Homework #2

DUE: MAR 1, 2024

1. Prove that $\big((a,b)\times\mathbb{S}^1, \mathrm{d}r^2 + f(r)^2\mathrm{d}\theta^2\big)$, where $f\colon (a,b)\to\mathbb{R}$ is a smooth positive function, embeds isometrically in $\mathbb{R}^3$ as a surface of revolution, i.e., via a map of the form $\phi\colon (a,b)\times\mathbb{S}^1\to\mathbb{R}^3$, $\phi(r,\theta) = (f(r)\cos\theta, f(r)\sin\theta, z(r))$, if and only if $|f'(r)|\leq 1$.

   From the above formula for $\phi\colon (a,b)\times\mathbb{S}^1\to\mathbb{R}^3$, we compute

$$\begin{aligned}\phi^*\mathrm{d}x &= f'(r)\cos\theta\,\mathrm{d}r - f(r)\sin\theta\,\mathrm{d}\theta\\ \phi^*\mathrm{d}y &= f'(r)\sin\theta\,\mathrm{d}r + f(r)\cos\theta\,\mathrm{d}\theta\\ \phi^*\mathrm{d}z &= z'(r)\,\mathrm{d}r\end{aligned}$$

   so $\phi^*(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2) = \big(f'(r)^2 + z'(r)^2\big)\mathrm{d}r^2 + f(r)^2\mathrm{d}\theta^2$. If $|f'(r)|\leq 1$, then setting

$$z(r) = \int_a^r \sqrt{1-f'(t)^2}\,\mathrm{d}t$$

   we have that ϕ becomes the desired isometric embedding. If $|f'(r_0)|>1$ for some r_0, then $\frac{\partial}{\partial r}$ has length >1 with respect to $\phi^*(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2)$ for r near r_0 for any choice of $z(r)$, so ϕ cannot be made isometric, as $\frac{\partial}{\partial r}$ has length 1 with respect to $\mathrm{d}r^2+f(r)^2\mathrm{d}\theta^2$.

2. Let $\pi\colon (M,\mathrm{g})\to(N,\mathrm{h})$ be a Riemannian submersion.

   a) Prove that if $\gamma\colon [a,b]\to M$ is a piecewise smooth curve, then $L_{\mathrm{h}}(\pi\circ\gamma)\leq L_{\mathrm{g}}(\gamma)$.

   b) Conclude that π does not increase distances, i.e., $\mathrm{dist}_{\mathrm{h}}(\pi(p),\pi(q))\leq\mathrm{dist}_{\mathrm{g}}(p,q)$.

   a) At each $p\in M$, let $\mathcal{V}_p = \ker\mathrm{d}\pi(p)$ be the vertical subspace and $\mathcal{H}_p = (\mathcal{V}_p)^{\perp_{\mathrm{g}}}$ be the horizontal subspace, so that $T_pM = \mathcal{H}_p\oplus\mathcal{V}_p$ is a g-orthogonal direct sum. Wherever it is defined, decompose $\dot\gamma(t)\in T_{\gamma(t)}M$ into horizontal and vertical components, say $\dot\gamma(t) = \dot\gamma(t)_{\mathrm{hor}} + \dot\gamma(t)_{\mathrm{ver}}$, with $\dot\gamma(t)_{\mathrm{hor}}\in\mathcal{H}_{\gamma(t)}$ and $\dot\gamma(t)_{\mathrm{ver}}\in\mathcal{V}_{\gamma(t)}$. Then,

$$\|(\pi\circ\gamma)'(t)\|_{\mathrm{h}} = \|\mathrm{d}\pi(\gamma(t))\dot\gamma(t)\|_{\mathrm{h}} = \|\dot\gamma(t)_{\mathrm{hor}}\|_{\mathrm{g}} \leq \|\dot\gamma(t)\|_{\mathrm{g}},$$

   where the second equality holds because π is a Riemannian submersion. Integrating the above over $t\in[a,b]$, it follows that $L_{\mathrm{h}}(\pi\circ\gamma)\leq L_{\mathrm{g}}(\gamma)$.

   b) Recall that distances are defined as follows:

$$\begin{aligned}\mathrm{dist}_{\mathrm{g}}(p,q) &= \inf\{L_{\mathrm{g}}(\gamma) : \gamma \text{ piecewise smooth path joining } p \text{ and } q\}\\ \mathrm{dist}_{\mathrm{h}}(\pi(p),\pi(q)) &= \inf\{L_{\mathrm{h}}(\alpha) : \alpha \text{ piecewise smooth path joining } \pi(p) \text{ and } \pi(q)\}\end{aligned}$$

