Bifurcating minimal surfaces with symmetries

Part 1: Symmetry reduction for minimal hypersurfaces

Renato G. Bettiol

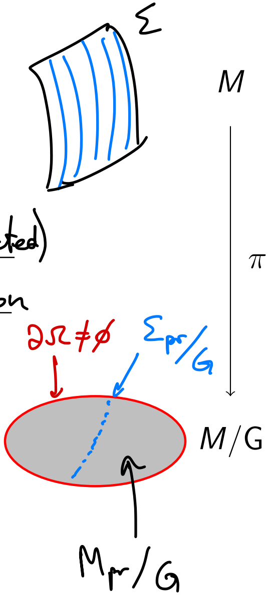
Supported by NSF Award DMS-1904342 and NSF CAREER Award DMS-2142575.

Outline of the talk

1. **Symmetry reduction for minimal hypersurfaces (Hsiang–Lawson)**
2. Basic tools from Bifurcation theory
3. Minimal spheres in 3-ellipsoids
4. Minimal tori in 3-ellipsoids

Problem

Let G $\curvearrowright (M, \mathrm{g})$ be an isometric action. Find G-invariant minimal hypersurfaces $\Sigma \subset M$.



Setup

- $\pi : M \longrightarrow M/\mathrm{G}$ quotient map
- $M_{pr} \subset M$ principal part (open, dense, connected)
- $\pi : (M_{pr}, \mathrm{g}) \longrightarrow (M_{pr}/\mathrm{G}, \check{\mathrm{g}})$ Riem. submersion
- Orbital volume function

$$V_{pr} : M_{pr}/\mathrm{G} \longrightarrow \mathbb{R}$$
$$x \longmapsto \mathrm{Vol}_M(\pi^{-1}(x))$$

is smooth, extends to cont. function $V : M/\mathrm{G} \longrightarrow \mathbb{R}$, such that $V|_{\partial(M/\mathrm{G})} \equiv 0$

- $\Omega := \left(M_{pr}/\mathrm{G}, V^{2/k}\check{\mathrm{g}}\right)$, $\quad k := \dim M_{pr}/G - 1$
$\quad = \dim \Sigma_{pr}/G$

Symmetry Reduction Theorem (Hsiang–Lawson)

$$\boxed{\begin{array}{c}\text{G-invariant hypersurface} \\ \Sigma \subset (M, \mathrm{g}) \text{ is minimal}\end{array}} \iff \boxed{\Sigma_{pr}/\mathrm{G} \subset \Omega \text{ is minimal}}$$

Proof

$\Sigma \subset M$ is a G-invariant hypersurface: $\Sigma_{pr} = \bigcup_{x \in \Sigma_{pr}/\mathrm{G}} \underline{\pi^{-1}(x)}$,

$$V(x) = \mathrm{Vol}_M(\pi^{-1}(x)), \quad k = \dim \Sigma_{pr}/\mathrm{G}$$

$$\mathrm{Vol}_M(\Sigma) = \int_{\Sigma_{pr}/\mathrm{G}} \underline{V(x)\, dx} = \int_{\Sigma_{pr}/\mathrm{G}} \underline{d\mathrm{vol}_{V^{2/k} g}} = \mathrm{Vol}_\Omega(\Sigma_{pr}/\mathrm{G})$$

Palais' Symm. Criticality principle

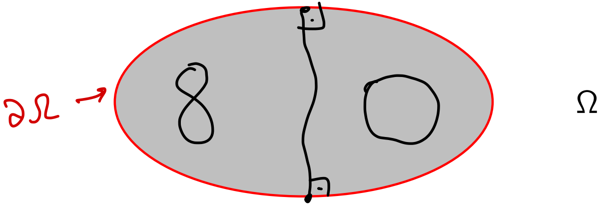
$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Vol}_M(\Sigma_t) = 0, \ \forall \Sigma_t}_{\Sigma \text{ minimal in } (M,g)} \iff \begin{array}{l}\underline{\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Vol}_M(\Sigma_t) = 0,} \\ \forall \ \underline{\Sigma_t \quad G\text{-invariant}}\end{array}$$

$$\iff \underbrace{\begin{array}{l}\underline{\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Vol}_\Omega((\Sigma_{pr}/G)_t) = 0,} \\ \forall \underline{(\Sigma_{pr}/G)_t}\end{array}}_{\Sigma/G \text{ minimal in } \Omega}$$

□

Actions of cohomogeneity 2

If $\dim \Omega = 2$, then we're looking for geodesics $\Sigma_{pr}/G \subset \Omega$!



$$\Sigma \subset M \text{ closed} \iff \begin{cases} \Sigma_{pr}/G \subset \Omega \text{ closed}, & \text{if } \partial\Omega = \emptyset \\ \Sigma_{pr}/G \subset \Omega \text{ free boundary}, & \text{if } \partial\Omega \neq \emptyset \end{cases}$$

$\Sigma \subset M$ embedded $\iff \Sigma_{pr}/G \subset \Omega$ embedded.

Applications

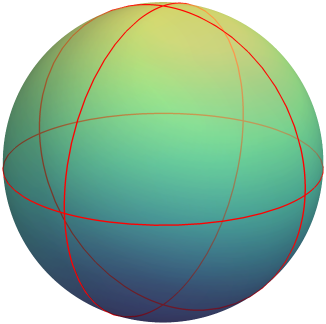
Chern 1969: "Spherical Bernstein Problem"

Let $\mathbb{S}^n$ be the unit **round sphere**. Is every embedded minimal $S^{n-1} \subset \mathbb{S}^n$ *planar*, i.e., congruent to an equator $\mathbb{S}^{n-1} \subset \mathbb{S}^n$?

