BIFURCATING CONFORMAL METRICS WITH CONSTANT Q-CURVATURE

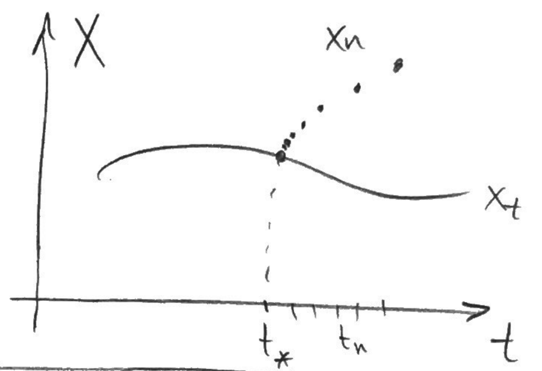
INSTITUTE FOR ADVANCED STUDY — APRIL 2019

BIFURCATION THEORY

- $f_t : X \to \mathbb{R}$ 1-PARAMETER FAMILY OF FUNCTIONALS
- $x_t \in X$ TRIVIAL BRANCH OF SOLUTIONS TO $df_t(x_t) = 0$

<u>DEF</u>: BIFURCATION OCCURS AT t_* IF
$\exists\, t_n \to t_*$, $\exists\, x_n \to x_{t_*}$ SUCH THAT:
$df_{t_n}(x_n) = 0$ AND $x_n \neq x_{t_n}$

(I.E., IMPLICIT FUNCTION THM FAILS AT x_{t_*}!)



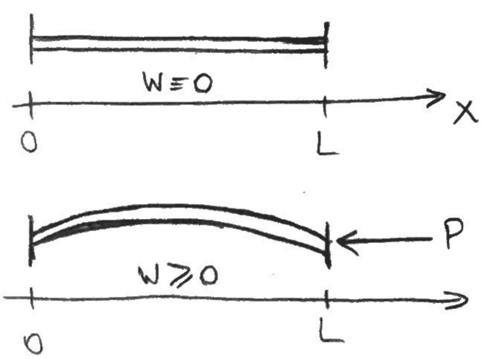
- <u>NECESSARY CONDITION</u>: x_{t_*} IS <u>DEGENERATE</u>, I.E. $\operatorname{Ker} J_{t_*} \neq \{0\}$, $d^2 f_t(x_t)(\psi,\psi) = \langle J_t\psi, \psi\rangle$ "JACOBI OPERATOR"
- A <u>SUFFICIENT CONDITION</u> IN TERMS OF $i_{Morse}(x_t) = \#\operatorname{Spec}(J_t) \cap (-\infty, 0)$

<u>THM</u> (KRASNOSEL'SKII). IF J_t IS FREDHOLM (OF INDEX ZERO) AND $\exists\, a < b$ SUCH THAT x_a AND x_b ARE NONDEGENERATE AND

$$i_{Morse}(x_a) \neq i_{Morse}(x_b)$$

THEN $\exists\, t_* \in (a,b)$ A BIFURCATION INSTANT.

<u>CLASSICAL EXAMPLE</u>: BUCKLING UNDER COMPRESSIVE STRESS



$x \in [0, L]$

$w(x)$ = DEFLECTION AT x

EULER (1757):

$$E \frac{d^2 w}{dx^2} + P\, w = 0$$

E: ELASTICITY CONSTANT; P: LOAD

- GENERAL SOLUTION: $w(x) = A\sin(\lambda x) + B\cos(\lambda x)$, $\lambda = \sqrt{\frac{P}{E}}$
- BOUNDARY CONDITIONS: PINNED ENDS

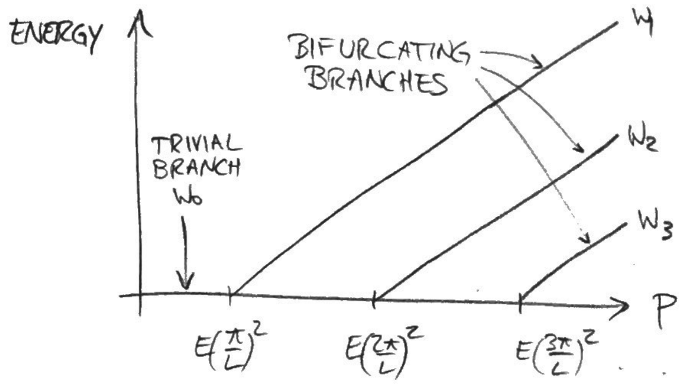
$$w(0) = 0 \implies B = 0$$

$$w(L) = 0 \implies A\sin(\lambda L) = 0 \overset{A \neq 0}{\implies} \lambda = \frac{n\pi}{L}, \quad n \in \mathbb{N}_0$$