   Thus, given $\varepsilon>0$, there exists a piecewise smooth path γ joining p and q such that $L_{\mathrm{g}}(\gamma) < \mathrm{dist}_{\mathrm{g}}(p,q)+\varepsilon$. By part a), we have that $\pi\circ\gamma$ is a piecewise smooth path joining $\pi(p)$ and $\pi(q)$ such that $L_{\mathrm{h}}(\pi\circ\gamma)\leq L_{\mathrm{g}}(\gamma)$. Since $\mathrm{dist}_{\mathrm{h}}(\pi(p),\pi(q))$ is the infimum of lengths of all such paths, $\mathrm{dist}_{\mathrm{h}}(\pi(p),\pi(q))\leq L_{\mathrm{h}}(\pi\circ\gamma) < \mathrm{dist}_{\mathrm{g}}(p,q)+\varepsilon$. The conclusion follows by letting $\varepsilon\searrow 0$.

3. Let M be a smooth manifold and ∇ be a connection on TM. Given vector fields X and Y on M, show that

$$(\nabla_X Y)(p) = \frac{\mathrm{d}}{\mathrm{d}t} P_t^{-1}(Y(\gamma(t)))\Big|_{t=0},$$

where γ is an integral curve of X with $\gamma(0) = p$, and $P_t\colon T_pM \to T_{\gamma(t)}M$ is the parallel transport along γ according to ∇. (Therefore, a connection ∇ determines parallel transport maps P; conversely, parallel transports P determine ∇.)

Since $\dot\gamma(t) = X(\gamma(t))$ for $t \in (-\varepsilon, \varepsilon)$ and $\gamma(0) = p$, we have that

$$(\nabla_X Y)(p) = \nabla_{X(p)} Y = \nabla_{X(\gamma(0))} Y = \nabla_{\dot\gamma(0)} Y = \frac{D}{\mathrm{d}t} Y(\gamma(t))\Big|_{t=0}, \tag{1}$$

where $\frac{D}{\mathrm{d}t}$ denotes the covariant derivative of vector fields along $\gamma(t)$. Let $\{\frac{\partial}{\partial x_i}\}$ be a basis of T_pM and set $E_i(t) := P_t\big(\frac{\partial}{\partial x_i}\big)$ for all $t \in (-\varepsilon, \varepsilon)$, which form a basis of $T_{\gamma(t)}M$. Note that $P_t^{-1}E_i(t) = E_i(0)$. There exist functions $y_i(t)$ such that, for $t \in (-\varepsilon, \varepsilon)$,

$$Y(\gamma(t)) = \sum_{i=1}^n y_i(t) E_i(t), \quad \text{and so} \quad P_t^{-1}(Y(\gamma(t))) = \sum_{i=1}^n y_i(t) E_i(0). \tag{2}$$

Moreover,

$$\frac{D}{\mathrm{d}t} Y(\gamma(t)) = \sum_{i=1}^n y_i'(t) E_i(t) + y_i(t) \frac{D}{\mathrm{d}t} E_i(t) = \sum_{i=1}^n y_i'(t) E_i(t). \tag{3}$$

Therefore, by (3) and (2), we have

$$\frac{D}{\mathrm{d}t} Y(\gamma(t))\Big|_{t=0} = \sum_{i=1}^n y_i'(0) E_i(0) = \frac{\mathrm{d}}{\mathrm{d}t} P_t^{-1}(Y(\gamma(t)))\Big|_{t=0}. \tag{4}$$

The conclusion follows from (1) and (4).

4. Let (M, g) be a Riemannian manifold, $\gamma\colon I \to M$ be a smooth path, where $0 \in I \subset \mathbb{R}$, and $P_t\colon T_{\gamma(0)}M \to T_{\gamma(t)}M$, $t \in I$, be the result of parallel transporting vectors along γ with respect to the Levi-Civita connection. Show that P_t is a linear isometry for all $t \in I$ and, if M is oriented, then P_t preserves orientation.

