CONVEX ALGEBRAIC GEOMETRY OF CURVATURE OPERATORS

JOINT WORK WITH M. KUMMER AND R. MENDES JULY/2019

GOAL: UNDERSTAND THE ALGEBRAIC STRUCTURE OF

$$\mathcal{R}_{\sec \geq k}(n) = \{ R : \Lambda^2 \mathbb{R}^n \to \Lambda^2 \mathbb{R}^n : \sec_R \geq k \}$$
$$(>, \leq, <)$$

WHY?

- BETTER/EASIER METHODS TO CERTIFY THAT (M^n, g) HAS $\sec > 0$.
  E.G., $P_2 \cong T_1 S^4$ [GROVE-VERDIANI-ZILLER, DEARRICOTT] AND HERE!
- APPLICATIONS TO GLOBAL STRUCTURE RESULTS,
  E.G., RANK RIGIDITY, MODIFICATIONS OF BOCHNER TECHNIQUE

HOW?

- CONVEX ALGEBRAIC GEOMETRY $\begin{cases} \cdot \text{ CONVEX GEOMETRY} \\ \cdot \text{ OPTIMIZATION} \\ \cdot \text{ (REAL) ALGEBRAIC GEOMETRY} \end{cases}$

1. A TASTE OF (REAL) ALGEBRAIC GEOMETRY; NONNEGATIVE v. S.O.S.

Q: IF $p(x) \in \mathbb{R}[x_1, \ldots, x_n]_{2d}$ IS A HOMOGENEOUS POLYNOMIAL, $\deg p = 2d$, AND $p(x) \geq 0$, $\forall x \in \mathbb{R}^n$, IS $p(x) = \sum_{i=1}^{N} q_i(x)^2$, $q_i(x) \in \mathbb{R}[x_1, \ldots, x_n]_d$?

A: (HILBERT, 1893) YES IF AND ONLY IF:

(i) $2d = 2$, $\forall n$ - QUADRATIC POLYNOMIALS
(SPECTRAL THM: SYMMETRIC $\Rightarrow$ DIAGONALIZABLE)

(ii) $n \leq 2$, $\forall d$ - UNIVARIATE (INHOMOGENEOUS) POLYNOMIALS
RMK: 2 SQUARES SUFFICE $(N=2)$!

(iii) $n = 3$, $2d = 4$ - TERNARY QUADRICS

SO, IN MOST CASES, IT IS FALSE!

EXAMPLE (MOTZKIN) $p(x,y,z) = z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2$ $\begin{pmatrix} n=3 \\ 2d=6 \end{pmatrix}$

- $p(x,y,z) \geq 0$ BY ARITHMETIC-GEOMETRIC INEQUALITY
- $p(x,y,z)$ IS NOT A S.O.S. BY DIRECT INSPECTION.

HILBERT'S 17TH PROBLEM: $p(x) \geq 0,\ \forall x \in \mathbb{R}^n \iff p(x) = \sum_{i=1}^{N} \left(\frac{q_i(x)}{r_i(x)}\right)^2$?

RATIONAL FUNCTIONS

THM (ARTIN, 1927). YES!

HILBERT'S 17TH PROBLEM ON VARIETIES

FINSLER'S LEMMA (1936): GIVEN A QUADRIC $X = \{x \in \mathbb{R}^n,\ \phi(x) = 0\}$, AND HOMOGENEOUS QUADRATIC POLYNOMIAL $p(x) \in \mathbb{R}[x]_2$,

$$p(x) \geq 0,\ \forall x \in X \iff \exists a \in \mathbb{R},\ p(x) = \sum_{i=1}^{N} q_i(x)^2 + a\,\phi(x)$$

LINEAR FUNCTIONS

NOTATION: X REAL PROJECTIVE VARIETY $\longleftarrow$ $X \subset \mathbb{CP}^n$ REAL, IRREDUCIBLE, FULL, $X(\mathbb{R}) \subset \mathbb{RP}^n$ Z.-DENSE

$$\Sigma_X = \{p \in \mathbb{R}[X]_2,\ p = \sum_{i=1}^{N} q_i^2,\ q_i \in \mathbb{R}[X]_1\}$$
$$\cap$$
$$P_X = \{p \in \mathbb{R}[X]_2,\ p(x) \geq 0,\ \forall x \in X\}$$

FINSLER'S LEMMA: $P_X = \Sigma_X$ IF X IS A QUADRIC

$\deg = 2$, $\text{codim} = 1$

THM [BLEKHERMAN-SMITH-VELASCO, JAMS 2016].

$$P_X = \Sigma_X \iff X \text{ HAS MINIMAL DEGREE}$$
$$\deg(X) = \text{codim}(X) + 1$$

