$ crystallographic groups are isomorphic if and only if π, π' are conjugate in $\mathrm{Aff}(\mathbb{R}^n) = \mathsf{GL}(n) \ltimes \mathbb{R}^n$;*
3. *For each n, there are only finitely many isomorphism types of crystallographic groups in $\mathsf{Iso}(\mathbb{R}^n)$.*

Structure of crystallographic groups

$$\rho\colon \mathsf{Iso}(\mathbb{R}^n) \cong \mathsf{O}(n) \ltimes \mathbb{R}^n \longrightarrow \mathsf{O}(n)$$
$$0 \longrightarrow L_\pi \longrightarrow \pi \xrightarrow{\rho} H_\pi \longrightarrow 0$$

$\ker \rho = L_\pi$ ⇜⇝ translations
$\operatorname{im} \rho = H_\pi$ ⇜⇝ rotations and reflections

Theorem (Bieberbach, 1911)

1. *$L_\pi \cong \mathbb{Z}^n$ is a lattice and $H_\pi < \mathsf{O}(n)$ is finite;*
2. *$\pi, \pi' < \mathsf{Iso}(\mathbb{R}^n)$ crystallographic groups are isomorphic if and only if π, π' are conjugate in $\mathrm{Aff}(\mathbb{R}^n) = \mathsf{GL}(n) \ltimes \mathbb{R}^n$;*
3. *For each n, there are only finitely many isomorphism types of crystallographic groups in $\mathsf{Iso}(\mathbb{R}^n)$.*

Solved (first third of) Hilbert's 18th problem

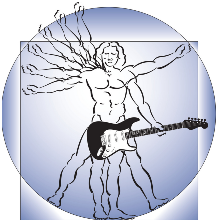
Classification in low dimensions

n	# Bieberbach groups	# Crystallographic groups
2	2	17

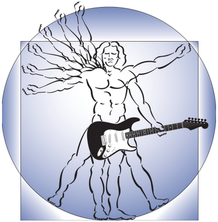
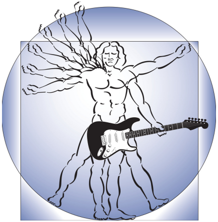
Classification in low dimensions

n	# Bieberbach groups	# Crystallographic groups
2	2	17

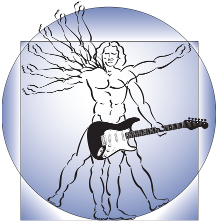
17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"

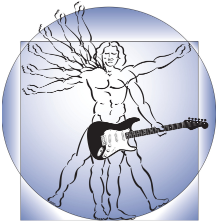


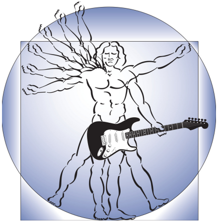
17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"

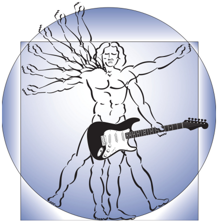
17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"

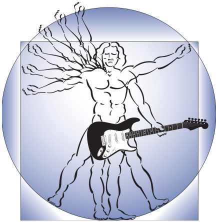


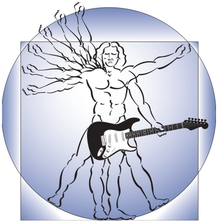


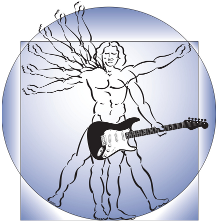


17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"

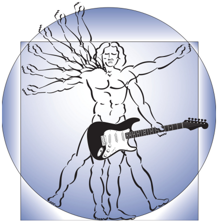
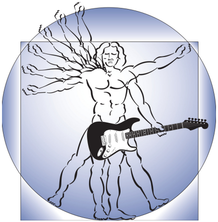
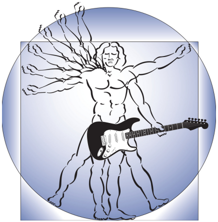
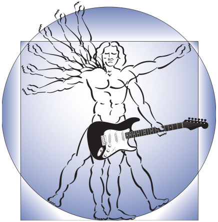
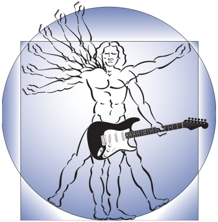




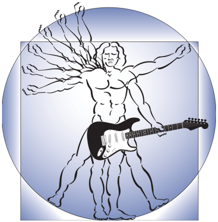
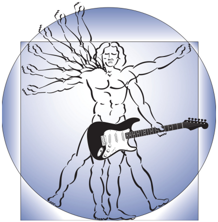
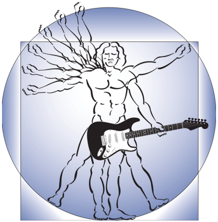
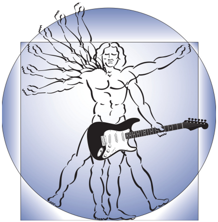
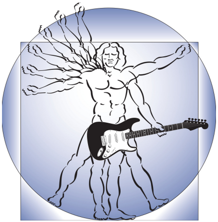
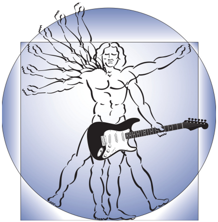




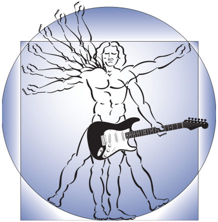
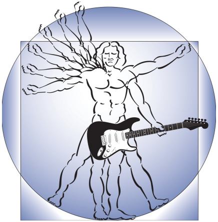
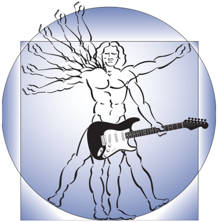
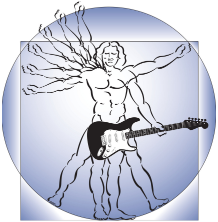
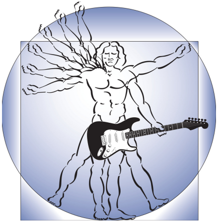
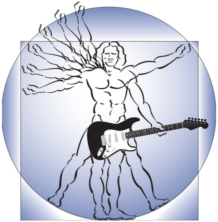
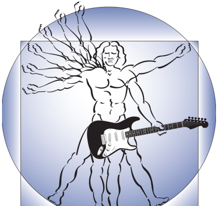
17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"