- Almgren 1966: Yes, if $n = 3$,
- Hsiang 1983: No, if $n =$ 4, 5, 6, 7, 8, 10, 12, 14, ...

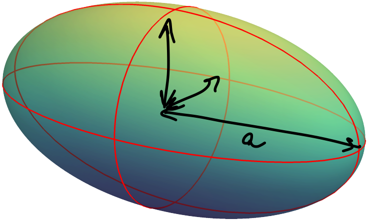
$G \curvearrowright S^n$

$\dim S^n/G = 2$



Yau 1987:

Are there any nonplanar embedded minimal spheres in $E(a,b,c,d) := \left\{ \vec{x} \in \mathbb{R}^4 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} + \frac{x_4^2}{d^2} = 1 \right\}$?



tot. geod.

Planar minimal spheres: $\Pi_i := E(a,b,c,d) \cap \{x_i = 0\}$

- Haslhofer–Ketover 2019: At least one, if a is large enough (using Min-Max theory and Mean Curvature Flow)

Part 3

- B.–Piccione 2021: Arbitrarily many, up to congruences if a is large enough, and either $b = c$ or $c = d$

Some other applications:

- Nonplanar minimal hypersurfaces of $\mathbb{R}^n$: Alencar 1993, Q.-M. Wang 1994, ...
- Nonplanar free boundary minimal surfaces in $\mathbb{B}^n$: Freidin–Gulian–McGrath 2017, Siffert–Wuzyk 2021
- Carlotto–Schulz 2022: Embedded minimal $T^3 \subset \mathbb{S}^4$

Bifurcating minimal surfaces with symmetries

Part 2: Basic tools from Bifurcation theory

Renato G. Bettiol

Supported by NSF Award DMS-1904342 and NSF CAREER Award DMS-2142575.

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"Bifurcation" (1885)

Topological change in the structure of a dynamical system when a parameter crosses a bifurcation value

SUR L'ÉQUILIBRE D'UNE MASSE FLUIDE

ANIMÉE D'UN MOUVEMENT DE ROTATION

PAR

H. POINCARÉ

À PARIS.

§ 1. ***Introduction.***

Quelles sont les figures d'équilibre relatif que peut affecter une masse fluide homogène dont toutes les molécules s'attirent conformément à la loi de NEWTON et qui est animée autour d'un certain axe d'un mouvement de rotation uniforme?

Quelles sont les conditions de stabilité de cet équilibre?

Tels sont les deux problèmes qui forment l'objet de ce mémoire.

On en connait depuis longtemps deux solutions: l'ellipsoïde de révolution et l'ellipsoïde à trois axes inégaux de JACOBI. Je me propose d'établir qu'il y en a une infinité d'autres.

Mais je vais avant d'aller plus loin signaler un certain nombre de résultats que l'on trouve dans le *Treatise on Natural Philosophy* de MM. TAIT et THOMSON, 2me édition, 778. Sir WILLIAM THOMSON énonce la plupart de ces propositions sans aucune démonstration; pour quelques unes d'entre elles, il renvoie à des mémoires plus étendus insérés aux Philosophical Transactions.

Voici ces résultats, qui doivent nous servir de point de départ.

(a). L'ellipsoïde de révolution aplati est une figure d'équilibre toujours stable, si on impose à la masse fluide la condition d'affecter la forme d'un ellipsoïde de révolution.

Acta mathematica. 7. Imprimé le 16 Septembre 1885.

§ 2. ***Equilibre de bifurcation.***

Considérons d'abord le cas où il s'agit d'un équilibre absolu et d'un système dont la position est définie par n quantités $x_1, x_2, \ldots, x_n$. Supposons qu'il y ait une fonction des forces $F(x_1, x_2, \ldots, x_n)$ de façon que l'équilibre ait lieu quand toutes les dérivées de cette fonction s'annulent et qu'il soit stable quand cette fonction est maximum. Je supposerai qu'outre les quantités $x_1, x_2, \ldots, x_n$, il entre dans la fonction F un paramètre variable y, de telle sorte que les valeurs des x qui correspondent à l'équilibre dépendent de ce paramètre y.

Henri Poincaré

How to find nontrivial solutions out of trivial ones?

Try deforming trivial solutions until they become *unstable*:

$f(a, s) = 0$ — a: parameter, s: variable

$$f(a, 0) = 0, \quad \forall a \in \mathbb{R}$$

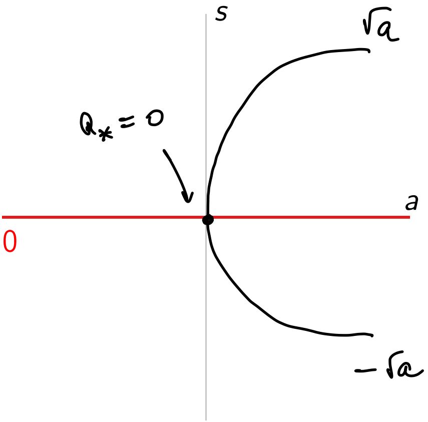
e.g., $f(a, s) = s^3 - as$ $\qquad S = 0$

$s^3 = as \quad$ if $s \neq 0$

$s^2 = a. \quad$ if $a > 0$, $s = \pm\sqrt{a}$

$$\frac{\partial f}{\partial s}(a, 0) = \underline{(3s^2 - a)\big|_{s=0}} = -a$$

$$\frac{\partial^2 f}{\partial a \partial s}(a, 0) = \underline{-1 \neq 0}$$



Local bifurcation

Theorem (Crandall–Rabinowitz)