Recall that $P_t(v) = V(t)$ is the unique vector field along $\gamma(t)$ solving the ODE $\frac{DV}{\mathrm{d}t} = 0$ with initial condition $V(0) = v$. Given $v, w \in T_{\gamma(0)}M$ and $\lambda \in \mathbb{R}$, let $V(t) = P_t(v)$ and $W(t) = P_t(w)$, and note that $\frac{D}{\mathrm{d}t}(V + \lambda W) = \frac{DV}{\mathrm{d}t} + \lambda \frac{DW}{\mathrm{d}t} = 0$, and $V(0) + \lambda W(0) = v + \lambda w$, so $P_t(v + \lambda w) = V(t) + \lambda W(t)$ by uniqueness of solutions to ODEs with same initial conditions. Moreover, by metric compatibility, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{g}_{\gamma(t)}(V(t), W(t)) = \mathrm{g}_{\gamma(t)}\left(\frac{DV}{\mathrm{d}t}, W\right) + \mathrm{g}_{\gamma(t)}\left(V, \frac{DW}{\mathrm{d}t}\right) = 0,$$

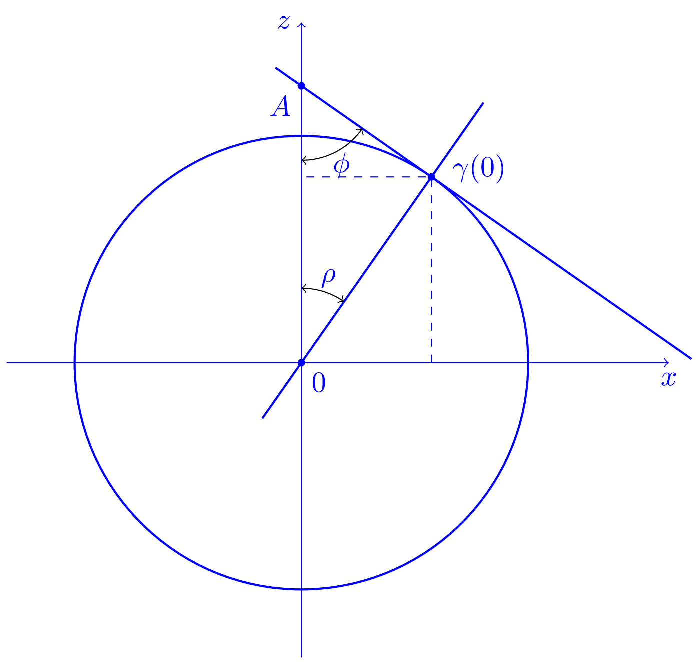
so $\mathrm{g}_{\gamma(t)}(P_t v, P_t w) = \mathrm{g}_{\gamma(t)}(V(t), W(t)) = \mathrm{g}_{\gamma(0)}(V(0), W(0)) = \mathrm{g}_{\gamma(0)}(v, w)$ for all $t \in I$, i.e., P_t is a linear isometry.

If M is oriented, let $\{e_1, \dots, e_n\}$ be a positively oriented basis of $T_{\gamma(0)}M$, e.g., set $e_i = \frac{\partial}{\partial x_i}\big|_{\gamma(0)}$ to be coordinate vector fields at $\gamma(0)$. There are smooth functions $a_{ij}(t)$ such that $P_t e_i = \sum_j a_{ij}(t) \frac{\partial}{\partial x_j}\big|_{\gamma(t)}$, and $a_{ij}(0) = \delta_{ij}$. The function $f \colon I \to \mathbb{R}$ given by $f(t) = \det(a_{ij}(t))$ is continuous and nowhere vanishing, because P_t is invertible for each $t \in I$, so $f(0) = 1$ implies that $f(t) > 0$ for all $t \in I$. Thus, P_t is an oriented isometry.