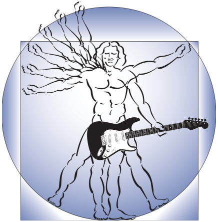
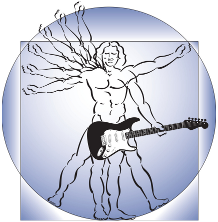
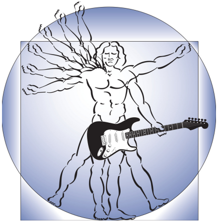
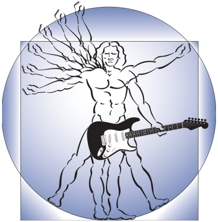
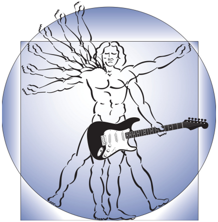
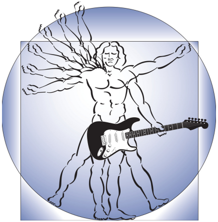
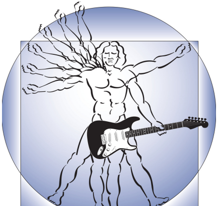
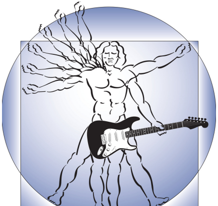
17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"

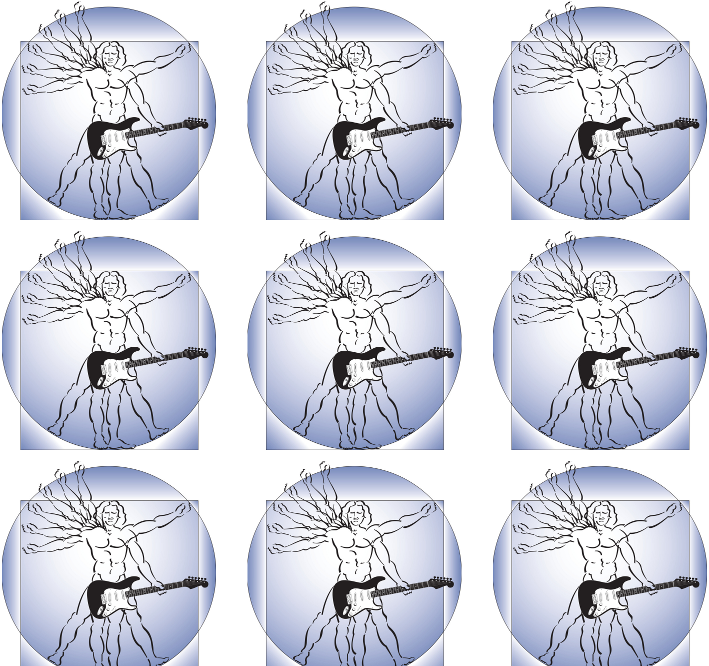
17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"

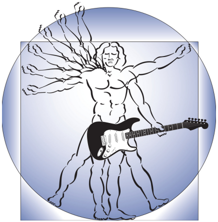
17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"

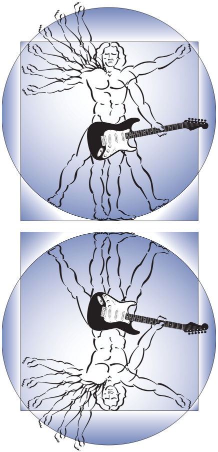
17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"



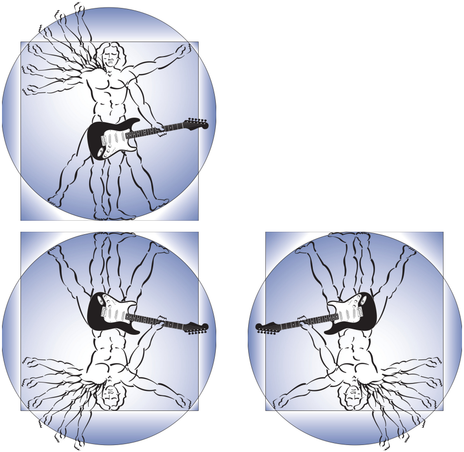
17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"



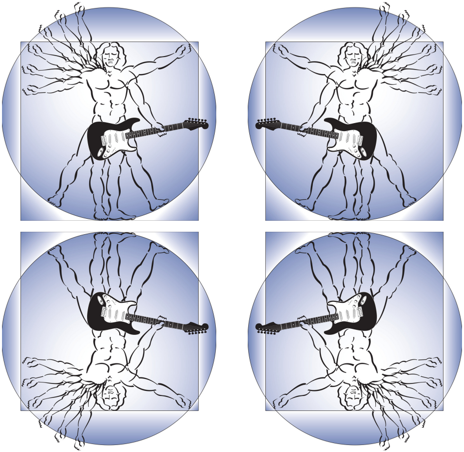
17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"



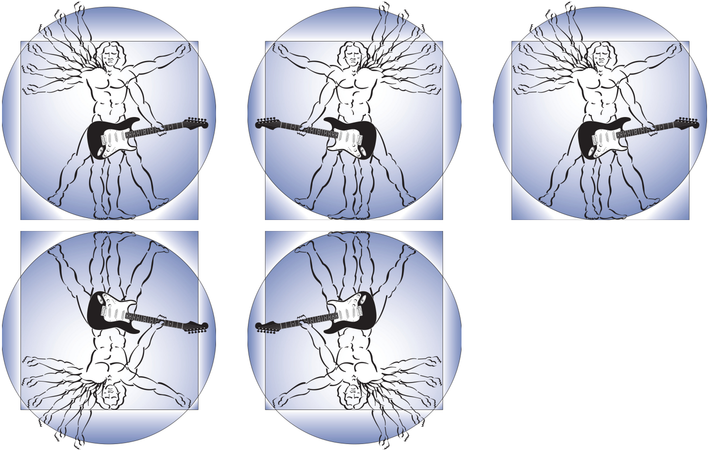
17 Crystallographic groups in $Iso(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"



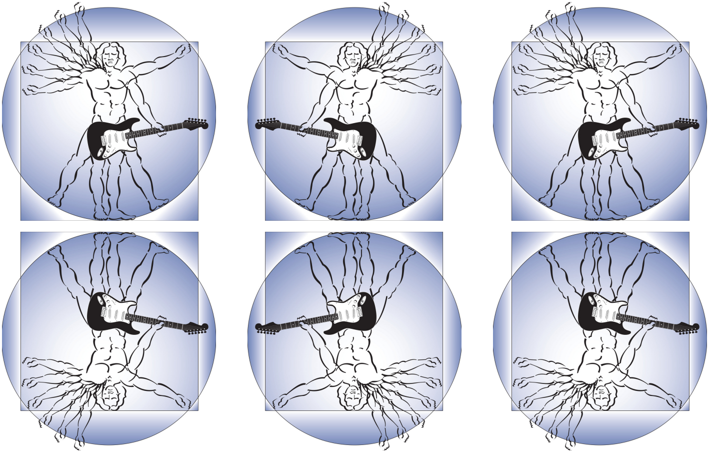
17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"