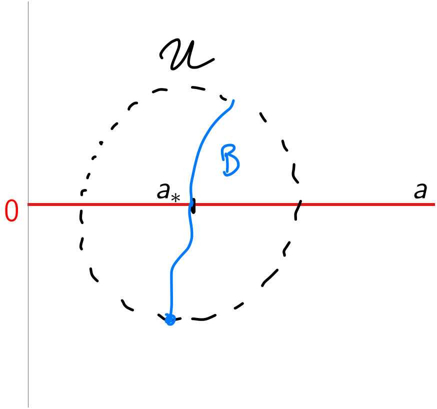
Suppose $f(a, 0) = 0$, $\forall a \in \mathbb{R}$, *and*

- $\dfrac{\partial f}{\partial s}(a_*, 0) = 0$
- $\dfrac{\partial^2 f}{\partial a \partial s}(a_*, 0) \neq 0$

Then $\exists U \ni (a_*, 0)$ *such that*

$$f^{-1}(0) \cap \mathcal{U} = \{(a,0) \in \mathcal{U}\} \cup \mathcal{B}$$

where $\mathcal{B}$ is a bifurcating branch.

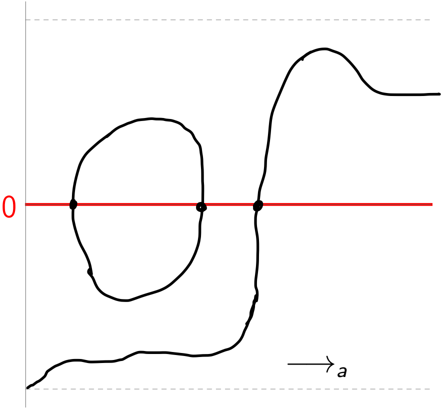


Proof.

Clever use of Implicit Function Thm. □

Global bifurcation

$f: (0, +\infty) \times I \to \mathbb{R}$,
I is an interval, $0 \in I$



Definition

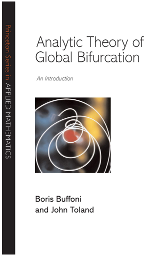
A bifurcating branch $\mathcal{B}$ is a connected component of $f^{-1}(0) \setminus \underbrace{\{(a, 0)\}}_{\mathcal{B}_{triv}}$.

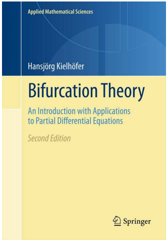
Theorem (Rabinowitz)

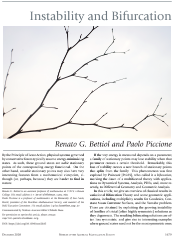
If restriction of $(a, s) \mapsto a$ to $f^{-1}(0)$ is proper, then every branch $\mathcal{B}$ either

- reconnects to trivial branch
- non compact

Commercial break







Instability and Bifurcation

Renato G. Bettiol and Paolo Piccione

By the Principle of Least Action, physical systems governed by conservative forces typically assume energy-minimizing states. As such, these ground states are *stable* stationary points of the corresponding energy functional. On the other hand, *unstable* stationary points may also have very interesting features from a mathematical viewpoint, although (or, perhaps, because) they are harder to find in nature.

If the way energy is measured depends on a parameter, a family of stationary points may lose stability when that parameter crosses a certain threshold. Remarkably, this loss of stability creates a new branch of stationary points that splits from the family. This phenomenon was first explored by Poincaré [Poi85], who called it a *bifurcation*, marking the dawn of a multifaceted theory with applications to Dynamical Systems, Analysis, PDEs, and, more recently, to Differential Geometry and Geometric Analysis.

In this article, we give an overview of classical results in variational Bifurcation Theory and some geometric applications, including multiplicity results for Geodesics, Constant Mean Curvature Surfaces, and the Yamabe problem. These are obtained by exploiting the growing instability of families of trivial (often highly symmetric) solutions as they degenerate. The resulting bifurcating solutions are often less symmetric, and give rise to interesting examples where ground states need not be the most symmetric ones.

Renato G. Bettiol is an assistant professor of mathematics at CUNY, Lehman College. His email address is r.bettiol@lehman.cuny.edu.

Paolo Piccione is a professor of mathematics at the University of São Paulo, Brazil, president of the Brazilian Mathematical Society, and member of the IMU Executive Committee. His email address is piccione@ime.usp.br.

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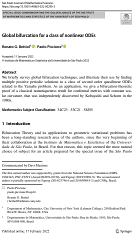

São Paulo Journal of Mathematical Sciences
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SPECIAL ISSUE COMMEMORATING THE GOLDEN JUBILEE OF THE INSTITUTE OF MATHEMATICS AND STATISTICS OF THE UNIVERSITY OF SÃO PAULO

Global bifurcation for a class of nonlinear ODEs

Renato G. Bettiol[1] · Paolo Piccione[2]

Accepted: 11 January 2022
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Abstract
We briefly survey global bifurcation techniques, and illustrate their use by finding multiple positive periodic solutions to a class of second order quasilinear ODEs related to the Yamabe problem. As an application, we give a bifurcation-theoretic proof of a classical nonuniqueness result for conformal metrics with constant scalar curvature, that was independently discovered by Kobayashi and Schoen in the 1980s.