$X = Gr_2\,\mathbb{R}^n \xrightarrow{\text{PLÜCKER}} \mathbb{RP}^{\binom{n}{2}-1}$ HAS $\deg = \frac{(2(n-2))!}{(n-2)!(n-1)!}$, $\text{codim} = \frac{(n-2)(n-3)}{2}$

SO MINIMAL DEGREE $\iff n \leq 4$

2. ALGEBRAIC CURVATURE OPERATORS

$$
\begin{tikzcd}
\Lambda^4 \mathbb{R}^n \arrow[d, hook] & & (I_{Gr_2 \mathbb{R}^n})_2 \arrow[d, hook] \\
\mathrm{Sym}^2(\Lambda^2 \mathbb{R}^n) \arrow[d] \arrow[rr, equal, "\text{IDENTIFY}"] & & \mathbb{R}[x_{ij}]_2 \arrow[d] \\
\mathrm{Sym}^2_b(\Lambda^2 \mathbb{R}^n) \arrow[rr, equal] & & \mathbb{R}[Gr_2 \mathbb{R}^n]_2 = \mathbb{R}[x_{ij}]_2 \big/ (I_{Gr_2\mathbb{R}^n})_2
\end{tikzcd}
$$

DEGREE 2 PART OF IDEAL THAT DEFINES $Gr_2 \mathbb{R}^n = \{\sigma \in \Lambda^2 \mathbb{R}^n : \sigma \wedge \sigma = 0, |\sigma| = 1\}$

$$\sigma \wedge \sigma = 0 \iff \langle \omega(\sigma), \sigma \rangle = 0 \quad \forall \omega \in \Lambda^4 \mathbb{R}^n$$

$$\mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) \ni R \longleftrightarrow q_R \in \mathbb{R}[x_{ij}]_2$$

$$q_R(x_{ij}) = \Big\langle R\Big(\sum_{i<j} x_{ij} e_i \wedge e_j\Big), \sum_{i<j} x_{ij} e_i \wedge e_j \Big\rangle$$

x_{ij}: PLÜCKER COORDINATES

$$\mathrm{Sym}^2_b(\Lambda^2\mathbb{R}^n) = \left\{ \begin{array}{c} R \in \mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) \\ b(R) = 0 \end{array} \right\}$$

$b(R)=0$: 1ST BIANCHI IDENTITY

$$\mathbb{R}[Gr_2\mathbb{R}^n]_2 \supset P_{Gr_2\mathbb{R}^n} \supset \Sigma_{Gr_2\mathbb{R}^n}$$

$\mathbb{R}[Gr_2\mathbb{R}^n]_2$: QUADRATIC FORMS ON THE GRASSMANNIAN

- UNDER THE ABOVE IDENTIFICATION:

$$P_{Gr_2\mathbb{R}^n} = \{ R \in \mathrm{Sym}^2_b(\Lambda^2\mathbb{R}^n) : \sec_R \geq 0 \} = \mathcal{R}_{\sec \geq 0}(n)$$

$$\Sigma_{Gr_2\mathbb{R}^n} = \{ R \in \mathrm{Sym}^2_b(\Lambda^2\mathbb{R}^n) : \exists\, \omega \in \Lambda^4\mathbb{R}^n, \ R + \omega \succeq 0 \}$$

$$\Big(\Sigma_{\Lambda^2\mathbb{R}^n} = \{ R \in \mathrm{Sym}^2_b(\Lambda^2\mathbb{R}^n) : R \succeq 0 \} \Big)$$

STRONGLY NONNEGATIVE CURVATURE

- FINSLER'S LEMMA: $P_{Gr_2\mathbb{R}^n} = \Sigma_{Gr_2\mathbb{R}^n}, \ \forall n \leq 4$ ("THORPE'S TRICK")
- [BSV 2016] OR [ZOLTEK, 1979]: $P_{Gr_2\mathbb{R}^n} \supsetneq \Sigma_{Gr_2\mathbb{R}^n}, \ \forall n \geq 5$

3. STRUCTURAL RESULTS ABOUT $\mathcal{R}_{\sec \geq 0}(n)$

DEF. A SPECTRAHEDRON IS A SET $S \subset \mathbb{R}^n$ OF THE FORM

$$S = \Big\{ x \in \mathbb{R}^n : A + \sum_{i=1}^{n} x_i B_i \succeq 0 \Big\}, \quad A, B_i \in \mathrm{Sym}^2(\mathbb{R}^d)$$

I.E., THE INTERSECTION OF $\mathrm{Sym}^2_{\geq 0}(\mathbb{R}^d)$ WITH AN AFFINE SUBSPACE.

EXAMPLE: POLYHEDRA $\leftrightsquigarrow$ A, B_i: DIAGONAL.