- $P < \underbrace{E\left(\frac{\pi}{L}\right)^2}_{\text{"CRITICAL LOAD"}} \implies \lambda = \sqrt{\frac{P}{E}} < \frac{\pi}{L} \implies \boxed{w_0 \equiv 0 \quad \text{(ONLY SOLUTION IS THE TRIVIAL ONE)}}$

- $P \geqslant E\left(\frac{N\pi}{L}\right)^2 \implies \boxed{w_n(x) = A\sin\left(\frac{n\pi x}{L}\right), \quad 0 \le n \le N \quad \text{(N NONTRIVIAL SOLUTIONS)}}$

- VARIATIONALLY:



$$f_P : W_0^{1,2}([0,L]) \longrightarrow \mathbb{R}$$

$$f_P(w) = \frac{1}{2}\int_0^L E(w')^2 - Pw^2 \, dx$$

$$df_P(w) = 0 \iff Ew'' + Pw = 0$$

$$d^2 f_P(w)(\phi,\phi) = -\int_0^L (E\phi'' + P\phi)\phi \, dx = \langle J_P \phi, \phi \rangle_{L^2}, \quad J_P = -E\frac{d^2}{dx^2} - P$$

$$i_{Morse}(w_0) = \#\,\mathrm{Spec}(J_P) \cap (-\infty, 0) = \#\left\{ j \in \mathbb{N} : E\left(\frac{j\pi}{L}\right)^2 - P < 0 \right\}$$

$w_0 \equiv 0$ IS THE TRIVIAL SOLUTION

INCREASES BY 1 AT EACH $P_j = E\left(\frac{j\pi}{L}\right)^2 \leftarrow$ BIFURCATION INSTANTS

<u>REMARK</u>: <u>CONTINUOUS</u> BIFURCATING BRANCHES BECAUSE EIGENVALUE CROSSINGS ARE <u>SIMPLE</u>: IF $\mathrm{Ker}\, d^2 f_{t_*}(x_{t_*}) = \mathrm{span}\{v_*\}$, $\frac{d}{dt} d^2 f_t(x_t)(v_*, v_*)\big|_{t=t_*} \neq 0$, THEN SOLUTIONS TO $df_t(x) = 0$ NEAR (t_*, x_{t_*}) FORM 2 CURVES INTERSECTING TRANSVERSELY ONLY AT (t_*, x_{t_*}) [CRANDALL-RABINOWITZ]

GEOMETRIC APPLICATIONS:

Q-CURVATURE AND PANEITZ OPERATOR [BRANSON '85, PANEITZ '83]

(M^n, g) CLOSED n-MANIFOLD, $n \geq 5$

SCHOUTEN TENSOR: $A_g = \frac{1}{n-2}\left(Ric_g - \frac{scal_g}{2(n-1)} g\right)$ $\quad R = A \owedge g + W$

Q-CURVATURE: $Q_g = \Delta_g \sigma_1(A_g) + 4\sigma_2(A_g) + \frac{n-4}{2}\sigma_1(A_g)^2$

$$\left(= \frac{1}{2(n-1)}\Delta_g scal_g - \frac{2}{(n-2)^2}\|Ric_g\|^2 + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2} scal_g^2\right)$$

PANEITZ OPERATOR: $P_g u = \Delta_g^2 u + div_g\left[\left(4A_g - (n-2)\sigma_1(A_g)g\right)(\nabla u, \cdot)\right] + \frac{n-4}{2} Q_g \cdot u$

$$\left(= \Delta_g^2 u + \frac{4}{n-2} div_g(Ric_g(\nabla u, e_i)e_i) - \frac{n^2-4n+8}{2(n-1)(n-2)} div_g(scal_g \nabla u) + \frac{n-4}{2} Q_g u\right)$$

