NON-UNIQUENESS OF CONFORMAL METRICS WITH CONSTANT Q-CURVATURE

9/2018
REGENSBURG, GERM

JOINT WORK WITH P. PICCIONE AND Y. SIRE

OUTLINE:

1. Q-CURVATURE
2. NON-UNIQUENESS VIA BIFURCATION
3. NON-UNIQUENESS VIA COVERINGS

1. Q-CURVATURE

(T. BRANSON, S. PANEITZ, 1980s). (M^n, g) RIEM. MFLD, $n \geq 5$

$$Q_g = \frac{1}{2(n-1)} \Delta_g \operatorname{scal}_g - \frac{2}{(n-2)^2} \|\operatorname{Ric}_g\|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} \operatorname{scal}_g^2$$

$$P_g \phi = \Delta_g^2 \phi + \frac{4}{n-2} \operatorname{div}_g\big(\operatorname{Ric}_g(\nabla\phi, e_i) e_i\big) - \frac{n^2 - 4n + 8}{2(n-1)(n-2)} \operatorname{div}_g(\operatorname{scal}_g \nabla \phi) + \frac{n-4}{2} Q_g$$

INSTEAD OF THE ABOVE, REMEMBER:

- $P_g = \Delta_g^2 + \text{L.O.T.}$, SUCH THAT $P_{u^{\frac{4}{n-4}} g}(\phi) = u^{-\frac{n+4}{n-4}} P_g(u\phi)$ "conformally covariant"
- $Q_g = \frac{2}{n-4} P_g(1)$

ANALOGOUS TO THE YAMABE PROBLEM:

CONSTANT Q-CURVATURE PROBLEM: GIVEN (M^n, g_0), $n \geq 5$, FIND COMPLETE CONFORMAL METRIC $g \in [g_0]$ SUCH THAT $Q_g = \text{const.}$

$\iff$ FIND $u : M \to \mathbb{R}_+$, $P_{g_0} u = \lambda \cdot u^{\frac{n+4}{n-4}}$, $\lambda = \frac{n-4}{2} Q_g$, $g = u^{\frac{4}{n-4}} \cdot g_0$

(WITH $u \nearrow +\infty$ FAST ENOUGH IF M IS NONCOMPACT)

EXISTENCE OF SOLUTIONS: KNOWN IN MANY (BUT NOT ALL) CASES
[A. CHANG, M. GURSKY, F. HANG, Y. LIN, P. YANG, ..

UNIQUENESS OF SOLUTIONS: FAILS IN LOTS OF WAYS
[G. LI, WEI - ZHAO ...] & TODAY
(Bubbling, $n \geq 25$)

MAIN TECHNICAL HURDLES:
- 4^{th} ORDER NONLINEAR ELLIPTIC PDE
- NO MAXIMUM PRINCIPLE (MINIMIZERS MIGHT NOT BE POSITIVE)
- CRITICAL SOBOLEV EXPONENT

$$W^{2,2}(M^n) \hookrightarrow L^{\frac{2n}{n-4}}(M^n) \text{ NONCOMPACT}$$

SO MINIMIZING SEQUENCES COULD CONVERGE (WEAKLY) TO ZERO

RELATED TO BEST CONSTANT

$$\|\varphi\|^2_{L^{\frac{2n}{n-4}}(M)} \leq \sigma_{2,n} \|\Delta\varphi\|_{L^2(M)}$$

JUST LIKE YAMABE IS RELATED TO

$$\|\varphi\|_{L^{\frac{2n}{n-2}}(M)} \leq \sigma_{1,n} \|\nabla\varphi\|_{L^2(M)}$$

2. NON-UNIQUENESS VIA BIFURCATION

FOR SIMPLICITY, LET US ONLY CONSIDER THE HOPF BUNDLES

$$S^1 \to S^{2q+1} \to \mathbb{C}P^q, \quad S^3 \to S^{4q+3} \to \mathbb{H}P^q \quad S^7 \to S^{15} \to S^8(\tfrac{1}{2})$$

$$\mathbb{C}P^1 \to \mathbb{C}P^{2q+1} \to \mathbb{H}P^q$$

WRITE AFTER

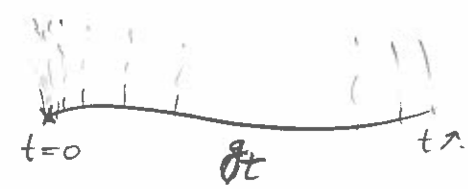
WITH BERGER METRICS $g_t = g|_{hor} \oplus t\, g|_{vert}$, WHERE $g_1 = g_{can}$