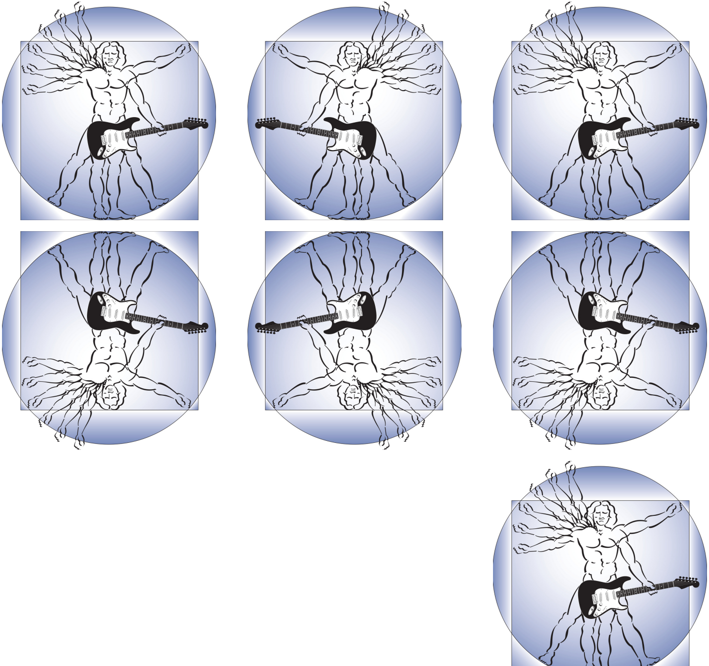
17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"



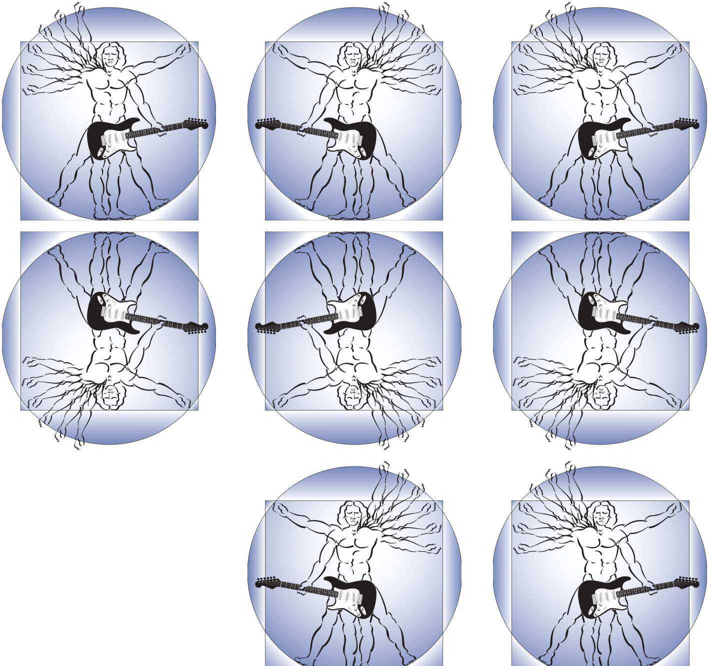
17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"



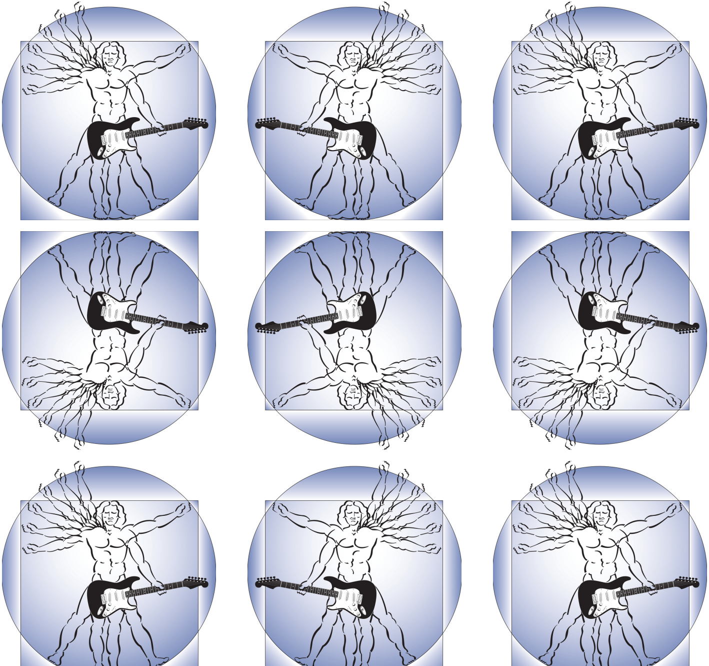
17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"



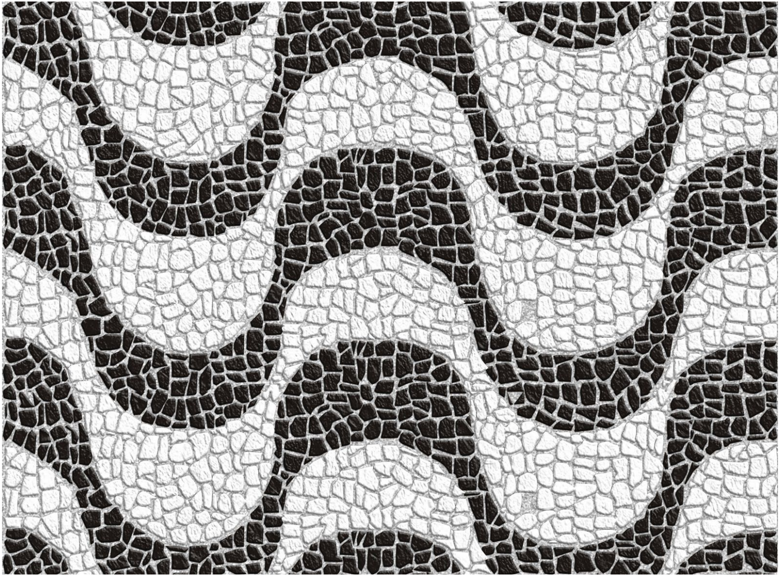
17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"



17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"



17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"

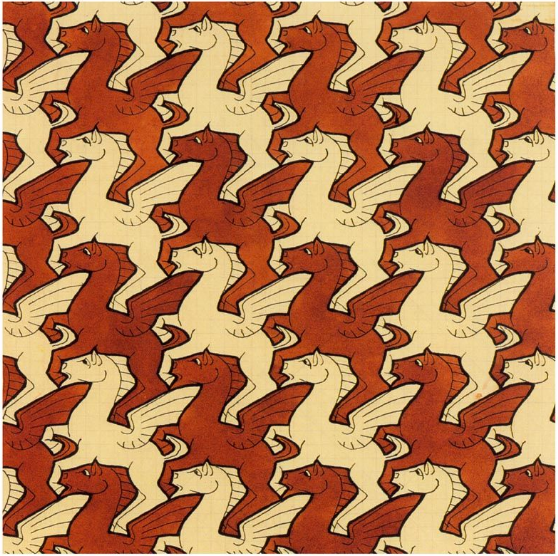


Copacabana, Rio de Janeiro (Brazil)

