I CURVATURE OPERATORS & RATIONAL COBORDISM

(joint w/ Goodman)

$\boxed{\text{Thm (Lichnerowicz '63). If } (M^n,g) \text{ is a closed Riem spin mfld w/ scal} > 0, \text{ then } \hat{A}(M)=0.}$

Pf: $D^2 = \nabla^*\nabla + \frac{\text{scal}}{4}$, $\quad$ scal $>0 \Rightarrow \operatorname{Ker} D = \{0\}$

$n = 4k$: $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$ w.r.t. $S = S^+ \oplus S^-$ so $\operatorname{Ker} D^\pm = \{0\}$

Atiyah–Singer: $\hat{A}(M) = \operatorname{ind}(D^+) = \dim \operatorname{Ker} D^+ - \dim \underbrace{\operatorname{coker} D^+}_{\operatorname{Ker} D^-} = 0$. $\square$

Ex: $M^4 = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{CP}^3 \mid x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}$ is spin, $\hat{A}(M) \neq 0$, so does not have scal >0.

II

KEEP

Twist: $\underset{\text{cplx vector bdl}}{E \to M} \rightsquigarrow D_E : S \otimes E \to S \otimes E$

$D_E^2 = \nabla^*\nabla + \mathcal{R}_E$

Atiyah–Singer: $\hat{A}(M,E) := \langle \hat{A}(TM) \cdot \operatorname{ch} E, [M] \rangle = \operatorname{ind}(D_E^+)$

$\mathbb{Q}$-lin. comb. of Pontryagin #'s

So $\boxed{\mathcal{R}_E > 0} \Longrightarrow \boxed{\hat{A}(M,E) = 0}$

??? $\quad$???

determined by curvature operator of (M,g) if E is built from TM.

Not very useful in this form!

KEEP!

Def: Let $\nu_1 \leq \dots \leq \nu_{\binom{n}{2}}$ be the eigenvalues of $R : \Lambda^2 TM \to \Lambda^2 TM$

$$C_p(R) := A_p \cdot \underbrace{\left(\nu_1 + \dots + \nu_{r_p}\right)}_{\Sigma(r_p, R)} - B_p \cdot \underbrace{\left(\nu_{\binom{n}{2} - r_p' + 1} + \dots + \nu_{\binom{n}{2}}\right)}_{-\Sigma(r_p', -R)} + \frac{\text{scal}}{8}$$

$(p \geq 2)$ $\qquad$ scal $= 2\operatorname{tr} R$

$$C_1(R) := A_1 \cdot \left(\nu_1 + \dots + \nu_{r_1}\right) - \mu + \frac{\text{scal}}{8} \qquad (A_p, B_p, r_p, r_p' > 0)$$

("easily COMPUTABLE") $\qquad$ μ: largest eigenvalue of Ric

(III)

<u>Thm</u> (B.– Goodman '22). If (M^n, g) is a closed Riem. spin mfld with $C_p(R) > 0$ and $E \subseteq TM^{\otimes p}$ is parallel, then $\hat{A}(M, E_{\mathbb{C}}) = 0$.

<u>Ex</u>: $(n=8, p=1)$ $C_1(R) = 3(\overbrace{\nu_1 + \dots + \nu_5}^{\Sigma(5,R)}) - \mu + \frac{\text{scal}}{8}$

$M^8 = \mathbb{H}P^2$ has $\hat{A}(M, TM_{\mathbb{C}}) \neq 0$ so does not admit $C_1(R) > 0$.

In particular, $\mathbb{H}P^2$ has no Einstein metric with $\Sigma(5,R) > 0$

<u>Note</u>: $(\mathbb{H}P^2, g_{FS})$ has $\Sigma(K,R) > 0$ only for $K \geq 19$.

$\Sigma(K,R)$ = sum of first K eigenvalues

(IV)

$C_p(R) > 0 \Longrightarrow \hat{A}(M, E_{\mathbb{C}}) = 0 \quad \forall E \underset{\text{parallel}}{\subseteq} TM^{\otimes p}$

e.g. $E = \mathrm{Sym}^p TM$, $\Lambda^p TM$ ...

$C_1(R) > 0 \Longrightarrow \hat{A}(M, TM_{\mathbb{C}}) = 0$

$\text{scal} > 0 \Longrightarrow \hat{A}(M) = 0$

<u>Nested</u>: If $C_p(R) > 0$, and $1 \leq q < p$ then $\frac{\text{scal}}{4} \geq C_q(R) \geq C_p(R) > 0$

cf.

$R > 0 \Longrightarrow M \underset{\text{diff}}{\cong} S^n/\Gamma$

$\sec > 0 \Longrightarrow \sum b_i \leq C(n)$

$\text{Ric} > 0 \overset{?}{\Longrightarrow} \varphi_W(M) = 0$ if M string

$\text{scal} > 0 \Longrightarrow \hat{A}(M) = 0$ if M spin

$\boxed{\text{LATER!}}$

(V)

Thm (B.–Goodman '22)

(i) Every nontorsion class in Ω_n^{SO}, $n \geq 10$, has a mfld with $C_1(R) > 0$ ("w/o spin condition, no restriction on cobordism class")

(ii) If M^n is spin, $n \geq 10$, and $\hat{A}(M) = \hat{A}(M, TM) = 0$, then $\#^{\ell} M$ is spin-cobordant to a mfld w/ $C_1(R) > 0$.

("w/ spin conditions, these are the only restrictions on cob. class")

(iii) $C_p(R) > 0$ is preserved under surgeries of codimension d if $(d-1)(d-3) > 8p(p+n-2)$.