IN PARTICULAR: $Q_g = \frac{2}{n-4} P_g(1)$

$$\boxed{\begin{gathered}\text{CONFORMAL COVARIANCE:} \quad g = u^{\frac{4}{n-4}} g_0 \\ P_g(\phi) = u^{-\frac{n+4}{n-4}} P_{g_0}(u\phi) \xRightarrow{\phi\equiv 1} Q_g = \frac{2}{n-4} u^{-\frac{n+4}{n-4}} P_{g_0}(u)\end{gathered}}$$

CF. CONFORMAL LAPLACIAN $L_g u = 4\frac{n-1}{n-2}\Delta_g u + scal_g u$, $g = u^{\frac{4}{n-2}} g_0$

$$L_g(\phi) = u^{-\frac{n+2}{n-2}} L_{g_0}(u\phi) \xRightarrow{\phi\equiv 1} scal_g = u^{-\frac{n+2}{n-2}} L_{g_0}(u)$$

Q-CURVATURE PROBLEM

- $P_{g_0} u = \lambda u^{\frac{n+4}{n-4}}$ $\quad (\lambda = \frac{n-4}{2} Q_g)$
- $g = u^{\frac{4}{n-4}} g_0$ HAS $Q_g \equiv const.$ AND $Vol(M,g) = 1$

$\Updownarrow$

g IS CRITICAL FOR $\mathcal{Q}: [g_0]_1 \to \mathbb{R}$

$\mathcal{Q}(g) = \int_M Q_g \, dV_g = \int_M u P_{g_0} u \, dV_{g_0}$

- JACOBI OPERATOR

$J_g \psi = \frac{1}{2} P_g \psi - \frac{n+4}{4} Q_g \psi$

YAMABE PROBLEM

- $L_{g_0} u = \lambda u^{\frac{n+2}{n-2}}$ $\quad (\lambda = scal_g)$
- $g = u^{\frac{4}{n-2}} g_0$ HAS $scal_g \equiv const$ AND $Vol(M,g) = 1$

$\Updownarrow$

g IS CRITICAL FOR $\mathcal{S}: [g_0]_1 \to \mathbb{R}$

$\mathcal{S}(g) = \int_M scal_g \, dV_g = \int_M u L_{g_0} u \, dV_{g_0}$

- JACOBI OPERATOR

$J_g \psi = \Delta_g \psi - \frac{scal_g}{n-1} \psi$

DEF: A RIEMANNIAN SUBMERSION $F \to (M,g) \xrightarrow{\pi} (B, g_B)$ IS HORIZONTALLY EINSTEIN IF $Ric_g = Ric_{\mathcal{H}} \oplus Ric_{\mathcal{V}}$ AND $Ric_{\mathcal{H}} = K \cdot \pi^* g_B$.

THM (B.–PICCIONE–SIRE, 2018) SUPPOSE $F \to (M^n, g_t) \xrightarrow{\pi_t} (B, g_B)$, $t \in [t_*-\varepsilon, t_*+\varepsilon]$ IS A 1-PARAMETER FAMILY OF HORIZONTALLY EINSTEIN RIEM. SUBMERSIONS WITH MINIMAL FIBERS, SUCH THAT $scal_{g_t}$ AND Q_{g_t} ARE CONSTANT FOR EACH t. LET

$$\alpha_t = \frac{n^2-4n+8}{4(n-1)(n-2)} scal_{g_t} - \frac{2}{n-2} K_t, \quad \beta_t = -2Q_t$$

KEEP ON THE BOARD

IF $\lambda \in Spec(\Delta_B)$ SATISFIES

$$\frac{1}{2}\lambda^2 + \alpha_{t_*}\lambda + \beta_{t_*} = 0 \quad \text{AND} \quad \alpha'_{t_*}\lambda + \beta'_{t_*} \neq 0$$

THEN METRICS WITH CONSTANT Q-CURVATURE BIFURCATE FROM g_{t_*}.