17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"

Alhambra, Granada (Spain)

17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"



M. C. Escher

17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"

M. C. Escher

17 Crystallographic groups in $\mathrm{Iso}(\mathbb{R}^2)$, a.k.a. "Wallpaper groups"

M. C. Escher

Classification in low dimensions

n	# Bieberbach groups	# Crystallographic groups
2	2	17

Classification in low dimensions

n	# Bieberbach groups	# Crystallographic groups
2	2	17
3	10	219

Classification in low dimensions

n	# Bieberbach groups	# Crystallographic groups
2	2	17
3	10	219
4	74	4,783

(Computer assisted)

Classification in low dimensions

n	# Bieberbach groups	# Crystallographic groups
2	2	17
3	10	219
4	74	4,783
5	1,060	222,018

(Computer assisted)

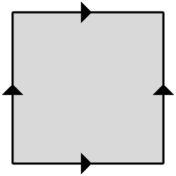
Classification in low dimensions

n	# Bieberbach groups	# Crystallographic groups
2	2	17
3	10	219
4	74	4,783
5	1,060	222,018
6	38,746	28,927,922

(Computer assisted)

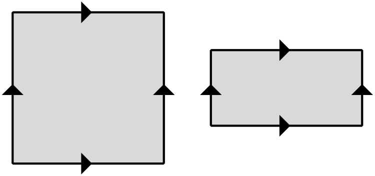
Can we always deform these spaces (nontrivially)?

T^2



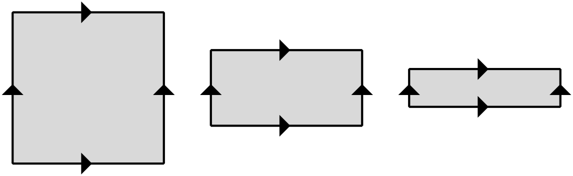
Can we always deform these spaces (nontrivially)?

T^2 – YES!



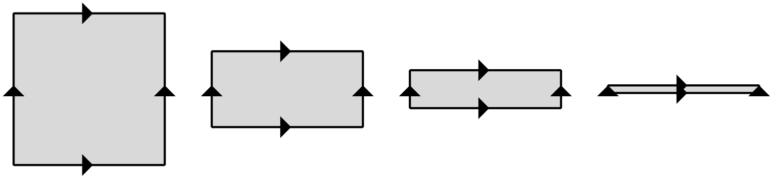
Can we always deform these spaces (nontrivially)?

T^2 – YES!



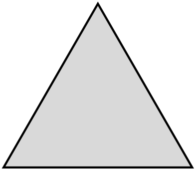
Can we always deform these spaces (nontrivially)?

T^2 – YES!



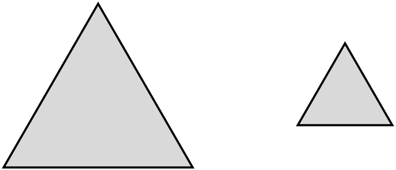
Can we always deform these spaces (nontrivially)?

$D^2(;3,3,3)$



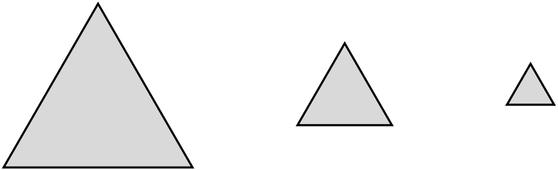
Can we always deform these spaces (nontrivially)?

$D^2(;3,3,3)$ – NO!



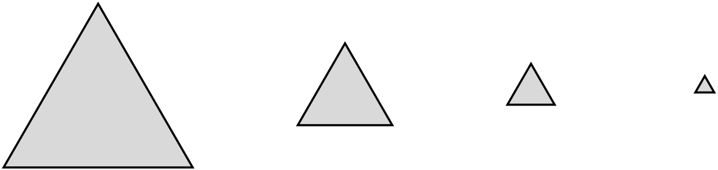
Can we always deform these spaces (nontrivially)?

$D^2(;3,3,3)$ – NO!



Can we always deform these spaces (nontrivially)?

$D^2(;3,3,3)$ – NO!



Flat deformations of $\mathbb{R}^n/\pi$ $\quad\leftrightsquigarrow\quad$ H_π-invariant subspaces of $\mathbb{R}^n$

Flat deformations of $\mathbb{R}^n/\pi$ $\rightsquigarrow$ H_π-invariant subspaces of $\mathbb{R}^n$

Theorem (Hiss, Szczepański, 1991)

$\pi < \mathrm{Iso}(\mathbb{R}^n)$ **Bieberbach group** $\Longrightarrow H_\pi \curvearrowright \mathbb{R}^n$ *is* **reducible**.

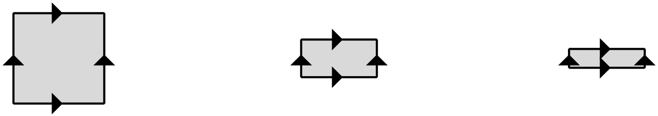
$\boxed{\text{Flat deformations of } \mathbb{R}^n/\pi} \quad \leftrightsquigarrow \quad \boxed{H_\pi\text{-invariant subspaces of } \mathbb{R}^n}$

Theorem (Hiss, Szczepański, 1991)

$\pi < \mathrm{Iso}(\mathbb{R}^n)$ **Bieberbach group** $\Longrightarrow$ $H_\pi \curvearrowright \mathbb{R}^n$ *is* **reducible**.

Corollary

All flat manifolds admit (nonhomothetic) flat deformations.



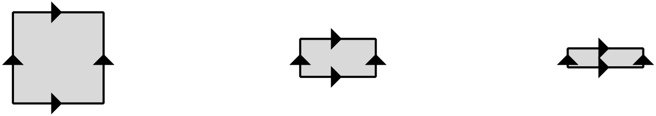
Flat deformations of $\mathbb{R}^n/\pi$	$\rightsquigarrow$	H_π-invariant subspaces of $\mathbb{R}^n$

Theorem (Hiss, Szczepański, 1991)

$\pi < \mathrm{Iso}(\mathbb{R}^n)$ **Bieberbach group** $\Longrightarrow H_\pi \curvearrowright \mathbb{R}^n$ *is* **reducible**.

Corollary

All flat manifolds admit (nonhomothetic) flat deformations.



But:

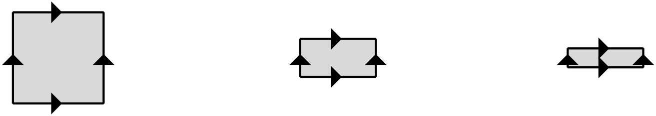
Flat deformations of $\mathbb{R}^n/\pi$ $\leftrightsquigarrow$ H_π-invariant subspaces of $\mathbb{R}^n$

Theorem (Hiss, Szczepański, 1991)

$\pi < \mathrm{Iso}(\mathbb{R}^n)$ **Bieberbach group** $\Longrightarrow$ $H_\pi \curvearrowright \mathbb{R}^n$ *is* **reducible**.