Mathematics Subject Classification 34C23 · 53C21 · 58J55

1 Introduction

Bifurcation Theory and its applications to geometric variational problems has been a long-standing research area of the authors, since the very beginning of their collaboration at the *Instituto de Matemática e Estatística* of the *Universidade de São Paulo*, in Brazil. For that reason, this topic seemed the most natural choice of subject for an article prepared for the special issue of the *São Paulo*

Communicated by Davi Maximo.

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Paolo Piccione
paolo.piccione@usp.br

Renato G. Bettiol
r.bettiol@lehman.cuny.edu

1 Department of Mathematics, City University of New York (Lehman College), 250 Bedford Park Blvd W, Bronx, NY 10468, USA

2 Departamento de Matemática, Universidade de São Paulo, Rua do Matão, 1010, São Paulo, SP 05508-090, Brazil

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Bifurcating minimal surfaces with symmetries

Part 3: Minimal spheres in 3-ellipsoids

Renato G. Bettiol

Supported by NSF Award DMS-1904342 and NSF CAREER Award DMS-2142575.

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1. Symmetry reduction for minimal hypersurfaces (Hsiang–Lawson)

2. Basic tools from Bifurcation theory

3. **Minimal spheres in 3-ellipsoids**

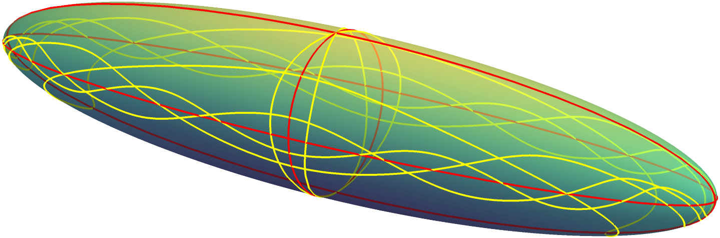
4. Minimal tori in 3-ellipsoids

From Part 1: $E(a,b,c,d) := \left\{ \vec{x} \in \mathbb{R}^4 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} + \frac{x_4^2}{d^2} = 1 \right\}$

Planar minimal spheres $\Pi_i := E(a,b,c,d) \cap \{x_i = 0\}$, $i = 1,2,3,4$

Theorem (B.–Piccione, 2021)

If b = c or c = d, *the number of distinct nonplanar embedded minimal spheres in* $E(a,b,c,d)$ goes to $+\infty$ as $a \nearrow +\infty$, *and they converge to* Π_1 *with increasing multiplicity.*



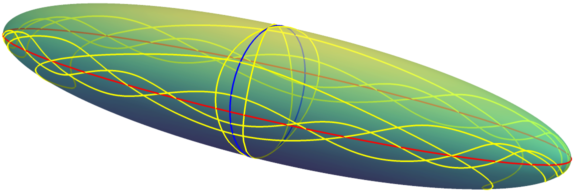
In fact, if $b = c > d$*, then*

$$\liminf_{a \nearrow +\infty} \frac{1}{a} \#\{\text{distinct nonplanar min. spheres in } E(a,b,b,d)\} \geq \frac{1}{2d}$$

Symmetry reduction setting:

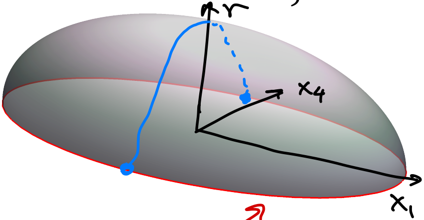
$$(S^3, \mathrm{g}) := E(a,b,b,d) = \left\{ \vec{x} \in \mathbb{R}^4 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{b^2} + \frac{x_4^2}{d^2} = 1 \right\}$$

- $\mathrm{Iso}(S^3, \mathrm{g}) \supseteq \underline{\mathbb{Z}_2 \times O(2) \times \mathbb{Z}_2}$. Set $\mathrm{G} = \underline{O(2)}$
- $\mathrm{G}(x) = \{(x_1, x_2\cos\theta - x_3\sin\theta, x_2\sin\theta + x_3\cos\theta, x_4) : \theta \in \mathbb{R}\}$
- $(S^3)^{\mathrm{G}} = \underline{S^3 \cap \{x_2 = x_3 = 0\}}$ $S^3_{pr} = \underline{S^3 \setminus (S^3)^G}$



$$\begin{array}{c} S^3 \\ \big\downarrow \pi \end{array}$$

- $(S^3/\mathrm{G}, \check{\mathrm{g}}) \cong \left\{ (x_1, r, x_4) \in \mathbb{R}^3 : \frac{x_1^2}{a^2} + \frac{r^2}{b^2} + \frac{x_4^2}{d^2} = 1, r \geq 0 \right\}$
- $\partial(S^3/\mathrm{G}) = \underline{(S^3)^G = \{r=0\}}$
- $V = 2\pi r = 2\pi b \sqrt{1 - \frac{x_1^2}{a^2} - \frac{x_4^2}{d^2}}$
- $\Omega_a := (S^3_{pr}/\mathrm{G}, \underline{V^2 \check{\mathrm{g}}})$



degenerate on $\partial\Omega \neq \emptyset$

$V \equiv 0$ on $\partial\Omega$

Symmetry Reduction Theorem:

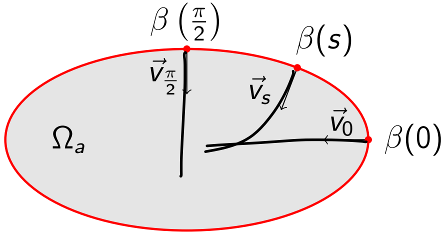
G-invariant minimal 2-sphere $\Sigma \subset (S^3, \mathrm{g})$	$\Longleftrightarrow$	$\Sigma_{pr}/\mathrm{G} \subset \Omega_a$ is a <u>free boundary geodesic</u>

