How to recognize the shape of a world from within it?

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What is the shape of the Earth?

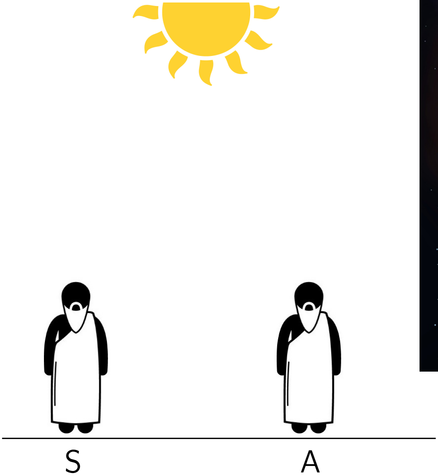
Maybe it is shaped like a Donut (a.k.a. torus)...

Light rays from Sun are nearly parallel

At noon of summer solstice:

Syene: Sun directly overhead

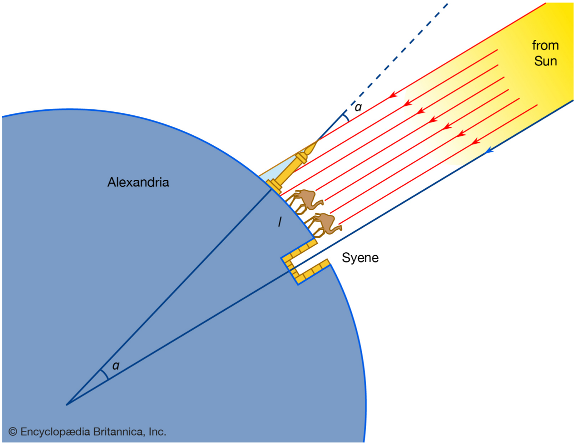
Alexandria: Sun casts shadows

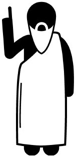


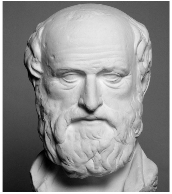
so the Earth cannot be flat...

Eratosthenes (Greece, 276 BC – 195 BC)

How to measure the Earth's circumference?



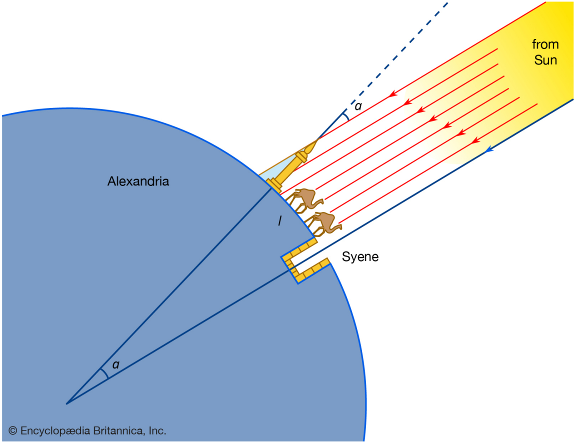




- Light rays from the Sun are nearly *parallel*
- Syene on the Tropic of Cancer, so Sun *directly overhead at noon of summer solstice*
- Alexandria and Syene on the *same meridian*, $l = 5000$ stadia apart

Eratosthenes (Greece, 276 BC – 195 BC)

How to measure the Earth's circumference?



Measure α at noon:
1/50th of circle

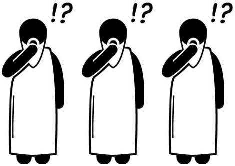
$$\alpha \cong \frac{\pi}{25} \cong 7.2^\circ$$

$$\frac{l}{c} = \frac{\alpha}{2\pi}$$

$$c = \frac{2\pi l}{\alpha} = \frac{2\pi \cdot 5000}{\pi/25} = 250{,}000 \text{ stadia} \cong 39{,}375 \text{ km}$$

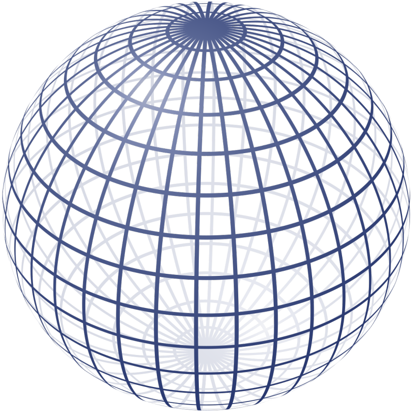
Actual value:
40,076 km

Definitely curved (not flat). But why spherical?