Corollary

All flat manifolds admit (nonhomothetic) flat deformations.



But:

$\exists\, \pi < \mathrm{Iso}(\mathbb{R}^n)$ **crystallographic** with $H_\pi \curvearrowright \mathbb{R}^n$ **irreducible.**

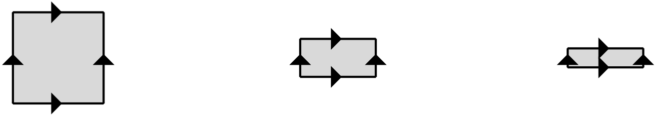
Flat deformations of $\mathbb{R}^n/\pi$ $\leftrightsquigarrow$ H_π-invariant subspaces of $\mathbb{R}^n$

Theorem (Hiss, Szczepański, 1991)

$\pi < \mathrm{Iso}(\mathbb{R}^n)$ **Bieberbach group** $\Longrightarrow$ $H_\pi \curvearrowright \mathbb{R}^n$ *is* **reducible**.

Corollary

All flat manifolds admit (nonhomothetic) flat deformations.

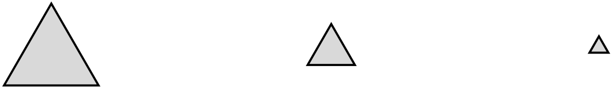


But:

$\exists\, \pi < \mathrm{Iso}(\mathbb{R}^n)$ **crystallographic** with $H_\pi \curvearrowright \mathbb{R}^n$ **irreducible.**

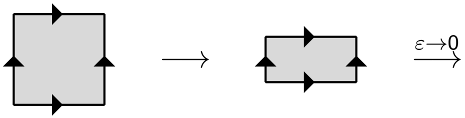
So:

Not all flat orbifolds have (nonhomothetic) flat deformations.

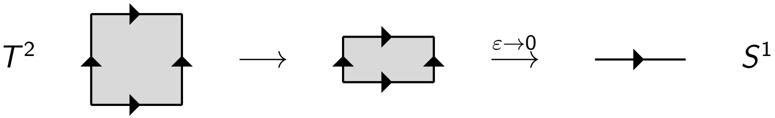


If it can be deformed, what is the limit?

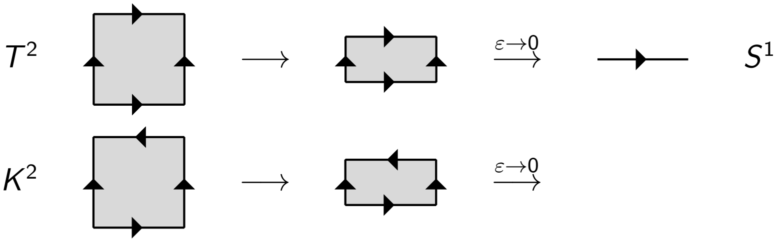
T^2



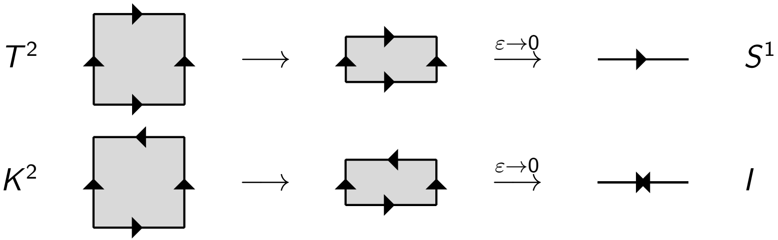
If it can be deformed, what is the limit?



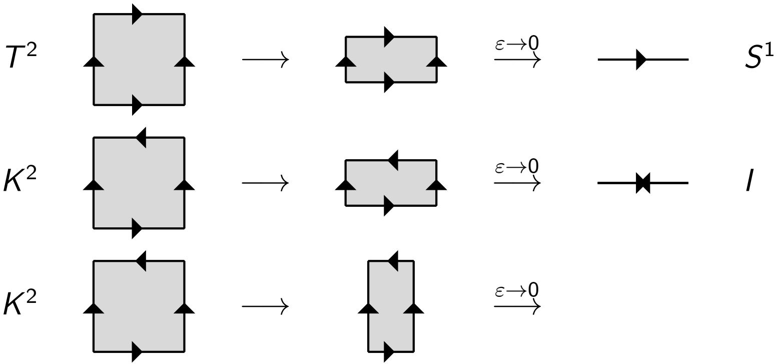
If it can be deformed, what is the limit?



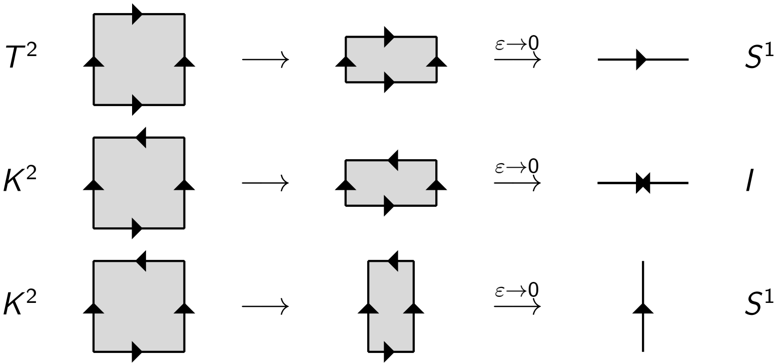
If it can be deformed, what is the limit?



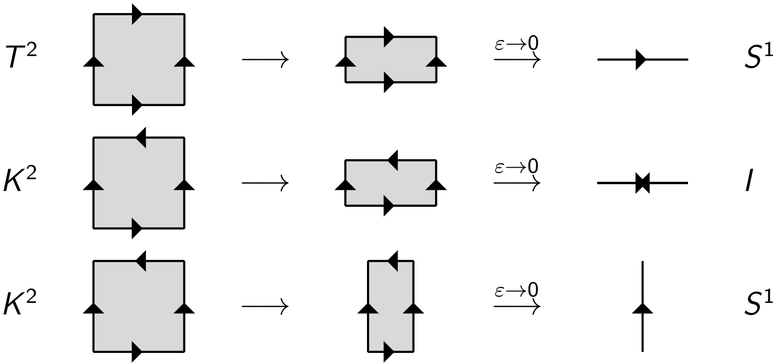
If it can be deformed, what is the limit?



If it can be deformed, what is the limit?