Geodesics on Ω_a

$$\beta\colon [-\pi, \pi) \longrightarrow \partial\Omega_a$$

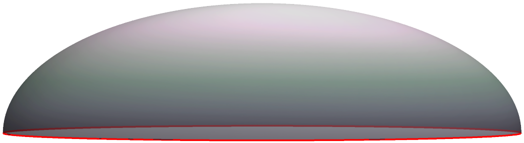
$$\beta(s) = \underline{(a \cos s, d \sin s)}$$

$$\vec{v}_s = \underline{\text{unit normal at}}\ \beta(s)$$



Theorem

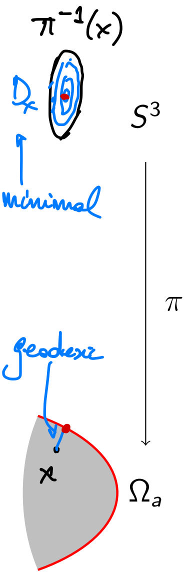
For each $s \in [-\pi, \pi)$, there is a unique maximal geodesic $\gamma_s\colon (0, \ell_s) \to \Omega_a$ that starts transversal to $\partial\Omega_a$ at $\beta(s)$. Moreover, it satisfies $\lim\limits_{t \searrow 0} \frac{\gamma_s'(t)}{|\gamma_s'(t)|_{\breve{g}}} = \underline{\vec{v}_s}$.



Proof

[Hass–Norbury–Rubinstein]

- Let $x \in \Omega_a$ be sufficiently close to $\partial\Omega_a$
- Orbit $\pi^{-1}(x) \subset S^3$ is extremal and real-analytic
- Plateau problem: least-area disk D_x with $\partial D_x = \pi^{-1}(x)$ exists and is
  unique, smooth, embedded, G-invariant
- $D_x \cap (S^3)^{\mathrm{G}} = \{p\}$, $\pi(p) = \beta(s)$ for some s
- Symmetry Reduction:
  $\pi(D_x \setminus \{p\}) \subset \Omega_a$ is a geodesic γ
- $\lim\limits_{t \searrow 0} \gamma(t) = \pi(p) = \beta(s) \in \partial\Omega_a$
- D_x smooth $\Longrightarrow \lim\limits_{t \searrow 0} \frac{\gamma'(t)}{|\gamma'(t)|_{\check{g}}} = \vec{v}_s$
- Removable Singularity Thm. & Maximum Princ.:
  Only geodesic starting transversal to $\beta(s)$ is γ.



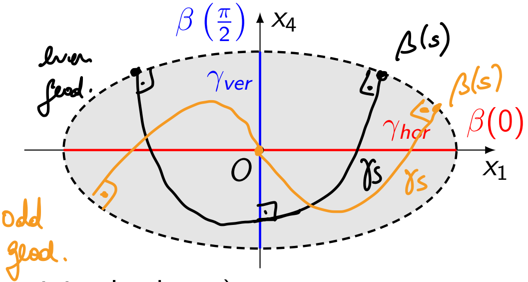
$\square$

Reflections on Ω_a

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \curvearrowright \Omega_a$$

$$\gamma_{hor} := \gamma_0$$

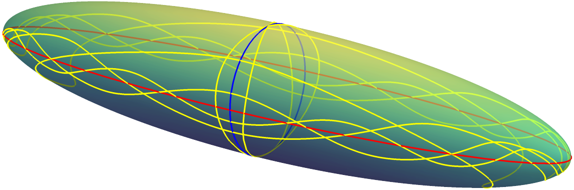
$$\gamma_{ver} := \gamma_{\frac{\pi}{2}}$$



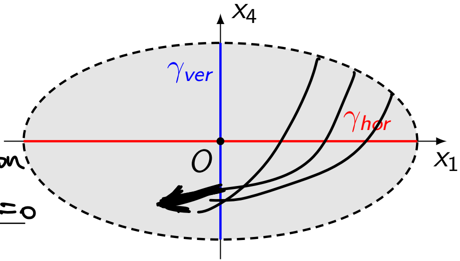
Trivial solutions (planar minimal spheres):

$$\Sigma_{hor} := \pi^{-1}(\gamma_{hor}) = \Pi_4, \quad \Sigma_{ver} := \pi^{-1}(\gamma_{ver}) = \Pi_1$$

- Find γ_s, $s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, meeting γ_{ver} orthogonally, or at O
- Reflect to get free boundary geodesic in Ω_a, then lift to M



- All geodesics γ_s, $s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,
  - intersect γ_{ver} transversely
  - do not self-intersect

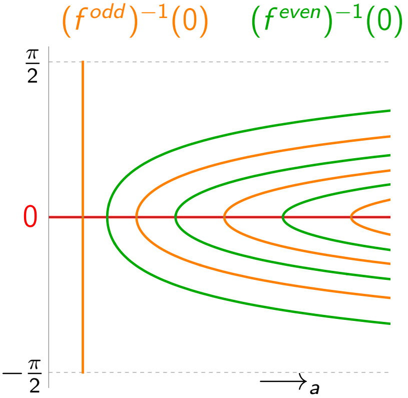


- Let $\gamma_s(\tau_s)$ be the first intersection point w/ γ_{ver}, i.e., $[\gamma_s(\tau_s)]_{x_1}=0$
- $s \mapsto \tau_s$, $s \mapsto \gamma_s$ are real-analytic
- Define the functions

