How to find nontrivial solutions out of trivial ones?

Renato G. Bettiol



$$x^{3} - ax = 0$$

- Easiest to find (trivial) solution: x = 0
- ► Other ones?

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$$x^{3} - ax = 0$$

- Easiest to find (trivial) solution: x = 0
- Other ones? ... if $x \neq 0$, then $x^2 - a = 0$, so $x = \pm \sqrt{a}$ (if $a \ge 0$)

$$x^3 - ax = 0$$



$$x^3 - ax = 0$$



• What if we only saw x = 0?

$$x^3 - ax = 0$$



• What if we only saw x = 0? What happens at a = 0?

$$f(a,x)=x^3-ax$$

$$f(a,x) = x^3 - ax$$

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• Special instant: $a_* = 0$

$$f(a,x) = x^3 - ax$$

- ▶ Trivial solution: f(a, 0) = 0 for all $a \in \mathbb{R}$
- Special instant: $a_* = 0$

$$\blacktriangleright \ \frac{\partial f}{\partial x}(a_*,\mathbf{0}) =$$

$$f(a,x) = x^3 - ax$$

- ▶ Trivial solution: f(a, 0) = 0 for all $a \in \mathbb{R}$

$$f(a,x) = x^3 - ax$$

Theorem (Crandall–Rabinowitz) Suppose f(a, 0) = 0 for all $a \in \mathbb{R}$, and $\stackrel{\partial f}{\partial x}(a_*, 0) = 0$ $\stackrel{\partial^2 f}{\partial a \partial x}(a_*, 0) \neq 0$

then a bifurcation branch issues at $(a_*, 0)$.



H. Poincaré. "L'Équilibre d'une masse fluide animée d'un mouvement de rotation". Acta Math., vol. 7, pp. 259-380, 1885.



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"Topological change in the structure of a dynamical system when a parameter crosses a bifurcation value"

Parameter: a

Bifurcation value: $a = a_*$

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H. Poincaré.

Il pourra d'ailleurs arriver qu'une même forme d'équilibre appartienne à la fois à deux ou plusieurs séries linéaires. Nous dirons alors que c'est une *forme de bifurcation*. On peut en effet, pour une valeur de y infiniment voisine de celle qui correspond à cette forme, trouver *deux* formes d'équilibre qui diffèrent infiniment peu de la forme de bifurcation.

Il neut arriver également que deux séries linéaires de formes d'équi-

Avant de démontrer ce résultat général, donnons quelques exemples. Soit:

$$F = Ax_1^2 + \frac{1}{3}x_2^3 - y^2x_2 - \alpha yx_2.$$

Il vient pour les équations d'équilibre:

$$x_1 = 0, \qquad x_2 = \pm \sqrt{y^2 + ay}$$

ďoù









$$t \in [0, L]$$

 $x(t) =$ lateral deflection at t
 $x(0) = x(L) = 0$ (pinned ends)

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$$\underbrace{\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{a}{E} x}_{f(a,x)} = 0$$



(1757) $t \in [0, L]$ x(t) =lateral deflection at t x(0) = x(L) = 0 (pinned ends) E = elasticity constant a =load (parameter) $\frac{d^2x}{dt} + \frac{a}{dt}x = 0$

$$\underbrace{\frac{\mathrm{d} x}{\mathrm{d} t^2} + \frac{a}{E} x}_{f(a,x)} = 0$$

 $f(a, \mathbf{0}) = \mathbf{0}$ for all $a \in \mathbb{R}$





x(0)=x(L)=0



 $x(0) = x(L) = 0 \Rightarrow B = 0,$






• If $a \ge a_n$, then n^{th} buckling mode appears: $x_n(t) = \sin\left(\frac{n\pi t}{L}\right)$





 If a ≥ a_n, then nth buckling mode appears: x_n(t) = sin (nπt/L)
 If a < a₁ = π²E/L², then there is no buckling: only solution is x₀(t) ≡ 0.