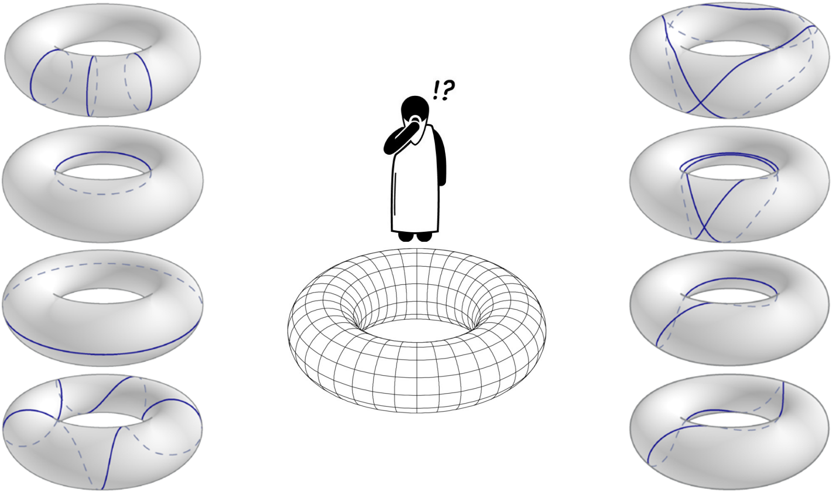


Walk around the whole Earth?

Little Prince (Antoine de Saint-Exupéry)



How long until you're back (on a torus Earth)?



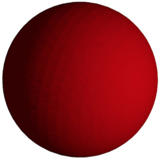
Practical matters

Since we can't actually *walk the whole Earth*, need to determine *global shape* using only *local measurements*.

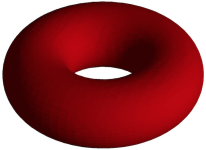
By the way, what is "shape"?

Topology

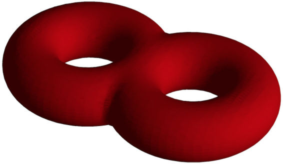
Surfaces are classified by their *genus*



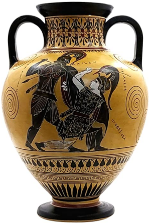
$g = 0$



$g = 1$



$g = 2$



Topological obstructions

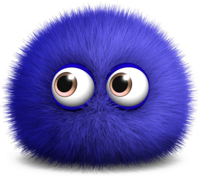
Can you comb a hairy ball?

What about a hairy torus?

The Hairy Ball Theorem

Definition

The *Euler Characteristic* of a surface M is $\chi(M) = 2 - 2g$.

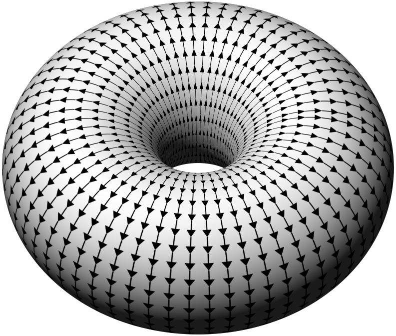


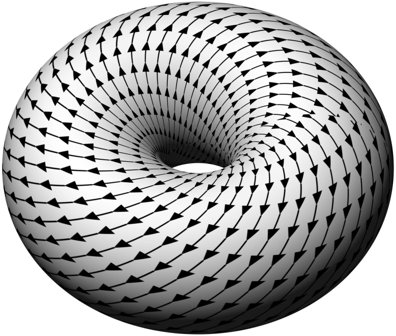
"can be combed" = admits a nonvanishing tangent vector field

Theorem (Poincaré, 1885)

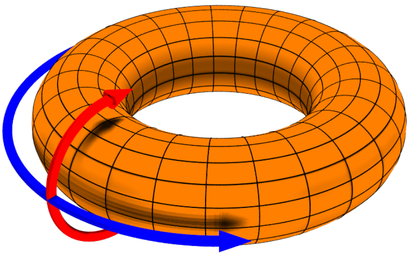
A surface M admits a nonvanishing tangent vector field if and only if $\chi(M) = 0$.