$$f\colon (0,+\infty) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$$

$$f^{\text{even}}(a,s) = [\gamma_s'(\tau_s)]_{x_4}$$

$$f^{\text{odd}}(a,s) = [\gamma_s(\tau_s)]_{x_4}$$



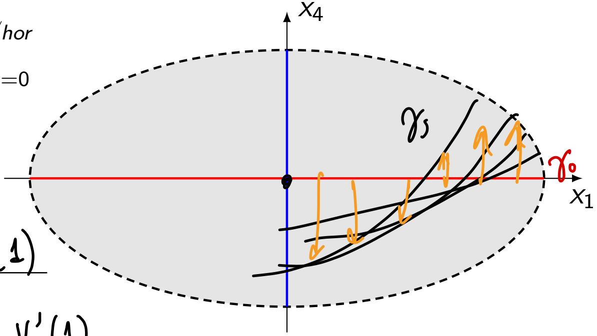
- $f(a,s) = 0 \;\rightsquigarrow\;$ $\pi^{-1}(\gamma_s) \subset S^3$ minimal 2-sphere
- Trivially: $f(a,0) = 0$, for all a
- Look for bifurcations from 0

Linearized problem

$$f : (0, +\infty) \times \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \longrightarrow \mathbb{R}$$

$$f^{\text{even}}(a, s) = [\gamma_s'(\tau_s)]_{x_4}, \quad f^{\text{odd}}(a, s) = [\gamma_s(\tau_s)]_{x_4}$$

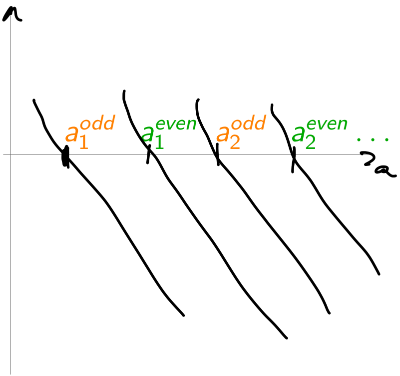
- Reparametrize so that $\tau_s \equiv 1, \forall a > 0$
- Jacobi field along γ_{hor}
  $$J(t) := \frac{\mathrm{d}}{\mathrm{d}s}\gamma_s(t)\Big|_{s=0}$$
- $v_a : [0, 1] \longrightarrow \mathbb{R}$
  $v_a(t) = [J(t)]_{x_4}$
- $\dfrac{\partial f^{\text{even}}}{\partial s}(a, 0) = \underline{V_a'(1)}$
- $\dfrac{\partial^2 f^{\text{even}}}{\partial a \partial s}(a, 0) = \underline{\dfrac{d}{da} V_a'(1)}$



Singular Sturm–Liouville analysis

$\dfrac{\partial f^\bullet}{\partial s}(a,s) = 0$ corresponds to $\lambda = 0$ being an eigenvalue of:

$$\begin{cases} -(p_a v_a')' + q_a v_a = \lambda\, p_a\, v_a, & 0 < t < 1 \\ a^2 v_a(0) + d^2 v_a'(0) = 0, & \text{(IC)} \\ v_a(1) = 0 \;/\; v_a'(1) = 0, & (\text{BC}, \bullet = \text{odd} \;/\; \text{even}) \end{cases}$$



Proposition

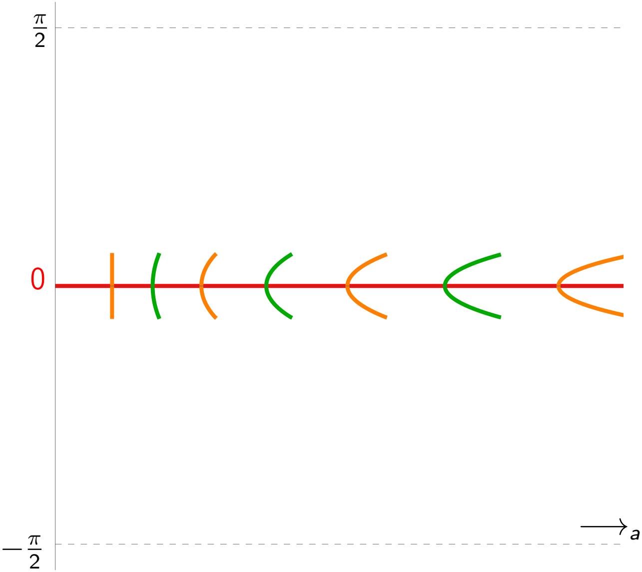
$\exists a_n^{odd}, a_n^{even} \nearrow +\infty$ such that

$$\frac{\partial f^\bullet}{\partial s}(a_n^\bullet, 0) = \underline{0}$$

$$\frac{\partial^2 f^\bullet}{\partial a \partial s}(a_n^\bullet, 0) = \underline{\frac{\partial}{\partial a}\lambda(a)\Big|_{a=a_n^\bullet} < 0}$$

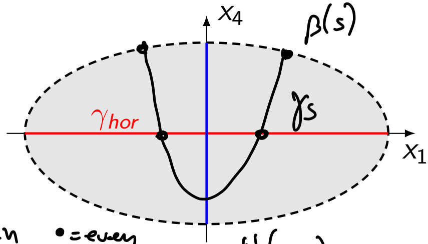
If $b = c = d = r$, then $a_1^{odd} = \underline{r}$.