(VI)

Cor. (M^{4k}, g) closed Riem. spin mfld, $\frac{scal}{8} - Ric > 0$

- $\Sigma(5, R) > 0$ if $k = 2$
- $\Sigma(2k+4, R) > 0$ if $k \geq 6$ even
- $\Sigma(2k+6, R) > 0$ if $k \geq 9$ odd

then M is rationally null cobordant, i.e., $\#^{\ell} M^n = \partial W^{n+1}$

Cf. [Petersen–Wink '21]: $\Sigma(n-p, R) > 0 \Rightarrow b_p M = b_{n-p} M = 0$

so $\Sigma(\lceil \frac{n}{2} \rceil, R) > 0 \Rightarrow M$ is rational homology sphere

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in words w/o writing:

Pf:

Get $C_p(R) > 0$ for large p.

$\Rightarrow$ lots of $\hat{A}(M, E) = 0$,

$\Rightarrow$ All Pontryagin #'s vanish

$\Rightarrow$ Rationally null cobordant

VII

Witten genus: $\varphi_W(M) \in \mathbb{Q}[[q]]$ w/ coeff $\hat{A}(M,-)$

$\hat{A}(M)$, $\hat{A}(M, TM_{\mathbb{C}})$, $\hat{A}(M, TM_{\mathbb{C}} \oplus \mathrm{Sym}^2 TM_{\mathbb{C}})$, ...

<u>Conj.</u> (Stolz). If (M^n, g) is a closed Riem spin mfld w/ $\frac{1}{2}p_1(TM) = 0$ and $\mathrm{Ric} > 0$, then $\varphi_W(M) = 0$.

<u>Cor.</u> If (M^{4k}, g) is a closed Riem. spin mfld, $k \geq 6$, $p_1(TM) = 0$ and $C_{\lfloor k/6 \rfloor}(R) > 0$, then $\varphi_W(M) = 0$.

<u>Note</u>: $\mathrm{Ric} > 0 \not\Rightarrow C_p(R) > 0$

<u>Cor</u> If (M^{4k}, g) is a closed Riem. spin mfld, $k \geq 2$, then $C_{\lfloor k/2 \rfloor}(R) > 0 \Longrightarrow \varphi(M) = 0 \Longrightarrow \mathrm{sign}(M) = 0$

(elliptic genus) (signature)

VIII <u>Ingredients for ~~proving~~ Main Theorem</u>

<u>Prop</u>: $\mathcal{R}_E = K(R, S \otimes E) - K(R, E) + \frac{\mathrm{scal}}{8}$ $\quad \left(D_E^2 = D^* D + \mathcal{R}_E\right)$

where $K(R,E)$ is the curvature term in Weitzenböck form.

$$\Delta = \nabla^* \nabla + K(R, E) \text{ for sections of } E \to TM$$

$\boxed{\pi: O(n) \to O(E) \text{ representation}}$ $\rightsquigarrow$ $\boxed{E_\pi = Fr(TM) \times_\pi E \downarrow M \text{ assoc. bdl.}}$

$$K(R, E_\pi) = -\sum_{a=1}^{\binom{n}{2}} d\pi(RX_a) \circ d\pi(X_a) \text{ where } \{X_a\} \text{ is o.n.b. of } \Lambda^2 \mathbb{R}^n \cong \mathfrak{so}(n).$$

$d\pi(X): \mathfrak{so}(n) \to \mathrm{End}(E)$

IX

Prop. If π is irred. w/ highest weight λ, then

$$K(R, E_\pi) \geq \|\lambda\|^2 \, \Sigma(r, R) \cdot \mathrm{Id}, \text{ where } r = \frac{\langle \lambda, \lambda + 2\rho \rangle}{\|\lambda\|^2}$$

half-sum of positive roots

Def. The Petersen–Wink invariant of π is:

$$PW(\pi) = \min \left\{ \frac{\langle \lambda, \lambda+2\rho \rangle}{\|\lambda\|^2} ; \lambda \neq 0 \text{ highest weight of irred. factors of } \pi \right\}$$

Then: $\Sigma(PW(\pi), R) > 0 \Rightarrow K(R, E_\pi) > 0.$

Ex: $PW(\Lambda^p \mathbb{R}^n) = n-p$ if $1 \leq p < \frac{n}{2}$. $\Delta = D^* D + K(R, \Lambda^p \mathbb{R}^n)$ is Hodge Laplacian on p-forms

so $\Sigma(n-p, R) > 0 \Rightarrow b_p M = 0$ [PW, 2021..]

X **Prop.** If π is subrep. of $(\mathbb{R}^n)^{\otimes p}$, then

$$PW(\pi) \geq PW(\mathrm{Sym}^p_0 \mathbb{R}^n) =: r_p' = \frac{n+p-2}{p}$$

$$PW(S \otimes \pi) \geq PW(S \otimes \mathrm{Sym}^p_0 \mathbb{R}^n) =: r_p = \frac{n^2 + (8p-1)n + 8p(p-1)}{n + 8p(p+1)}$$

"$\pi = \mathrm{Sym}^p$ is worst case scenario"

End of Proof:

$$\mathcal{R}_{E_\pi} = K(R, S \otimes E_\pi) - K(R, E_\pi) + \frac{\mathrm{scal}}{8}$$

$$\overset{\text{Prop}}{\geq} \Big(\underbrace{A_p \cdot \Sigma(r_p, R) + B_p \cdot \Sigma(r_p', -R) + \frac{\mathrm{scal}}{8}}_{C_p(R)} \Big) \mathrm{Id}$$

So $C_p(R) > 0 \Rightarrow \mathcal{R}_{E_\pi} > 0 \Rightarrow \hat{A}(M, (E_\pi)_{\mathbb{C}}) = 0.$ $\square$

N.B. $-K(R, E_\pi) = K(-R, E_\pi) \geq \|\lambda\|^2 \cdot \Sigma(r, -R)$