- g_t HOMOGENEOUS $\Rightarrow Q_{g_t} = \text{const}$. (BUT ALSO $scal_{g_t} = \text{const}$)

THM [B.-PICCIONE-SIRE, 2018]. THERE ARE INFINITELY MANY SEQUENCES OF BIFURCATING BRANCHES OF METRICS ON M WITH $Q = \text{const}$. THAT ISSUE FROM g_t AS $t \searrow 0$ OR $t \nearrow +\infty$ AS FOLLOWS:

AND WITHOUT $scal = \text{const}$

$F \to M \to B$	$t \searrow 0$		$t \nearrow +\infty$	
$S^1 \to S^{2q+1} \to \mathbb{C}P^q$	NO	NO	$q \geq 6$	NO
$S^3 \to S^{4q+1} \to \mathbb{H}P^q$	$q \geq 1$	$q \geq 1$	$q \geq 2$	NO
$\mathbb{C}P^1 \to \mathbb{C}P^{2q+1} \to \mathbb{H}P^q$	$q \geq 2$	$q \geq 1$	$q \geq 3$	NO
$S^7 \to S^{15} \to S^8(\tfrac{1}{2})$	YES	YES	YES	NO



(YAMABE PROBLEM)

COMPARE WITH YAMABE PROBLEM [B.-PICCIONE '13, OTOBA-PETEAN

REMARKS: • AS $t \nearrow +\infty$, $scal_{g_t} \searrow -\infty$ (HENCE $Y(M,[g_t])<0$) AN

$Q_{g_t} \searrow -\infty$ IN LOW DIMENSIONS

$Q_{g_t} \nearrow +\infty$ IN HIGH DIMENSIONS

S^{2q+1}, $6 \leq q \leq$

S^{4q+1}, $q=1$

$\mathbb{C}P^{2q+1}$, $q=2$

SO CONSTANT Q-CURVATURE METRICS BIFURCATE EVEN IF

THIS IMPLIES UNIQUENESS ON YAMABE PROBLEM (BY MAX. PRINCIPLE)

$scal_g < 0$, $Y(M,[g]) < 0$, $Q_g < 0$

• GLOBAL EXAMPLES OF g WITH $Q_g = \text{const}$ AND WITHOUT $scal = \text{const}$, ALSO AS $t \searrow 0$ (I.E. $scal > 0$)

SKETCH OF PROOF

STRATEGY ADAPTED FROM [OTOBA - PETEAN, 2016]

• $\pi : M \longrightarrow B$ RIEMANNIAN SUBMERSION

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "\overline{\phi}"] & \\ B \arrow[r, "\phi"'] & \mathbb{R} \end{tikzcd}$$

• MINIMAL FIBERS: $\overline{\Delta_B \phi} = \Delta_M \overline{\phi}$

• HORIZONTALLY EINSTEIN: $Ric_g = Ric_H \oplus Ric_V$

$Ric_H = K \cdot \pi^*(g_B)$

• g_t "CANONICAL VARIATION": $g_t = g|_H \oplus t\, g|_V$

• DEFINE:

$$\alpha_t := \frac{(n^2-4n+8)\, scal_{g_t} - 8K_t(n-1)}{4(n-1)(n-2)}, \quad \beta_t := -2Q_t$$

$$\boxed{P_t(\phi) = \Delta_{g_B}^2 \phi + 2\alpha_t \Delta_{g_B} \phi - \frac{n-4}{4} \beta_t \phi, \quad \phi \in C^\infty(B).}$$

POLYNOMIAL ON Δ_{g_B}

• THEN $P_{g_t}(\overline{\phi}) = \overline{P_t \phi}$

subcritical exponent: b/c dim $B < n$

AND $\boxed{g = \overline{\phi}^{\frac{4}{n-4}} g_t \text{ HAS } Q_g = C \text{ (const.)} \Longleftrightarrow P_t \phi = \frac{n-4}{2} C \phi^{\frac{n+4}{n-4}}}$

• VARIATIONALLY, SUCH $\phi : B \to \mathbb{R}$ ARE CRITICAL POINTS OF

$$\mathcal{E}_t(\phi) = \int_B \phi P_t \phi, \quad \text{SUBJECT TO CONSTRAINT } \|\phi\|_{L^{\frac{2n}{n-4}}(B)} = \text{const}$$