By Crandall–Rabinowitz (Part 2), both $(f^{odd})^{-1}(0)$ and $(f^{even})^{-1}(0)$ have infinitely many branches:



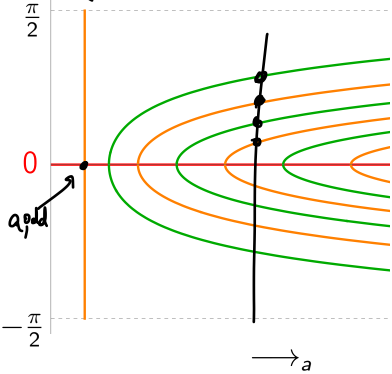
we still need to show that branches "persist" as $a \nearrow +\infty$

Last steps



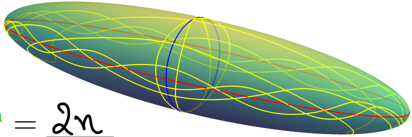
$N(a,\gamma)=2$

- $N(a,\gamma) := \#\{\gamma \cap \gamma_{hor} \subset \Omega_a^\bullet\}$
- $N(a,\gamma_s)$ is locally constant
- Let $\mathcal{B}_n^\bullet \subset (f^\bullet)^{-1}(0) \setminus \{(a,0)\}$ be the branch issuing from $(a_n^\bullet, 0)$
- If $(a,s) \in \mathcal{B}_n^\bullet$, then $N(a,\gamma_s) = \begin{cases} 2n & \bullet = \text{even} \\ 2n-1 & \bullet = \text{odd} \end{cases}$
- Thus $\mathcal{B}_n^\bullet$ are disjoint, hence noncompact by Rabinowitz (Part 2)
- Each $\{a\} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $a > a_1^{odd}$ can intersect only finitely many $\mathcal{B}_n^\bullet$'s
- $|s| \nearrow \frac{\pi}{2}$ along $\mathcal{B}_n^\bullet$ as $a \nearrow +\infty$, so minimal spheres converge to Σ_{ver} with multiplicity $2n-1$, or $2n$.
- $\mathcal{B}_n^{\text{even}}$ yield free bdy min. disks



Bonus: When do new minimal spheres bifurcate?

- Q: How to compute the bifurcation instants $a_n^\bullet$?
- No closed formula, only an arithmetic equation involving infinite continued fractions (Heun functions)!
- If $b = c = d = 1$, then we know $a_1^{\mathrm{odd}} = 1$. What is a_1^{even}? $= 2$?
- We have strong numeric evidence for:



Conjecture

If $b = c = d = 1$, then
$a_n^{\mathrm{odd}} = \underline{2n-1}$ *and* $a_n^{\mathrm{even}} = \underline{2n}$

- Similar to bifurcation of closed geodesics on 2-ellipsoids: instants at which $\mathrm{Area}(\Pi_i) = \frac{4\pi}{3}a$ of planar minimal 2-spheres $\Pi_i \subset E(a,1,1,1)$, $i = 2,3,4$, is an integer multiple of $\mathrm{Area}(\Pi_1) = \frac{4\pi}{3}$.

Bifurcating minimal surfaces with symmetries

Part 4: Minimal tori in 3-ellipsoids

Renato G. Bettiol

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Outline of the talk

1. Symmetry reduction for minimal hypersurfaces (Hsiang–Lawson)

2. Basic tools from Bifurcation theory

3. Minimal spheres in 3-ellipsoids

4. **Minimal tori in 3-ellipsoids**

Similarly to Part 3, let

$$(S^3, \mathrm{g}) := E(a,a,b,b) = \left\{ \vec{x} \in \mathbb{R}^4 : \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} + \frac{x_4^2}{b^2} = 1 \right\}$$

- $\mathrm{Iso}(S^3, \mathrm{g}) \supseteq \underline{O(2) \times O(2)}$. Set $\mathrm{G} = \underline{O(2)}$
- Consider the trivial minimal torus

$$\Sigma_a := \left\{ x \in E(a,a,b,b) : \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} = \frac{x_3^2}{b^2} + \frac{x_4^2}{b^2} = \frac{1}{2} \right\}$$

Theorem (B.–Piccione, 2022)

For all $\underline{0 < a < b/\sqrt{3}}$*, there is at least one* G*-invariant embedded minimal torus in* $E(a,a,b,b)$ *not congruent to* Σ_a.

About minimal tori in spheres

Lawson Conjecture, proved by Brendle 2013

If $a = b = c = d$, then every *embedded* minimal torus in $E(a, a, a, a)$ is congruent to the Clifford torus Σ_a.

Theorem (White, 1989)

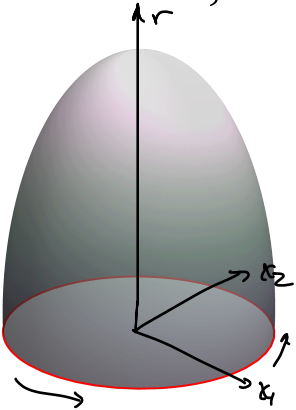
Every metric on S^3 with $\mathrm{Ric} > 0$ has at least 1 embedded minimal torus.

Conjecture (White, 1989)

Every metric on S^3 admits at least 5 embedded minimal tori.

Symmetry reduction setting:

- $(S^3/\mathrm{G}, \check{\mathrm{g}}) \cong \left\{(x_1, x_2, r) \in \mathbb{R}^3 : \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{r^2}{b^2} = 1, r \geq 0\right\}$
- $\partial(S^3/\mathrm{G}) = \underline{(S^3)^G = \{r = 0\}}$
- $V = 2\pi r = 2\pi b\sqrt{1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{a^2}}$
- $\Omega_a := (S^3_{pr}/\mathrm{G}, V^2\check{\mathrm{g}})$
- Note Ω_a is rotationally symmetric, so geodesic ODE has a first integral!