Q: In how many ways can you comb a hairy torus?





A: Two basic hairstyles



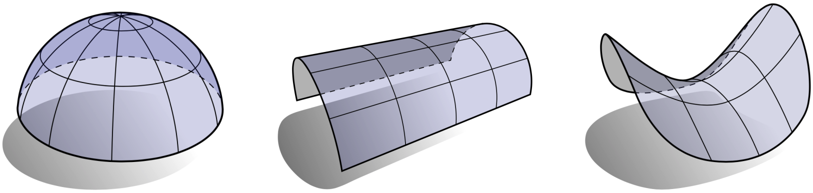
1. Around the hole
2. Through the hole

All others are *linear combinations* of the above two.

$$b_1(T^2) = \dim H^2(T^2, \mathbb{R}) = 2$$

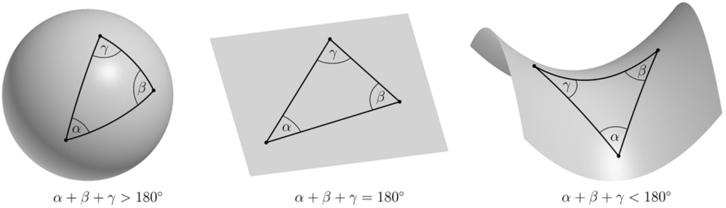
Back to our quest for the shape of the Earth:
What "local measurements" can we use?

Curvature



How can we mathematically express the difference?

1) Sum of inner angles in a triangle T

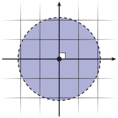


Definition (Sectional curvature)

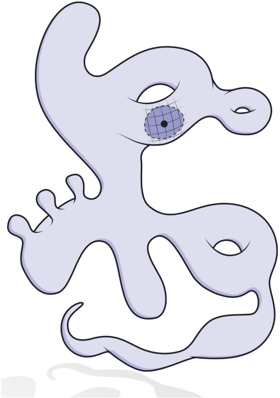
$$\text{sec} = \frac{\alpha + \beta + \gamma - \pi}{\text{Area}(T)}$$

2) Area inside circles

Farbstudie - Quadrate und konzentrische Ringe (Wassily Kandinsky)



$A(r) = \pi r^2$



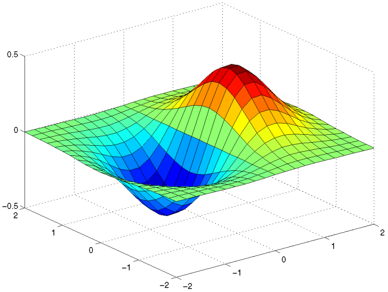
$\widetilde{A}(r)$

$$\sec = \lim_{r \searrow 0} 12 \, \frac{A(r) - \widetilde{A}(r)}{\pi r^4}$$

3) Fancy formulas using Calculus

If surface is a graph, then:

$$\text{sec} = \frac{\dfrac{\partial^2 F}{\partial x^2}\dfrac{\partial^2 F}{\partial y^2} - \left(\dfrac{\partial^2 F}{\partial x \partial y}\right)^2}{\left(1 + \left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2\right)^2}$$



$z = F(x, y)$

Curvature meets Topology, at last

Theorem (Gauss–Bonnet)