• JACOBI OPERATOR: $J_t \phi = \frac{1}{2} P_t \phi - \frac{n+4}{4} Q_t \phi$

$$= \frac{1}{2} \Delta^2_{g_B} \phi + \alpha_t \Delta^2_{g_B} \phi + \beta_t$$

$$\mathrm{Spec}(J_t) = \left\{ \tfrac{1}{2}\lambda^2 + \alpha_t \lambda + \beta_t : \lambda \in \mathrm{Spec}(\Delta_{g_B}) \right\}$$

• MORSE INDEX: $i_{Morse}(\phi) = \#\left(\mathrm{Spec}(J_t) \cap (-\infty, 0)\right)$

• MAIN STEP:

$\boxed{\begin{array}{l}\text{IF, WHEN } t = t_*, \\ \exists \lambda \in \mathrm{Spec}(\Delta_{g_B}) \text{ S.T.} \\ \begin{cases} \frac{1}{2}\lambda^2 + \alpha_{t_*}\lambda + \beta_{t_*} = 0 \\ \alpha'_{t_*}\lambda + \beta'_{t_*} \neq 0 \end{cases}\end{array}} \Longrightarrow \boxed{\begin{array}{c} i_{Morse}(1) \text{ JUMPS} \\ \text{AT } t = t_* \end{array}} \xRightarrow{\text{STANDARD VARIATIONAL BIFURCATION THEORY}} \boxed{\begin{array}{l} t = t_* \text{ IS A} \\ \text{BIFURCATION INSTANT} \end{array}}$

TO CONCLUDE: COMPUTE α_t, β_t AND VERIFY WHEN THE ABOVE APPLIES (INFINITELY OFTEN) AS $t \searrow 0$ OR $t \nearrow +\infty$ $\square$

REMARKS: • $\exists t_q \searrow 0$ SEQUENCE OF BIF. INSTANTS FOR $Q = $ const. (WITHOUT scal = const) WHENEVER $F \to M \to B$ IS OF THE FORM

$$K/H \longrightarrow G/H \longrightarrow G/K, \qquad H < K < G \text{ LIE GROUPS}$$

WITH:

- K NON ABELIAN
- H-ISOTROPY REP. ON K/H AND G/K ARE INEQUIVALENT
- $\dim K/H \geq 2$ & $\dim G/H \geq 9$, or $\dim K/H \geq 3$ & $5 \leq \dim G/H \leq$

e.g., $\begin{cases} G = SO(n) \times S^1 \\ K = SO(n) \\ H = SO(n-1) \end{cases}$, $n \geq 5$

$\Rightarrow M_r = S^{n-1} \times S^1(r)$

has # of sol $\nearrow +\infty$ as $r \searrow 0$ [cf. Schoen]

40 min

3. NON-UNIQUENESS VIA COVERINGS

W/ scal ≡ co[nst]

THM [B.-PICCIONE-SIRE, 2018] LET (C,g) BE A CLOSED MANIFOL[D], (N,h) A SIMPLY-CONNECTED SYMMETRIC SPACE OF NONCOMPACT OR EUCLIDEAN TYPE, S.T. $(C\times N, g\oplus h)$ HAS $\dim \geq 5$, $\mathrm{scal} \geq 0$ AN[D] $Q \geq 0$ (BUT NOT $Q\equiv 0$), THEN $(C\times N, g\oplus h)$ HAS INFINITELY MAN[Y] NONHOMOTHETIC PERIODIC CONFORMAL METRICS WITH $Q = \mathrm{const.}\ (>0)$

Cf. (ANALOGOUS) RESULT FOR THE YAMABE PROBLEM, [B.-PICCIONE, 2018]

COR: THERE ARE INFINITELY MANY (COMPLETE) METRICS WITH $Q = \mathrm{const.}\ (>$ ON $S^n \setminus S^k$, $0 \leq k < \frac{n-4}{2}$, CONFORMAL TO THE (INCOMPLETE) ROUND M[ETRIC]

$\cong S^{n-k-1}\times \mathbb{H}^{k+1}$ MAXIMAL RANGE?