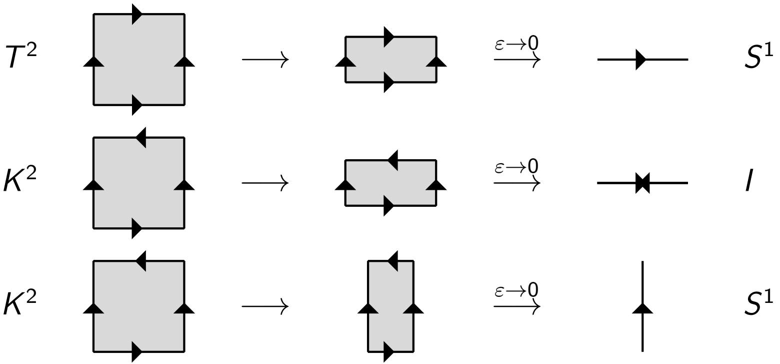
If it can be deformed, what is the limit?



Theorem (B., Derdzinski, Piccione, 2018)

The Gromov-Hausdorff limit of flat manifolds is a flat orbifold.

If it can be deformed, what is the limit?



Theorem (B., Derdzinski, Piccione, 2018)

The Gromov-Hausdorff limit of flat manifolds is a flat orbifold. Conversely, every flat orbifold is the Gromov-Hausdorff limit of flat manifolds.

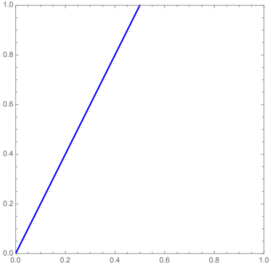
Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t\searrow 0} M_t$.

Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t \searrow 0} M_t$.

Collapsing an H_π-invariant subspace $V \subset \mathbb{R}^n$:

Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t\searrow 0} M_t$.

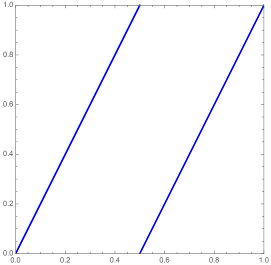
Collapsing an H_π-invariant subspace $V \subset \mathbb{R}^n$:



$V \subset \mathbb{R}^2$ rational slope

Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t\searrow 0} M_t$.

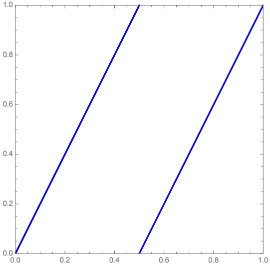
Collapsing an H_π-invariant subspace $V \subset \mathbb{R}^n$:



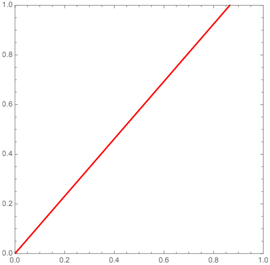
$V \subset \mathbb{R}^2$ rational slope

Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t\searrow 0} M_t$.

Collapsing an H_π-invariant subspace $V \subset \mathbb{R}^n$:



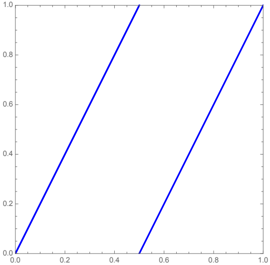
$V \subset \mathbb{R}^2$ rational slope



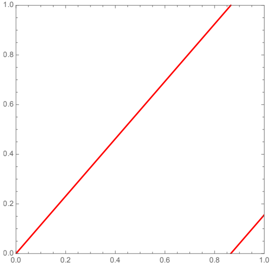
$V \subset \mathbb{R}^2$ irrational slope

Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t\searrow 0} M_t$.

Collapsing an H_π-invariant subspace $V \subset \mathbb{R}^n$:



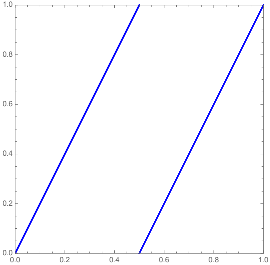
$V \subset \mathbb{R}^2$ rational slope



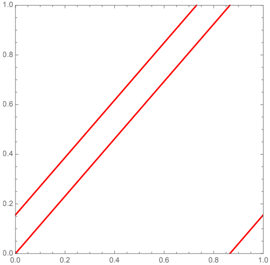
$V \subset \mathbb{R}^2$ irrational slope

Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t\searrow 0} M_t$.

Collapsing an H_π-invariant subspace $V \subset \mathbb{R}^n$:



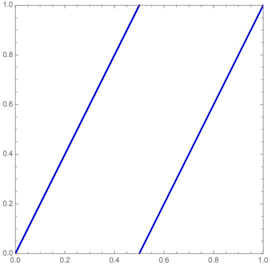
$V \subset \mathbb{R}^2$ rational slope



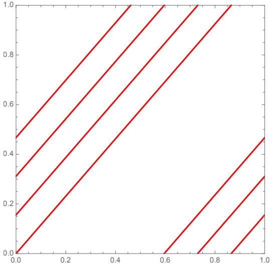
$V \subset \mathbb{R}^2$ irrational slope

Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t\searrow 0} M_t$.

Collapsing an H_π-invariant subspace $V \subset \mathbb{R}^n$:



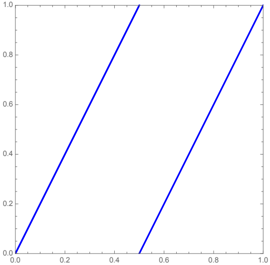
$V \subset \mathbb{R}^2$ rational slope



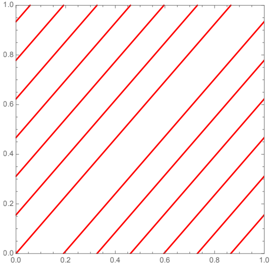
$V \subset \mathbb{R}^2$ irrational slope

Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t \searrow 0} M_t$.

Collapsing an H_π-invariant subspace $V \subset \mathbb{R}^n$:



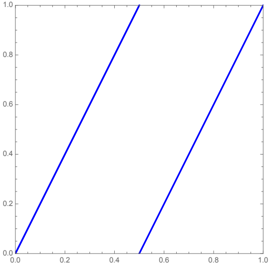
$V \subset \mathbb{R}^2$ rational slope



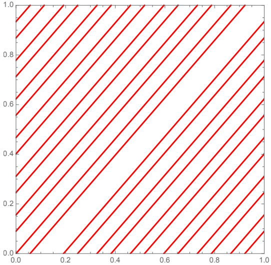
$V \subset \mathbb{R}^2$ irrational slope

Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t\searrow 0} M_t$.

Collapsing an H_π-invariant subspace $V \subset \mathbb{R}^n$:



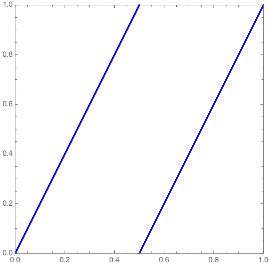
$V \subset \mathbb{R}^2$ rational slope



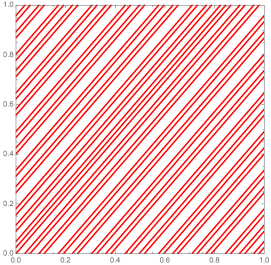
$V \subset \mathbb{R}^2$ irrational slope

Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t\searrow 0} M_t$.