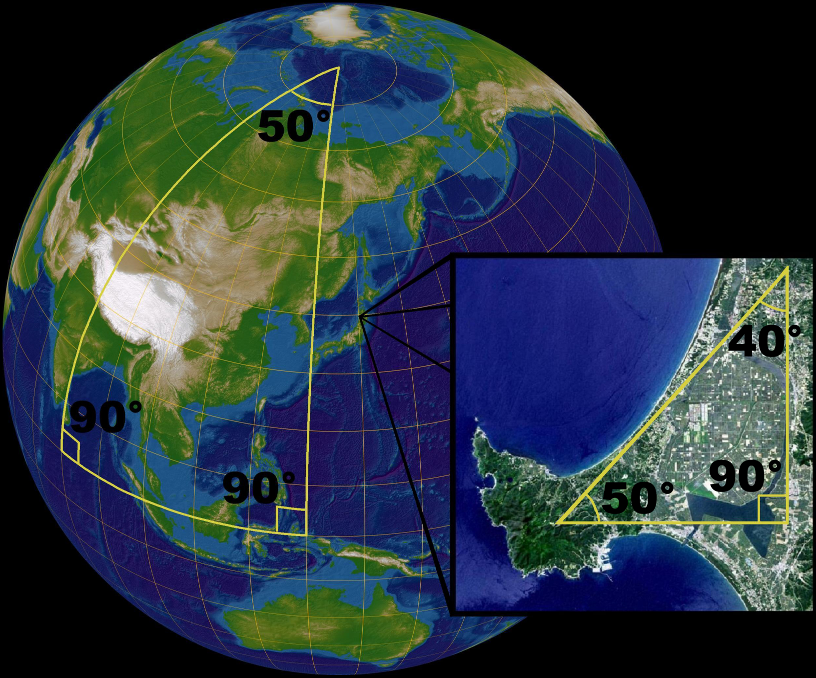
$$\int_M \sec = 2\pi\chi(M)$$

Corollary

If a surface has $\sec > 0$*, then it must be a sphere.*

Proof.

$$0 < \int_M \sec = 2\pi\chi(M) = 2\pi(2 - 2g) \implies g = 0 \qquad \square$$


50°
90°
90°
40°
90°
50°

"Almost" local measurement

Need sufficiently large triangle, or circle, otherwise too close to "0/0".

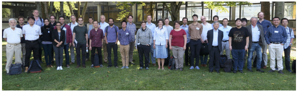
$$\text{sec} = \frac{\alpha + \beta + \gamma - \pi}{\mathrm{Area}(T)}$$

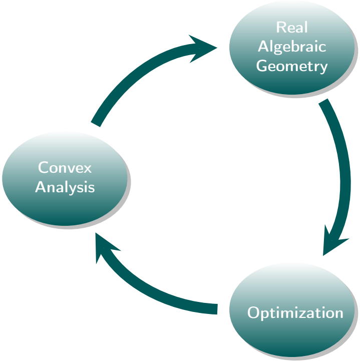
$$\text{sec} = \lim_{r \searrow 0} 12 \, \frac{A(r) - \widetilde{A}(r)}{\pi r^4}$$

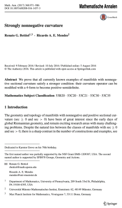
My current research

How do Curvature and Topology interact in higher dimensions?

- Topological obstructions to $\sec > 0$?
- If unobstructed, construct examples!
- Which deformations preserve $\sec > 0$?
- How rigid are shape optimizers?





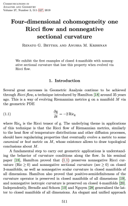


Math. Ann.

Mathematische Annalen

Strongly nonnegative curvature

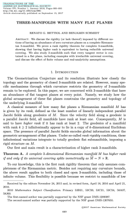
Renato G. Bettiol · Ricardo A. E. Mendes



Communications in Analysis and Geometry

Four-dimensional cohomogeneity one Ricci flow and nonnegative sectional curvature

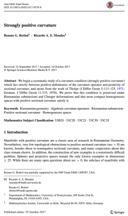
Renato G. Bettiol and Anusha M. Krishnan



Transactions of the American Mathematical Society

THREE-MANIFOLDS WITH MANY FLAT PLANES

Renato G. Bettiol and Benjamin Schmidt



Ann Glob Anal Geom

Strongly positive curvature

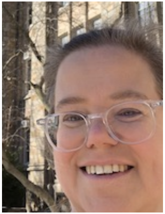
Renato G. Bettiol · Ricardo A. E. Mendes

Some of my colleagues whose research is related


J. Behrstock


C.-Y. Lin


R. Schneiderman


C. Sormani


M. Zeinalian

We hope you consider joining us!

LEHMAN COLLEGE

Thank you for your attention!