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► Crandall-Rabinowitz Thm: $f(a, x) = \frac{d^2x}{dt^2} + \frac{a}{E}x$ $f(a, 0) = 0, \quad \forall a \in \mathbb{R}$



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$$\quad \bullet \ \frac{\partial f}{\partial x}(a_n,0)v = \frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + \frac{a_n}{E}v = 0 \text{ if } v \in \operatorname{span} x_n$$



• Crandall-Rabinowitz Thm:

$$f(a, x) = \frac{d^2x}{dt^2} + \frac{a}{E}x$$

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$$\begin{aligned} & \frac{\partial f}{\partial x}(a_n, \mathbf{0})\mathbf{v} = \frac{\mathrm{d}^2 \mathbf{v}}{\mathrm{d}t^2} + \frac{a_n}{E}\mathbf{v} = \mathbf{0} \text{ if } \mathbf{v} \in \operatorname{span} x_n \\ & \mathbf{b} \left. \frac{\partial^2 f}{\partial a \partial x}(a_n, \mathbf{0}) = \frac{\partial}{\partial a} \lambda(a) \right|_{a=a_n} \end{aligned}$$



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How realistic are these?



Sunkink on train tracks



Can we *buckle/bifurcate* interesting geometric objects into new ones?

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1. Constant Mean Curvature surfaces

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Applications to Geometric Analysis

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In this article, we give an correleve of clusical reach in its nutriational Bifurcation Theory and some geometric applications, including multiplicity results for Condetex, Contant Massa Characterization and the Transles problem, sing theory of the start of the

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Constant Mean Curvature surfaces

 $\Sigma^n \subset \mathbb{R}^{n+1}$ hypersurface

Principal curvatures:

 κ_1



Constant Mean Curvature surfaces

- $\Sigma^n \subset \mathbb{R}^{n+1}$ hypersurface
- Principal curvatures:
- κ_1 , κ_2



Constant Mean Curvature surfaces $\Sigma^{\prime\prime} \subset \mathbb{R}^{n+1}$

- hypersurface
- Principal curvatures:
- $\kappa_1, \kappa_2, \ldots, \kappa_n.$



Principal curvatures:

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Definition $\Sigma^n \subset \mathbb{R}^{n+1}$ has Constant Mean Curvature (CMC) if



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 $\begin{array}{l} \begin{array}{l} {\sf Definition} \\ {\Sigma}^n \subset {\mathbb R}^{n+1} \text{ has} \\ {\sf Constant Mean} \\ {\sf Curvature (CMC) if} \end{array}$

$$\underbrace{\kappa_1 + \dots + \kappa_n}_{H(\Sigma)} = c$$



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▶ Soap bubbles in \mathbb{R}^3 are CMC surfaces: round spheres

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Soap bubbles in R³ are CMC surfaces: round spheres
 General *isoperimetric regions* have CMC boundary

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Soap bubbles in R³ are CMC surfaces: round spheres
 General *isoperimetric regions* have CMC boundary
 Center of Mass in General Relativity: talk to Dan Lee!




































Theorem (Delaunay, 1841)

Surface of revolution $\Sigma \subset \mathbb{R}^3$ has CMC



Profile curve of Σ is the roulette of a conic section.

Delaunay





C.-E. Delaunay

Southeast side of the Eiffel tower:



Delaunay surfaces











Conics of varying eccentricity





Theorem (Mazzeo–Pacard, 2002)

There are infinitely many families of CMC surfaces in \mathbb{R}^3 that bifurcate from nodoids as their eccentricity goes to $+\infty$.



http://service.ifam.uni-hannover.de/~geometriewerkstatt/gallery/0003.html





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Symmetry-breaking:

Bifurcating surfaces are not of revolution!



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Symmetry-breaking: Bifurcating surfaces are not of revolution!

Theorem (B.-Piccione, 2016)

There are infinitely many families of CMC surfaces in cohomogeneity one manifolds that bifurcate from homogeneous surfaces.











https://www.ams.org/journals/notices/202011/rnoti-p1679.pdf

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Minimal surfaces

Definition

A surface with constant mean curvature H = 0 is *minimal*.

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Trivial example on round spheres: equators



Minimal surfaces

Definition

A surface with constant mean curvature H = 0 is *minimal*.

Trivial example on round spheres: equators



Question (Yau, 1987)

Are all minimal spheres in ellipsoids planar?

$$E(a, b, c, d) := \left\{ ec{x} \in \mathbb{R}^4 : rac{x_1^2}{a^2} + rac{x_2^2}{b^2} + rac{x_3^2}{c^2} + rac{x_4^2}{d^2} = 1
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$$E(a,b,c,d) := \left\{ \vec{x} \in \mathbb{R}^4 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} + \frac{x_4^2}{d^2} = 1 \right\}$$

Theorem (B.–Piccione, 2022) As a $\nearrow +\infty$ in the 3-dimensional ellipsoid E(a, b, c, d), nonplanar minimal spheres bifurcate from (planar) equators.



Thank you for your attention



Come join us at CUNY GC!