Collapsing an H_π-invariant subspace $V \subset \mathbb{R}^n$:



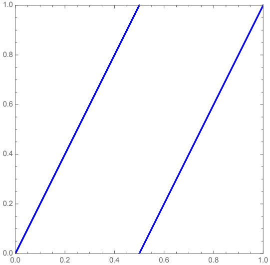
$V \subset \mathbb{R}^2$ rational slope



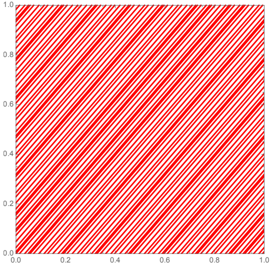
$V \subset \mathbb{R}^2$ irrational slope

Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t\searrow 0} M_t$.

Collapsing an H_π-invariant subspace $V \subset \mathbb{R}^n$:



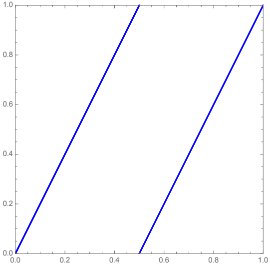
$V \subset \mathbb{R}^2$ rational slope



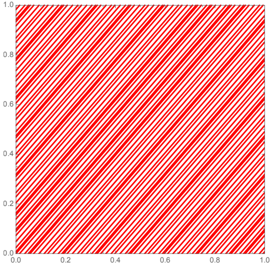
$V \subset \mathbb{R}^2$ irrational slope

Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t\searrow 0} M_t$.

Collapsing an H_π-invariant subspace $V \subset \mathbb{R}^n$:



$V \subset \mathbb{R}^2$ rational slope

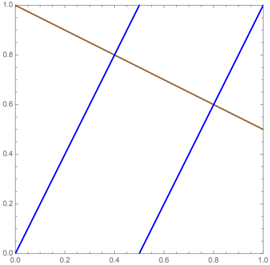


$V \subset \mathbb{R}^2$ irrational slope

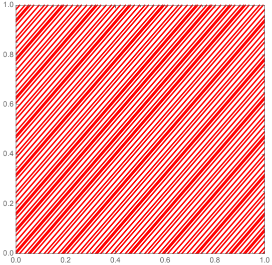
- $\overline{V}$ smallest H_π-invariant, spanned by elements of L_π

Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t \searrow 0} M_t$.

Collapsing an H_π-invariant subspace $V \subset \mathbb{R}^n$:



$V \subset \mathbb{R}^2$ rational slope

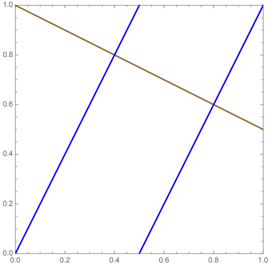


$V \subset \mathbb{R}^2$ irrational slope

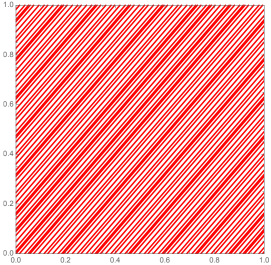
- $\overline{V}$ smallest H_π-invariant, spanned by elements of L_π
- $\exists\, W$ H_π-invariant, spanned by elements of L_π, $\overline{V} \oplus W = \mathbb{R}^n$

Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t\searrow 0} M_t$.

Collapsing an H_π-invariant subspace $V \subset \mathbb{R}^n$:



$V \subset \mathbb{R}^2$ rational slope

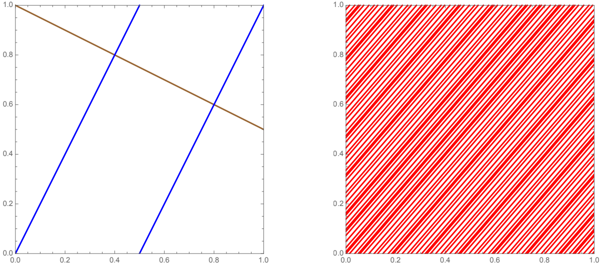


$V \subset \mathbb{R}^2$ irrational slope

- $\overline{V}$ smallest H_π-invariant, spanned by elements of L_π
- $\exists\, W$ H_π-invariant, spanned by elements of L_π, $\overline{V} \oplus W = \mathbb{R}^n$
- $H_\pi^W = H_\pi / \{A \in H_\pi : A|_{\overline{V}} = \mathrm{Id}\}$

Q: Given a family of flat n-manifolds $M_t = \mathbb{R}^n/\pi_t$ that collapses as $t \searrow 0$, compute the resulting orbifold $\mathcal{O} = \lim_{t\searrow 0} M_t$.

Collapsing an H_π-invariant subspace $V \subset \mathbb{R}^n$:



$V \subset \mathbb{R}^2$ rational slope $V \subset \mathbb{R}^2$ irrational slope

- $\overline{V}$ smallest H_π-invariant, spanned by elements of L_π
- $\exists\, W$ H_π-invariant, spanned by elements of L_π, $\overline{V} \oplus W = \mathbb{R}^n$
- $H_\pi^W = H_\pi / \{A \in H_\pi : A|_{\overline{V}} = \mathrm{Id}\}$

Theorem (B., Derdzinski, Mossa, Piccione, 2018)

Collapsing $M = \mathbb{R}^n/\pi$ along V results in $\mathcal{O}_V = (L_\pi \cap W)\backslash W / H_\pi^W$.

Q: Given a flat n-orbifold $\mathcal{O}$, what is the **smallest** k such that a flat $(n+k)$-manifold collapses to $\mathcal{O}$?

Q: Given a flat n-orbifold $\mathcal{O}$, what is the **smallest** k such that a flat $(n+k)$-manifold collapses to $\mathcal{O}$?

Theorem (B., Derdzinski, Piccione, 2018)

The Gromov-Hausdorff limit of a sequence of flat 3-manifolds must be one of:

Q: Given a flat n-orbifold $\mathcal{O}$, what is the **smallest** k such that a flat $(n+k)$-manifold collapses to $\mathcal{O}$?

Theorem (B., Derdzinski, Piccione, 2018)

The Gromov-Hausdorff limit of a sequence of flat 3-manifolds must be one of:

- *Flat 0-orbifold:* $\{p\}$;

Q: Given a flat n-orbifold $\mathcal{O}$, what is the **smallest** k such that a flat $(n+k)$-manifold collapses to $\mathcal{O}$?

Theorem (B., Derdzinski, Piccione, 2018)

The Gromov-Hausdorff limit of a sequence of flat 3-manifolds must be one of:

- *Flat 0-orbifold:* $\{p\}$;
- *Flat 1-orbifolds:* I, S^1;

Q: Given a flat n-orbifold $\mathcal{O}$, what is the **smallest** k such that a flat $(n+k)$-manifold collapses to $\mathcal{O}$?