DEF. A SPECTRAHEDRAL SHADOW IS A SET $S \subset \mathbb{R}^n$ OF THE FORM

$$S = \left\{ x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m,\ A + \sum_{i=1}^{n} x_i B_i + \sum_{j=1}^{m} y_j C_j \succeq 0 \right\},\ A, B_i, C_j \in \mathrm{Sym}^2(\mathbb{R}^d)$$

I.E, LINEAR PROJECTION OF A SPECTRAHEDRON.

IMPORTANCE: FEASIBLE REGIONS OF SEMIDEFINITE PROGRAMMING.

THM A. (B.-KUMMER-MENDES, 2019)

(i) $\mathcal{R}_{\sec\geq 0}(n)$, $n \geq 5$, IS NOT A SPECTRAHEDRAL SHADOW

(ii) $\mathcal{R}_{\sec\geq 0}(4)$ IS A SPECTRAHEDRAL SHADOW, BUT NOT A SPECTRAHEDRON

(iii) $\mathcal{R}_{\sec\geq 0}(2)$ AND $\mathcal{R}_{\sec\geq 0}(3)$ ARE SPECTRAHEDRA (TRIVIAL)

NOT ALL IS LOST IF $n \geq 5$...

PROP: $\mathcal{R}_{\sec\geq 0}(n)$ IS A SEMIALGEBRAIC SET, THAT IS,

$$\mathcal{R}_{\sec\geq 0}(n) = \left(\bigcup_{j=1}^{N} \right) \left\{ R \in \mathrm{Sym}^2_b(\Lambda^2 \mathbb{R}^n) : P_{i,(j)}(R) \geq 0,\ 1 \leq i \leq N_{(j)} \right\}$$

PF: TARSKI-SEIDENBERG THM/QUANTIFIER ELIMINATION

(QUANTIFIED)	$\exists x \in \mathbb{R},\ ax^2 + bx + c = 0$, $a \neq 0$	$\forall \sigma \in \mathrm{Gr}_2 \mathbb{R}^n,\ \sec_R(\sigma) \geq 0$
$\Updownarrow$		
(QUANTIFIER-FREE)	$b^2 - 4ac \geq 0$	$P_{i,(j)}(R) \geq 0,\ 1 \leq i \leq N_{(j)}$ (FOR SOME j)

- FINDING $P_{i(j)}(R)$ EXPLICITLY IS HARD/HOPELESS/USELESS, IN GENERAL.
- HOWEVER, IF $n = 4$, SOMETHING CAN BE DONE:

THM B (B.-KUMMER-MENDES, 2019). $\mathcal{R}_{\sec\geq 0}(4) = \overline{\mathcal{C}}$, WHERE $\mathcal{C}$ IS THE CONNECTED COMPONENT WITH $\mathrm{Id} \in \overline{\mathcal{C}}$ OF THE SET WHERE $P(R) = \mathrm{disc}_x(\det(R + x *)) > 0$; I.E., "ALGEBRAIC INTERIOR W/ MIN. DEF. POLY. $P(R)$".

- SIMILAR RESULT (WITH SAME $P(R)$) BY DAN FODOR.
- $\exists$ EFFICIENT ALGORITHMS TO TEST $\sec \geq 0$ AND $\sec > 0$ IN $n=4$.
- ALGEBRAIC BOUNDARY OF $\mathcal{R}_{\sec\geq 0}(4)$ IS ZERO SET OF $P(R)$.

ZARISKI-CLOSURE OF TOPOLOGICAL BOUNDARY

IRREDUCIBLE!

4. OUTLINE OF PROOFS

THM A

EXPANDING ON [NEMIROVSKI, ICM 2006]

CONJ. (HELTON-NIE, SIAM 2009). EVERY CONVEX SEMIALGEBRAIC SET $S \subset \mathbb{R}^n$ IS A SPECTRAHEDRAL SHADOW.

THM (SCHEIDERER, 2018) ABOVE CONJECTURE IS:
- TRUE, IF $\dim S \leq 2$;
- FALSE, IN GENERAL (COUNTER-EXAMPLES KNOWN WITH $n \geq 14$).

$\Downarrow$

SCHEIDERER'S CRITERION:

A NECESSARY AND SUFFICIENT CRITERION IS ALSO AVAILABLE (BUT WE DON'T NEED IT).

- $L \subset \mathbb{R}[x_1, \dots, x_n]$ FINITE-DIMENSIONAL SUBSPACE, $1 \in L$.
- $f \in L$, $f \geq 0$ BUT NOT S.O.S., S.T. $\forall y \in \mathbb{R}^n$, THE COEFFICIENTS OF HOMOGENEIZATION $f^h(t, x_1 - y_1, \dots, x_n - y_n)$, AS POLYNOMIALS IN t, BELONG TO L.

