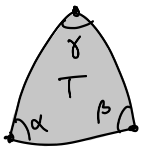
GRAD STUDENT COLLOQUIUM: POSITIVE CURVATURE

CUNY GC 2022

This is 1 of 2 main areas of research in which I work.

$\boxed{\dim M = 2}$ ✓ Surfaces



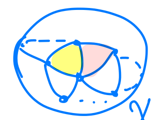
Angle defect: $\mathrm{def}(T) = \alpha + \beta + \gamma - \pi$

(Gauss) curvature: $K(x) = \lim\limits_{\substack{T \ni x \\ T \to 0}} \dfrac{\mathrm{def}(T)}{\mathrm{area}(T)}$

(Can also define with some derivatives...)

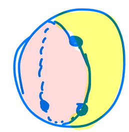
Thus, $\int_T K(x)\, dA = \mathrm{def}(T)$.

Euler characteristic: $\chi(M) = |V| - |E| + |T|$

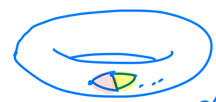


Vertices
Edges
Triangles (faces)

$\chi(S^2) = 2$



$|V| = |E| = 3$
$|T| = 2$



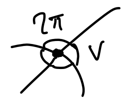
$\chi(T^2) = 0$

Thm (Gauss-Bonnet). $\underbrace{\int_M K\, dA}_{\text{Geometry}} = 2\pi \underbrace{\chi(M)}_{\text{Topology}}$

Only proof today

Pf: $\int_M K\, dA = \sum_T \int_T K\, dA = \sum_T \mathrm{def}(T)$

$= 2\pi|V| - \pi|T|$

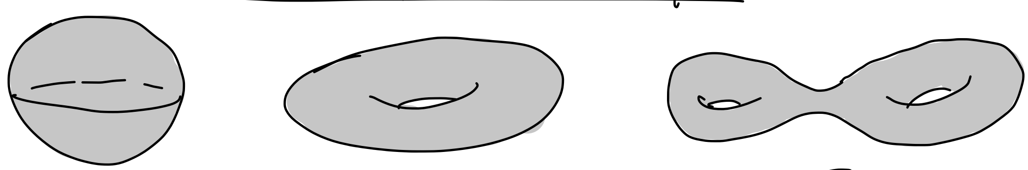


2π, v

$2|E| = 3|T|$

$\overset{?}{=} 2\pi|V| - 2\pi|E| + 2\pi|T| = 2\pi\,\chi(M)$. $\square$

Classification of (orientable) closed surfaces:



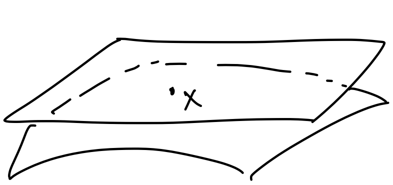
S^2	T^2	Σ	...
$\chi > 0$	$\chi = 0$	$\chi < 0$	
$K > 0$	$K = 0$	$K < 0$	

$\chi(M) = 2 - 2g$

g = # of handles

Thm (Uniformization). Every closed surface admits a metric with constant Gauss curvature $K \equiv c$. (Only one you "see" in $\mathbb{R}^3$ is S^2!)

Curvature in higher dimensions:

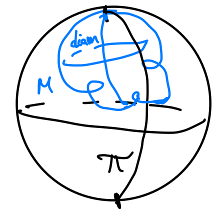


$\Sigma = \exp(\sigma)$ surface tangent to 2-plane σ.

$\sec(\sigma) = K(x)$, $x \in \Sigma$.

"sectional curvature"

Facts in general dimension $n \geq 2$:



Bonnet–Myers: $\sec \geq 1 \implies \operatorname{diam}(M) \leq \pi = \operatorname{diam}(S^n)$

$\implies \pi_1(M)$ is finite.

Synge: $\sec > 0$, n even $\implies \pi_1(M) = \begin{cases} 1 & \text{if } M \text{ is orientable} \\ \mathbb{Z}_2 & \text{if } M \text{ is non-orientable} \end{cases}$

n odd $\implies M$ is orientable.

Geometry

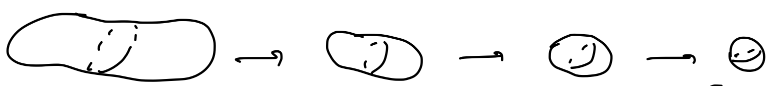
Topology

$\boxed{\dim M = 3}$ ✓

- Thm (Hamilton, 1982). If M^3 has $\sec > 0$ and $\pi_1 M = 1$, then Ricci flow evolves it to constant curvature, hence $M^3 \underset{\text{diffeo}}{\cong} S^3$.

"heat flow for curvature"

M^3



$\sec \equiv 1$

$\boxed{\dim M = 4}$??

A lot of my research lately

- Known examples ($\pi_1 M = 1$): S^4, $\mathbb{CP}^2$. (Conjecturally this list is complete!)
- Hopf Question 1: Does $S^2 \times S^2$ admit $\sec > 0$? (Some of my work explores this...)

$\boxed{\dim M \geq 5}$???

- Hopf Question 2: Does M^{2n}, $\sec > 0$, have $\chi(M) > 0$? (Known if $\dim M = 4$)
- Other than S^n, $\mathbb{CP}^n$, $\mathbb{HP}^n$, $\mathbb{C}a\mathbb{P}^2$, only known examples w/ $\sec > 0$ in dimensions 6, 7, 12, 13, 24

(∞) (∞)

Q: Do they exist in other dimensions too?

So what to do?

1) Grove Symmetry program: (2000–...) Classify manifolds w/ sec > 0 and large symmetry group.

Landmarks:

- Hsiang–Kleiner Thm: If M^4 has sec > 0 and continuous symmetries, then $M^4 \cong S^4$ or $\mathbb{CP}^2$.
- Classification of homogeneous spaces w/ sec > 0 (cohomogeneity 0).
- New example in dim $M = 7$: exotic $T_1 S^4$ w/ cohomogeneity 1.

- - -

Tools: Alexandrov Geometry, critical point theory for distance functions, ...

Will explore these in next semester's Comparison Geometry (& Bonnet-Myers, Synge, Bochner, etc.)

2) New tools from Optimization / Real Algebraic Geometry (2020-...)

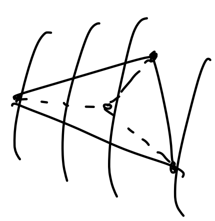
My recent work:

- Effective bounds on topology of 4-manifolds w/ $\delta \leq \sec \leq \Delta$ "pinched curvature"

$$\chi(M) = \int_M p_\chi(R)\, dV, \qquad \tau(M) = \int_M p_\tau(R)\, dV$$

(signature)

Key: Domain of p_χ, p_τ if $\delta \leq \sec \leq \Delta$ has nice convex algebro-geometric structure



$\to$ Find min/max of integrands, hence bounds on $\chi(M)$ and $\tau(M)$.

- Ongoing / Future work:
  1) search for topological obstructions to sec > 0 and related curvature conditions in higher dimensions using these tools.
  2) try to construct new examples!

Issues:

- Convex algebro-geometric structure is less (obviously) nice
- Computations are harder.