NOTE: $\alpha'_{t_*} = \frac{d}{dt}\alpha_t\big|_{t=t_*}$, $\beta'_{t_*} = \frac{d}{dt}\beta_t\big|_{t=t_*}$

REMARK: IF $\lambda \neq \frac{scal_{g_{t_*}}}{n-1}$, THEN BIFURCATING METRICS NEAR g_{t_*} DO NOT HAVE $scal \equiv const$

COROLLARY: THERE ARE INFINITELY MANY BIFURCATING BRANCHES OF METRICS WITH $Q \equiv const$ (BUT WITHOUT $scal \equiv const$) ON THE FOLLOWING:

$tF \to (M, g_t) \to B$	$t \searrow 0$	$t \nearrow +\infty$
$tS^1 \to (S^{2q+1}, g_t) \to \mathbb{C}P^q$	NO	$q \geq 6$
$tS^3 \to (S^{4q+3}, g_t) \to \mathbb{H}P^q$	$q \geq 1$	$q \geq 2$
$tS^7 \to (S^{15}, g_t) \to S^8(1/2)$	YES	YES
$t\mathbb{C}P^1 \to (\mathbb{C}P^{2q+1}, g_t) \to \mathbb{H}P^q$	$q \geq 2$	$q \geq 3$

$\left(g_t = g|_{hor} \oplus t\, g|_{ver} \right.$ "BERGER METRICS"$\left.\right)$

REMARKS:
- AS $t \nearrow +\infty$, $scal_{g_t} \searrow -\infty$ AND $Q_{g_t} \searrow -\infty$ ON S^{2q+1} $(6 \leq q \leq 9)$, S^{4q+3} $(q=1)$, $\mathbb{C}P^{2q+1}$ $(q=2)$.
- THIS GIVES NONUNIQUENESS OF SOLUTIONS ON CONFORMAL CLASSES WITH NEGATIVE YAMABE CONSTANT AND $Q_g < 0$.
- THIS GIVES GLOBAL EXAMPLES OF METRICS WITH $Q \equiv const$ BUT WITHOUT $scal \equiv const$.

DIFFERENT FROM YAMABE

COROLLARY: IF (F, g_F) AND (B, g_B) ARE CLOSED EINSTEIN MANIFOLDS WITH POSITIVE EINSTEIN CONSTANT AND DIMENSION ≥ 3, THEN $g_t = g_B \oplus t g_F$ ON $M = B \times F$ HAS INFINITELY MANY BIFURCATIONS FOR THE CONSTANT Q-CURVATURE PROBLEM AS $t \searrow 0$ AND $t \nearrow +\infty$.

(———— SHORT BREAK ————)

BIFURCATION RESULTS FOR THE YAMABE PROBLEM:

[O. KOBAYASHI, 1985]; [SCHOEN, 1989]; [LIMA-PICCIONE-ZEDDA, 2012]; [B.-PICCIONE, 2013], ...

THM (OTOBA-PETEAN, 2016). SUPPOSE $F \to (M^n, g_t) \xrightarrow{\pi_t} (B, g_B)$, $t \in [t_* - \varepsilon, t_* + \varepsilon]$ IS A 1-PARAMETER FAMILY OF RIEM. SUBMERSIONS WITH MINIMAL FIBERS, SUCH THAT scal_{g_t} IS CONSTANT FOR EACH t. IF $\lambda \in \mathrm{Spec}(\Delta_B)$ SATISFIES

$$\lambda = \frac{\mathrm{scal}_{g_{t_*}}}{n-1} \qquad \text{AND} \qquad \mathrm{scal}'_{g_{t_*}} \neq 0$$

THEN METRICS WITH CONSTANT SCALAR CURVATURE BIFURCATE FROM g_{t_*}.

PF: $\pi_t : (M^n, g_t) \to (B, g_B)$ RIEM. SUBM. W/ MIN. FIBERS $\Rightarrow$ $\overline{\Delta_B u} = \Delta_M \overline{u}$

$$\begin{tikzcd} M^n \arrow[d, "\pi"'] \arrow[dr, "\overline{u}"] & \\ B \arrow[r, "u"'] & \mathbb{R} \end{tikzcd}$$

- LET $L_t : C^\infty(B) \to C^\infty(B)$

$$L_t(u) = 4\frac{n-1}{n-2}\Delta_B u + \mathrm{scal}_{g_t} u$$

- $\overline{L_t(u)} = L_{g_t}(\overline{u})$ IS THE CONF. LAPLACIAN (ON BASIC FUNCTIONS) ON (M, g_t)

- $g = \overline{u}^{\frac{4}{n-2}} g_t$ HAS $\mathrm{scal}_g \equiv c \iff L_t(u) = c\, u^{\frac{n+2}{n-2}}$

$\iff u \in C^\infty(B)$ IS A CRITICAL POINT OF $f_t : W^{1,2}(B) \to \mathbb{R}$, $f_t(u) = \int_B u\, L_t(u)\, dV_{g_B}$ SUBJECT TO THE CONSTRAINT $\|u\|_{L^{\frac{2n}{n-2}}(B)} = 1$.