Theorem (B., Derdzinski, Piccione, 2018)

The Gromov-Hausdorff limit of a sequence of flat 3-manifolds must be one of:

- *Flat 0-orbifold:* $\{p\}$*;*
- *Flat 1-orbifolds:* I, S^1*;*
- *Flat 2-orbifolds (10 out of 17):* T^2, K^2, $S^1 \times I$, M^2, $D^2(4;2)$, $D^2(3;3)$, $D^2(2,2;\,)$, $S^2(3,3,3;\,)$, $S^2(2,2,2,2;\,)$, $\mathbb{R}P^2(2,2;\,)$*;*

Q: Given a flat n-orbifold $\mathcal{O}$, what is the **smallest** k such that a flat $(n+k)$-manifold collapses to $\mathcal{O}$?

Theorem (B., Derdzinski, Piccione, 2018)

The Gromov-Hausdorff limit of a sequence of flat 3-manifolds must be one of:

- *Flat 0-orbifold:* $\{p\}$;
- *Flat 1-orbifolds:* I, S^1;
- *Flat 2-orbifolds (10 out of 17):* T^2, K^2, $S^1 \times I$, M^2, $D^2(4;2)$, $D^2(3;3)$, $D^2(2,2;)$, $S^2(3,3,3;)$, $S^2(2,2,2,2;)$, $\mathbb{R}P^2(2,2;)$;
- *Flat 3-manifold (trivial).*

Q: Given a flat n-orbifold $\mathcal{O}$, what is the **smallest** k such that a flat $(n+k)$-manifold collapses to $\mathcal{O}$?

Theorem (B., Derdzinski, Piccione, 2018)

The Gromov-Hausdorff limit of a sequence of flat 3-manifolds must be one of:

- *Flat 0-orbifold:* $\{p\}$;
- *Flat 1-orbifolds:* I, S^1;
- *Flat 2-orbifolds (10 out of 17):* T^2, K^2, $S^1 \times I$, M^2, $D^2(4;2)$, $D^2(3;3)$, $D^2(2,2;)$, $S^2(3,3,3;)$, $S^2(2,2,2,2;)$, $\mathbb{R}P^2(2,2;)$;
- *Flat 3-manifold (trivial).*

A: When $n = 2$, $k = 1$ for 10 out of 17 flat 2-orbifolds.

Moduli space and Teichmüller space

Moduli space of flat metrics:
$\mathcal{M}_{flat}(\mathcal{O}) := \{\text{flat metrics on } \mathcal{O}\}/\{\text{isometries}\}$

Moduli space and Teichmüller space

Moduli space of flat metrics:
$\mathcal{M}_{\mathit{flat}}(\mathcal{O}) := \{\text{flat metrics on } \mathcal{O}\}/\{\text{isometries}\}$

Theorem (Wolf, Thurston, Baues, ...)

There exists a Teichmüller space $\mathcal{T}_{\mathit{flat}}(\mathcal{O}) \cong \mathbb{R}^d$ *such that*

$$\mathcal{M}_{\mathit{flat}}(\mathcal{O}) = \mathcal{T}_{\mathit{flat}}(\mathcal{O})/\mathrm{MCG}(\mathcal{O}),$$

where $\mathrm{MCG}(\mathcal{O}) = \mathrm{Diff}(\mathcal{O})/\mathrm{Diff}_0(\mathcal{O})$ *is the mapping class group, which is countable and discrete.*

Moduli space and Teichmüller space

Moduli space of flat metrics:
$\mathcal{M}_{flat}(\mathcal{O}) := \{\text{flat metrics on } \mathcal{O}\}/\{\text{isometries}\}$

Theorem (Wolf, Thurston, Baues, . . .)

There exists a Teichmüller space $\mathcal{T}_{flat}(\mathcal{O}) \cong \mathbb{R}^d$ such that

$$\mathcal{M}_{flat}(\mathcal{O}) = \mathcal{T}_{flat}(\mathcal{O})/\mathrm{MCG}(\mathcal{O}),$$

where $\mathrm{MCG}(\mathcal{O}) = \mathrm{Diff}(\mathcal{O})/\mathrm{Diff}_0(\mathcal{O})$ is the mapping class group, which is countable and discrete.

$\mathcal{T}_{flat}(\mathcal{O})$ is the *space of flat deformations*.

Algebraic characterization of Teichmüller space

- $\mathcal{O} = \mathbb{R}^n/\pi$ flat n-orbifold

Algebraic characterization of Teichmüller space

- $\mathcal{O} = \mathbb{R}^n/\pi$ flat n-orbifold
- $H_\pi \curvearrowright \mathbb{R}^n = \bigoplus_{i=1}^{\ell} W_i$ isotypical components of holonomy

Algebraic characterization of Teichmüller space

- $\mathcal{O} = \mathbb{R}^n/\pi$ flat n-orbifold
- $H_\pi \curvearrowright \mathbb{R}^n = \bigoplus_{i=1}^{\ell} W_i$ isotypical components of holonomy
- W_i direct sum of m_i copies of irreducible of type $\mathbb{K}_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$

Algebraic characterization of Teichmüller space

- $\mathcal{O} = \mathbb{R}^n/\pi$ flat n-orbifold
- $H_\pi \curvearrowright \mathbb{R}^n = \bigoplus_{i=1}^{\ell} W_i$ isotypical components of holonomy
- W_i direct sum of m_i copies of irreducible of type $\mathbb{K}_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$

Theorem (B., Derdzinski, Piccione, 2018)

$$\mathcal{T}_{flat}(\mathcal{O}) = \prod_{i=1}^{\ell} \frac{\mathsf{GL}(m_i, \mathbb{K}_i)}{\mathsf{O}(m_i, \mathbb{K}_i)}$$

Algebraic characterization of Teichmüller space

- $\mathcal{O} = \mathbb{R}^n/\pi$ flat n-orbifold
- $H_\pi \curvearrowright \mathbb{R}^n = \bigoplus_{i=1}^{\ell} W_i$ isotypical components of holonomy
- W_i direct sum of m_i copies of irreducible of type $\mathbb{K}_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$

Theorem (B., Derdzinski, Piccione, 2018)

$$\mathcal{T}_{flat}(\mathcal{O}) = \prod_{i=1}^{\ell} \frac{\mathsf{GL}(m_i, \mathbb{K}_i)}{\mathsf{O}(m_i, \mathbb{K}_i)}$$

$$\frac{\mathsf{GL}(m_i, \mathbb{K}_i)}{\mathsf{O}(m_i, \mathbb{K}_i)} \cong \mathbb{R}^{d_i}, \quad d_i = \begin{cases} \frac{1}{2} m_i(m_i+1), & \text{if } \mathbb{K}_i = \mathbb{R}, \\ m_i^2, & \text{if } \mathbb{K}_i = \mathbb{C}, \\ m_i(2m_i - 1), & \text{if } \mathbb{K}_i = \mathbb{H}. \end{cases}$$

“Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not. "Immortality" may be a silly word, but probably a mathematician has the best chance of whatever it may mean.”

G. H. Hardy

Thank you for your attention!