  UNIQUE HOM. POLY. WITH $f^h(1, x_1, \dots, x_n) = f(x_1, \dots, x_n)$
- THEN $K = \{g \in L : g(x) \geq 0,\ \forall x \in \mathbb{R}^n\}$ IS NOT A SPECTR. SHADOW.

- [BSV, 2016] OR [ZOLTEK, 1979]: $\forall n \geq 5$, $\exists\, q \in P_{\mathrm{Gr}_2\mathbb{R}^n} \setminus \Sigma_{\mathrm{Gr}_2\mathbb{R}^n} \subset \mathbb{R}[\mathrm{Gr}_2\mathbb{R}^n]_2$
- FEED DEHOMOGENEIZATION f OF q INTO SCHEIDERER'S CRITERION TO PROVE $P_{\mathrm{Gr}_2\mathbb{R}^n} = \mathcal{R}_{\sec\geq 0}(n)$ IS NOT A SPECTRAH. SHADOW. $\square$

"COMPOSITION OF q WITH AN AFFINE CHART IN $\mathrm{Gr}_2\mathbb{R}^n$"

CLAIM THAT $\mathcal{R}_{\sec\geq 0}(4)$ IS NOT A SPECTRAHEDRON PROVED WITH THM B.

THM B

$$P: \mathrm{Sym}^2_b(\Lambda^2 \mathbb{R}^4) \longrightarrow \mathbb{R}$$

$$P(R) = \mathrm{disc}_x\left(\det(R + x *)\right)$$

$\boxed{\text{CLAIM 1: } P|_{\partial \mathcal{R}_{sec\geq 0}(4)} = 0}$

PF: $R \in \partial \mathcal{R}_{sec\geq 0}(4) \iff \exists\, a_0 \in \mathbb{R},\ R + a_0 * \geq 0,\ \mathrm{Ker}(R + a_0 *) \neq \{0\}.$

(so $\det(R + a_0 *) = 0$)

NOTE: $\det(R + x*) = -\det(TRT + x\,\mathrm{Id})$, $\quad T = \left(\begin{array}{c|c} \mathrm{Id}_{3\times 3} & \\ \hline & \sqrt{-1}\,\mathrm{Id}_{3\times 3} \end{array}\right)$

So $T^2 = *$

$$\Rightarrow \underbrace{\mathrm{disc}_x(\det(R + x*))}_{P(R)} = \mathrm{disc}(TRT)$$

MATRIX DISCRIMINANT IS THE DISCRIMINANT OF ITS CHAR. POLY.

- IF $a_0 \in \mathbb{R}$ IS A SIMPLE ROOT, THEN:
  - $-a_0$ IS AN EIGENVALUE OF TRT WITH MULTIPLICITY 1.

  $\Rightarrow \dim \mathrm{Ker}(R + a_0 *) = \dim \mathrm{Ker}\left(T^{-1}(TRT + a_0\,\mathrm{Id})\,T\right) = 1$

  $\Rightarrow R + a_0 * \geq 0$ MUST HAVE 5 POSITIVE EIGENVALUES

  $\Rightarrow$ FOR $x \approx a_0$, $R + x * \geq 0 \iff \det(R + x*) > 0$

  $\Rightarrow \exists\, x \approx a_0$ WITH $R + x * > 0$, A CONTRADICTION!
- THUS $a_0 \in \mathbb{R}$ IS A ROOT WITH MULTIPLICITY ≥ 2, HENCE $P(R) = 0$.

$\boxed{\text{CLAIM 2: } \mathcal{R}_{sec\geq 0}(4) \setminus \{P = 0\} \text{ IS CONNECTED.}}$

PF: SHOW $\{P=0\}$ HAS CODIM ≥ 2 INSIDE $\mathcal{R}_{sec\geq 0}(4)$.

$\boxed{\text{CLAIM 3: } P(R) \text{ IS IRREDUCIBLE.}}$

PF: SHOW DISCRIMINANT OF SYMMETRIC $n \times n$ ($n \geq 3$) MATRICES IS IRREDUCIBLE.

COMPLETING PROOF OF THM A:

- $P(R) = \mathrm{disc}_x(\det(R + x*))$ IS IRREDUCIBLE (HENCE THE MINIMAL DEF. POLY.) AND $P(\mathrm{Id}) = 0$
- BY [HELTON-VINNIKOV, CPAM 2007], THE MINIMAL DEFINING POLYNOMIAL OF A SPECTRAHEDRON (AS AN ALGEBRAIC INTERIOR) DOES NOT VANISH IN THE INTERIOR, CONTRADICTING THE ABOVE. THUS $\mathcal{R}_{sec\geq 0}(4)$ IS NOT A SPECTR. $\square$