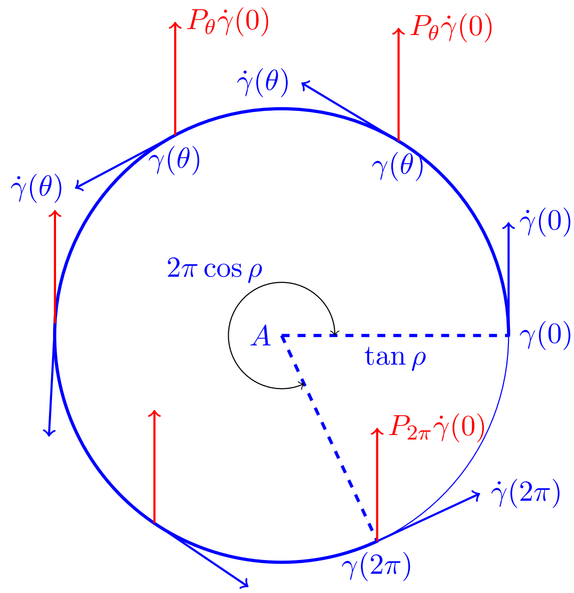
5. (Foucault) Let γ be any latitude on the unit round sphere $\mathbb{S}^2$, i.e., there exists $0 < \rho < \pi$ such that, in polar coordinates, $\gamma(\theta) = (\rho, \theta)$ for $\theta \in [0, 2\pi]$. Describe explicitly the map $P_\theta \colon T_{\gamma(0)}\mathbb{S}^2 \to T_{\gamma(\theta)}\mathbb{S}^2$ given by parallel transport of vectors along γ with respect to the Levi-Civita connection of $\mathbb{S}^2$. By Problem 4, we know $P_{2\pi} \colon T_{\gamma(0)}\mathbb{S}^2 \to T_{\gamma(0)}\mathbb{S}^2$ is an orientation-preserving isometry of $T_{\gamma(0)}\mathbb{S}^2 \cong \mathbb{R}^2$, hence a rotation of angle $\alpha(\rho)$. Compute $\alpha(\rho)$ explicitly in terms of ρ.

   Hint: Embed $\mathbb{S}^2$ isometrically in $\mathbb{R}^3$, and note that there is a unique cone $C \subset \mathbb{R}^3$ tangent to $\mathbb{S}^2$ along the latitude γ. Show that parallel transport along γ is the same, whether with respect to the Levi-Civita connection on $\mathbb{S}^2$ or the flat connection on C.

   First, consider the northern hemisphere $0 < \rho < \frac{\pi}{2}$, and let C be the unique cone in $\mathbb{R}^3$ that is tangent to the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ along $\gamma(\theta) = (\rho, \theta)$. Note C is the cone with apex $A = (0, 0, \sec\rho)$ and aperture angle $\phi = \frac{\pi}{2} - \rho$, obtained rotating around the z-axis the straight line through A and $\gamma(0)$ in the figure below.



In particular, C is isometric to an Euclidean wedge in $\mathbb{R}^2$ with apex at the origin and angle $2\pi \sin\phi = 2\pi\cos\rho$, see figure below (where the dotted lines indicate points that are identified). The circular arc of radius $\tan\rho$ in this wedge (shown as the thick blue curve below) is mapped to $\gamma(\theta)$, and C is tangent to $\mathbb{S}^2$ along the image of this curve.



Parallel transport along γ only depends on the Riemannian metric along γ up to first derivatives. Since the round metric of $\mathbb{S}^2$ and the Euclidean metric on C are tangent to one another along γ, their metrics agree up to first derivatives. Thus, parallel transport along γ is the same, whether with respect to the Levi-Civita connection on $\mathbb{S}^2$ or the Euclidean connection on C. Parallel transport with respect to the latter is the constant map $P_\theta v = v$. This is illustrated by the red vectors above, which are the result $P_\theta\dot\gamma(0)$ of parallel transporting the vector $\dot\gamma(0)$ along $\gamma(\theta)$, $\theta \in [0, 2\pi]$, which traces the arc $[0, 2\pi\cos\rho] \ni t \mapsto (\cos t, \sin t) \in \mathbb{R}^2$, with initial velocity $(0, 1)$. In particular, $P_\theta\dot\gamma(0)$ is a (clockwise) rotation of $\dot\gamma(\theta)$ by the angle $\theta\cos\rho$, for each $\theta \in [0, 2\pi]$. From the previous exercise, we know that $P_\theta \colon T_{\gamma(0)}\mathbb{S}^2 \to T_{\gamma(\theta)}\mathbb{S}^2$ is an oriented isometry, therefore it is a (counterclockwise) rotation of angle $-\theta\cos\rho$ on every vector.

The southern hemisphere, where $\frac{\pi}{2} < \rho < \pi$, is isometric to the northern hemisphere via the reflection $\rho \mapsto \pi - \rho$ about the equator, which reverses orientations. Thus, for a latitude with $\frac{\pi}{2} < \rho < \pi$, the parallel transport map P_θ is a counterclockwise rotation of angle $\theta\cos(\pi - \rho) = -\theta\cos\rho$. Altogether, for any $0 < \rho < \pi$, the parallel transport map $P_\theta \colon T_{\gamma(0)}\mathbb{S}^2 \to T_{\gamma(\theta)}\mathbb{S}^2$ is a (counterclockwise) rotation of angle $-\theta\cos\rho$ on every vector. In particular, $\alpha(\rho) = -2\pi\cos\rho$.