OTHER EXAMPLES: $S^m \times \mathbb{R}^d$, $m\geq 4,\ d\geq 1$
$(S^m \times \mathbb{H}^d$, $2\leq d\leq m-3)$

MAIN INPUT: (REVERSED) "AUBIN-TYPE" INEQUALITY FOR

$$\Theta_4(M,g_0) = \sup_{f\in L^{\frac{2n}{n+4}}(M)} \frac{\int_M G_P f\cdot f}{\|f\|^2_{L^{\frac{2n}{n+4}}(M)}} < \infty$$

HARDY-LITTLEWOOD-SOBOLEV INEQUALITY

WHERE G_P IS THE INTEGRAL OPERATOR INVERSE OF PANEITZ P,

$$(G_P f)(p) = \int_M G_P(p,q) f(q)\, dq, \quad G_P(\cdot,\cdot) \text{ GREEN'S FUNCTION FOR } P$$

(i.e. $P(G_P(x,y)) = \delta_x(y)$)

THM [HANG-YANG, 2016]. IF (M^n, g_0) IS A CLOSED RIEM. MFLD, $n\geq 5$, $Y(M,[g_0])>0$, $Q_{g_0}\geq 0$ (BUT NOT $Q_{g_0}\equiv 0$), THEN:

$$\Theta_4(M,g_0) = \frac{2}{n-4}\sup_{g\in[g_0]} \frac{\int_M Q_g \,\mathrm{vol}_g}{\|Q_g\|^2_{L^{\frac{2n}{n+4}}(M)}} \geq \Theta_4(S^n, g_{round})$$

AND THE SUPREMUM IS ACHIEVED AT A SMOOTH CONFORMAL METRIC W/ $Q = \mathrm{const.}$

SKETCH OF PROOF:

[BOREL 1963]: $\exists\ \Sigma_0 = N/\Gamma_0$ COMPACT QUOTIENT

$\pi_1(\Sigma_0) = \Gamma_0$ INFINITE AND RESIDUALLY FINITE.

LET $(\underbrace{C\times\Sigma_0, g\oplus h_0}_{=(M_0,g_0)}) \leftarrow (\underbrace{C\times\Sigma_1, g\oplus h_1}_{=(M_1,g_1)}) \leftarrow \cdots \leftarrow (\underbrace{C\times\Sigma_j, g\oplus h_j}_{=(M_j,g_j)}) \leftarrow \cdots \leftarrow (C\times N, g\oplus$

BE AN INFINITE SEQ. OF FINITE-SHEETED RIEMANNIAN COVERINGS

- $Y(M_j, g_j) \geq Y(M_0, g_0) \geq 0, \quad \forall j \geq 1$

  [AKUTAGAWA-NEVES] $\quad$ $scal_{C\times N} \equiv const \geq 0.$

- $Q_{g_j} \geq 0 \ (\not\equiv 0)$ BECAUSE ALL (M_j, g_j) ARE LOC. ISOMETRIC

> THUS, BY [HANG-YANG],
>
> $$\overline{\Theta}(M_j, g) := \frac{\int_{M_j} Q_g \, vol_g}{\|Q_g\|^2_{L^{\frac{2n}{n+4}}(M_j)}}$$
>
> ATTAINS ITS MAXIMUM AT $\tilde{g}_j \in [g_j], \ \forall j \geq 1.$

$$\overline{\Theta}(M_j, \underbrace{\pi_j^*(\tilde{g}_1)}_{\text{PULLBACK } \tilde{g}_1 \text{ TO } M_j}) = \frac{\int_{M_j} Q_{\tilde{g}_1} \, vol_{\pi_j^*(\tilde{g}_1)}}{\left(\int_{M_j} Q_{\tilde{g}_1}^{\frac{2n}{n+4}} \, vol_{\pi_j^*(\tilde{g}_1)}\right)^{\frac{n+4}{n}}} = Q_{\tilde{g}_1} \cdot Vol(M_j, \pi_j^*(\tilde{g}_1))^{-\frac{4}{n}} \searrow 0$$

AS $j \nearrow +\infty$

SO $\frac{2}{n-4}\overline{\Theta}(M_j, \pi_j^*(\tilde{g}_1)) < \Theta_4(S^n, g_{round}), \ \forall j \geq j_0$ FOR SOME $j_0 \geq 1$

$\Rightarrow \pi_j^*(\tilde{g}_1)$ NOT MAXIMIZER OF $\overline{\Theta}$ IN $[\pi_{j_0}^*(\tilde{g}_1)] = [g_{j_0}]$.

$\overset{[HY]}{\Longrightarrow}$ $\exists$ ANOTHER SOLUTION $g'_{j_0} \in [g_{j_0}]$ WHICH MAXIMIZES $\overline{\Theta}$.

ITERATE, REPLACING (M_1, g_1) WITH (M_{j_0}, g'_{j_0}); PULLBACK TO $C\times N$ $\square$

60