- JACOBI OPERATOR

$$d^2 f_t(1)(\psi, \psi) = \langle J_t \psi, \psi \rangle_{L^2(B)}$$

$$J_t \psi = \Delta_B \psi - \frac{\mathrm{scal}_{g_t}}{n-1}\psi = L_t \psi - \frac{n+2}{n-2}\mathrm{scal}_{g_t}\psi$$

- $i_{\mathrm{Morse}}(1) \overset{\downarrow}{=} \# \mathrm{Spec}(\Delta_B) \cap \left(-\infty, \frac{\mathrm{scal}_{g_t}}{n-1}\right)$ JUMPS AT $t = t_*$, HENCE BIFURCATION AT g_{t_*}. $\square$

<u>CANONICAL VARIATION</u> OF RIEM. SUBMERSION $\pi:(M,g)\to(B,g_B)$ IS

$$\pi:(M,g_t)\to(B,g_B),\qquad g_t=g|_{hor}\oplus t\, g|_{ver},\qquad t>0$$

- ALSO A RIEM. SUBMERSION, W/ MINIMAL/TOT. GEOD. FIBERS ACCORDING TO THOSE OF $\pi:(M,g)\to(B,g_B)$
- $\mathrm{scal}_{g_t}=\frac{1}{t}\mathrm{scal}_{g_F}+\mathrm{scal}_{g_B}-t|A|^2$ IS <u>CONSTANT</u> FOR EACH $t>0$ IF FIBERS ARE TOT. GEODESIC, scal_{g_F} AND scal_{g_B} ARE CONSTANT.

> <u>COROLLARY</u>: IF $F\to M\xrightarrow{\pi}B$ HAS TOTALLY GEODESIC FIBERS, $\mathrm{scal}_{g_F}>0$ AND scal_{g_B} ARE CONSTANT, THEN THERE ARE INFINITELY MANY BIFURCATIONS FOR THE YAMABE PROBLEM ON g_t AS $t\searrow 0$.

<u>PF</u>: • $\mathrm{scal}_{g_t}=\frac{1}{t}\underbrace{\mathrm{scal}_{g_F}}_{>0}+\mathrm{scal}_{g_B}-t|A|^2\nearrow+\infty$ AS $t\searrow 0$

- $\mathrm{Spec}(\Delta_B)=\{\lambda_1<\lambda_2<\dots<\lambda_n\nearrow+\infty\}$ INFINITE SEQUENCE
- APPLY PREVIOUS THEOREM EACH TIME $\frac{\mathrm{scal}_{g_t}}{n-1}\in\mathrm{Spec}(\Delta_B)$. □

<u>RMK</u>: • IF ALSO $\mathrm{scal}_B>0$, THEN $(B\times F,[g_B\oplus t\,g_F])$ ALSO BIFURCATES AS $t\nearrow+\infty$ INFINITELY MANY TIMES

- RECOVERS EARLIER WORK OF [B.-PICCIONE] ON BERGER SPHERES.

<u>PF OF THEOREM [B.P.S]</u>: LET $P_t: C^\infty(B)\to C^\infty(B)$

$$P_t(u)=\Delta^2_{g_B}u+2\alpha_t\Delta_{g_B}u-\frac{n-4}{4}\beta_t u$$

- $\overline{P_t(u)}=P_{g_t}(\bar u)$ IS THE <u>PANEITZ OPERATOR</u> (ON BASIC FUNCTIONS) ON (M,g_t)
- $g=\bar u^{\frac{4}{n-4}}g_t$ HAS $Q_g\equiv c \iff P_t u=\frac{n-4}{2}c\cdot u^{\frac{n+4}{n-4}}$

$\iff u\in C^\infty(B)$ IS A <u>CRITICAL POINT</u> OF $f_t: W^{2,2}(B)\to\mathbb{R}$, $f_t(u)=\int_B u P_t u\, dV_{g_B}$ SUBJECT TO THE CONSTRAINT $\|u\|_{L^{\frac{2n}{n-4}}(B)}=1$.