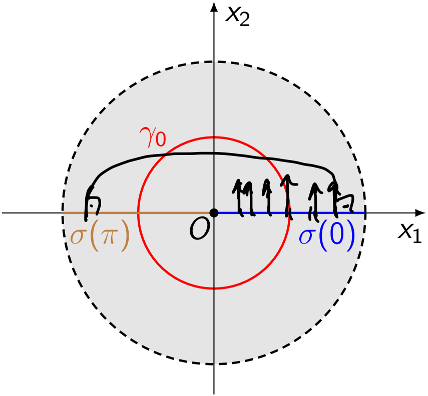


Symmetry Reduction Theorem:

$\boxed{\text{G-invariant minimal torus } \Sigma \subset (S^3, \mathrm{g})} \iff \boxed{\Sigma_{pr}/\mathrm{G} \subset \Omega_a \text{ is a closed geodesics}}$

Bifurcation setup

- Polar coordinates in Ω_a: (ρ, θ)
- $\sigma(\theta) :=$ radial segment at angle θ with x_1 axis
- $\gamma_s(t) :=$ geodesic with $\gamma_s(0) = (\rho(s), 0)$, $\dot\gamma_s(0) = \frac{\partial}{\partial\theta}$, where $I \ni s \mapsto (\rho(s), 0) \in \sigma(0)$ is so that γ_0 is trivial closed geodesic
- Trivial solution: $\Sigma_a := \pi^{-1}(\gamma_0)$

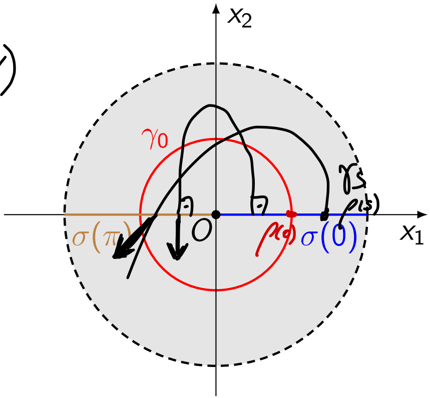


- Find γ_s meeting $\sigma(\pi)$ orthogonally
- Reflect and lift back to S^3

- Write metric as $\mathrm{d}\rho^2 + \varphi(\rho)^2 \mathrm{d}\theta^2$
- $(\rho(t), \theta(t))$ is a geodesic iff

  $\ddot{\rho} - \varphi(\rho)\varphi'(\rho)\,\dot{\theta}^2 = 0$

  $\ddot{\theta} + 2\frac{\varphi'(\rho)}{\varphi(\rho)}\,\dot{\rho}\,\dot{\theta} = 0$
- Conserved quantity $\dot{\theta}\,\varphi(\rho)^2 = c(\gamma)$
- θ is constant or monotonic, so can reparametrize $\gamma_s(t)$ as $\gamma_s(\theta)$



- Define the function

$$f\colon (0, +\infty) \times I \longrightarrow \mathbb{R}$$
$$f(a, s) = [\gamma_s'(\pi)]_{\frac{\partial}{\partial\rho}}$$

- $f(a, s) = 0 \rightsquigarrow$ $\pi^{-1}(\gamma_s) \subset S^3$ minimal torus
- Trivially: $f(a, 0) = 0$, for all a

Linearized problem

$$f : (0, +\infty) \times I \longrightarrow \mathbb{R}$$
$$f(a, s) = [\gamma_s'(\pi)]_{\frac{\partial}{\partial \rho}}$$

- Jacobi field along γ_0

$$J_a(\theta) := \frac{\mathrm{d}}{\mathrm{d}s}\gamma_s(\theta)\Big|_{s=0} = (R_a(\theta), T_a(\theta))$$

- Jacobi equation:

$$\begin{cases} \ddot{R}_a + \frac{4a^2}{a^2+b^2} R_a = 0 \\ (\pi ab)^2 \, \dot{T}_a = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} c(\gamma_\varepsilon)\Big|_{\varepsilon=0} \end{cases}$$

- $\frac{\partial f}{\partial s}(a, 0) = \underline{\dot{R}_a(\pi)}$

- $\frac{\partial^2 f}{\partial a \partial s}(a, 0) = \underline{\frac{d}{da} \dot{R}_a(\pi)}$

With our boundary conditions,

$$R_a(\theta) = \underline{\cos\left(\frac{2a}{\sqrt{a^2+b^2}}\,\theta\right)}$$

$$\underline{\dot{R}_a(\pi) = 0} \iff \underline{\frac{2a}{\sqrt{a^2+b^2}} = m \in \mathbb{Z}}$$

$$\iff \underline{\frac{2a}{\sqrt{a^2+b^2}} = 1}$$

$$\iff \underline{a = b/\sqrt{3}}$$

$$\underline{\frac{d}{da} \dot{R}_a(\pi)\Big|_{a = b/\sqrt{3}} > 0}$$

Upshot

$$\frac{\partial f}{\partial s}\left(\frac{b}{\sqrt{3}}, 0\right) = 0 \qquad \frac{\partial^2 f}{\partial a \partial s}\left(\frac{b}{\sqrt{3}}, 0\right) > 0$$

Thus:

- Local bifurcation at $a = \frac{b}{\sqrt{3}}$, by Crandall–Rabinowitz (Part 2)
- This is the **only** bifurcation branch, hence noncompact by Rabinowitz (Part 2)
- Branch persists for all $0 < a < \frac{b}{\sqrt{3}}$ as it cannot cross $a = b$ by Brendle's proof of Lawson conjecture □