$d^2f_t(1)(\psi,\psi)=\langle J_t\psi,\psi\rangle_{L^2(B)}$

- JACOBI OPERATOR $J_t\psi=\frac{1}{2}P_t\psi-\frac{n+4}{4}Q_t\psi=\frac{1}{2}\Delta^2_{g_B}\psi+\alpha_t\Delta_{g_B}\psi+\beta_t\psi$
- $i_{Morse}(1)=\#\,\mathrm{Spec}(J_t)\cap(-\infty,0)$ JUMPS AT $t=t_*$ HENCE <u>BIFURCATION</u> AT g_{t_*}. □

LEMMA: IF $F^\ell \to M^n \xrightarrow{\pi} B$ IS A RIEM. SUBMERSION WITH TOT. GEOD. FIBERS, $\mathrm{Ric}_{g_B} = \Lambda_B g_B$, $\mathrm{Ric}_{g_F} = \Lambda_F g_F$, AND $\exists\, \varsigma, \eta \in \mathbb{R}$ SUCH THAT

$$(A_X, A_Y) = \sum_{i=1}^{n-\ell} g(A_X X_i, A_Y X_i) = \varsigma\, g(X,Y)$$

$$(AU, AV) = \sum_{i=1}^{n-\ell} g(A_{X_i} U, A_{X_i} V) = \eta \cdot g(X,Y)$$

(HERE, $\{X_i\}$ IS O.N.B. OF HORIZONTAL SPACE)

THEN THE CANONICAL VARIATION (M, g_t) IS HORIZONTALLY EINSTEIN, WITH $K_t = \Lambda_B - 2\varsigma t$, AND HAS CONSTANT SCALAR CURVATURE, AND Q-CURVATURE

$$Q_t = \frac{2(n-\ell)(\Lambda_B - 2\varsigma t)^2}{(n-2)^2} - \frac{2\ell}{(n-2)^2}\left(\frac{\Lambda_F}{t} + \eta t\right)^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2}\left(\frac{\ell \Lambda_F}{t} + (n-\ell)\Lambda_B - \eta \ell t\right)^2$$

COROLLARY 0: IF $\Lambda_F > 0$ AND EITHER $5 \le n \le 8$ AND $\ell \geq 3$ OR $n \geq 9$ AND $\ell \geq 2$, THEN THERE ARE INFINITELY MANY BIFURCATIONS FOR THE CONSTANT Q-CURVATURE PROBLEM AS $t \downarrow 0$.

COROLLARY ∞: IF $\varsigma > 0$, $\eta > 0$ AND $\dfrac{\varsigma}{\eta} > c_{n,\ell} \left(= \dfrac{8(n-1)\sqrt{n-\ell}}{\sqrt{(n^3-4n^2+16n-16)\ell^2 - 16(n-1)^2 \ell}} \right)$ AND EITHER $5 \le n \le 8$ AND $\ell \geq 3$ OR $n \geq 9$ AND $\ell \geq 2$ OR $n \geq 21$ AND $\ell = 1$, THEN THERE ARE INFINITELY MANY BIFURCATIONS FOR THE CONST. Q-CURV. PROBLEM AS $t \nearrow +\infty$.

METHOD OF PROOF: ANALYZE ASYMPTOTICS OF α_t AND β_t AS $t \downarrow 0$ OR $t \nearrow +\infty$.

HARD COMPUTATIONS!

APPLICATIONS: • HOPF BUNDLES (PREVIOUSLY STATED COROLLARY)

• HOMOGENEOUS FIBRATIONS WITH 2 ISOTROPY SUMMANDS

$$K/H \to G/H \to G/K, \quad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus \mathfrak{p}$$

H-IRREDUCIBLE REPRESENTATIONS (NON-ISOMORPHIC)

FINAL REMARKS $\longrightarrow$

<u>REMARK</u>:
- ON BOTH COROLLARIES 0 AND ∞, BIFURCATING METRICS WITH CONSTANT Q-CURVATURE <u>DO NOT</u> HAVE $\mathrm{scal} \equiv \mathrm{const}$.
- COROLLARY 0 APPLIES TO <u>PRODUCT MANIFOLDS</u> $M = B \times F$, WHERE $\gamma = \eta = 0$, WHILE COROLLARY ∞ <u>DOES NOT</u>!