Alternatively, setting $(a, b) = (0, \pi)$ and $f(r) = \sin r$ in the computation of Christoffel symbols from the next exercise, we see that the vector field $V(\theta) = v_1(\theta)\frac{\partial}{\partial r} + v_2(\theta)\frac{\partial}{\partial\theta}$

is parallel along $\gamma(\theta) = (\rho, \theta)$ if and only if

$$v_1'(\theta) = \sin\rho\cos\rho\, v_2(\theta)$$
$$v_2'(\theta) = -\cot\rho\, v_1(\theta).$$

Thus, $v_2''(\theta) = -\cot\rho\, v_1'(\theta) = -\cos^2\rho\, v_2(\theta)$, which implies that, if $v_2'(0) = 0$, then

$$v_2(\theta) = v_2(0)\cos(\theta\cos\rho).$$

Thus, given the initial condition $V(0) = \frac{\partial}{\partial\theta}$, the unique solution to the above is

$$P_\theta V(0) = V(\theta) = \sin\rho\sin(\theta\cos\rho)\tfrac{\partial}{\partial r} + \cos(\theta\cos\rho)\tfrac{\partial}{\partial\theta},$$

which directly shows that P_θ is a counterclockwise rotation of angle $-\theta\cos\rho$.

The name Foucault is attached to this exercise due to its relation with *Foucault's pendulum*[1] as explained in Shifrin's notes[2] (p. 71); transcribed below with our notation:

> Foucault observed in 1851 that the swing plane of a pendulum located on the latitude circle $\gamma(\theta) = (\rho, \theta)$ precesses with a period of $T = 24/\cos\rho$ hours. We can use the result [above] to explain this. We imagine the Earth as fixed and "transport" the swinging pendulum once around the circle in 24 hours. If we make the pendulum very long and the swing rather short, the motion [of the tip] will be "essentially" tangential to the surface of the Earth. If we move slowly around the circle, the forces will be "essentially" normal to the sphere. In particular, letting R denote the radius of the Earth (approximately 3960 mi), the tangential component of the centripetal acceleration is
>
> $$(R\sin\rho)\cos\rho\left(\frac{2\pi}{24}\right)^2 \leq \frac{2\pi^2 R}{24^2} \cong 135.7\ \text{mi/hr}^2 \cong 0.0553\ \text{ft/sec}^2 \cong 0.17\%\,\text{g}.$$
>
> Thus, the "swing vector field" is, for all practical purposes, parallel along the curve. Therefore, it turns through an angle of $\alpha(\rho) = -2\pi\cos\rho$ in one trip around the circle, so it takes $\frac{2\pi}{(2\pi\cos\rho)/24} = \frac{24}{\cos\rho}$ hours to return to its original swing plane.

6. (Clairaut) Let $\big((a,b)\times\mathbb{S}^1, \mathrm{d}r^2 + f(r)^2\mathrm{d}\theta^2\big)$, where $0 \leq a < b \leq +\infty$, $\mathbb{S}^1 = [0, 2\pi]/\sim$, and $f\colon (a,b) \to \mathbb{R}$ is a positive smooth function.

   a) Compute the Christoffel symbols Γ_{ij}^k, $1 \leq i, j, k \leq 2$, of the Levi-Civita connection.

   b) Explicitly write the geodesic equation for a curve γ in the coordinates (r, θ), i.e., $\gamma(t) = (r(t), \theta(t))$, as a coupled system of second order ODEs on $r(t)$ and $\theta(t)$.

[1] https://en.wikipedia.org/wiki/Foucault_pendulum

[2] https://math.franklin.uga.edu/sites/default/files/inline-files/ShifrinDiffGeo.pdf

c) Use b) to obtain two special types of geodesics: all *meridians* $\gamma(t) = (t, \theta_0)$ for fixed θ_0, and, possibly, some parallels $\gamma(t) = (r_0, t)$ for fixed $r_0 \in (a, b)$. What does such an r_0 need to satisfy?

d) Multiply the equation $\ddot{\theta}(t) + \cdots = 0$ by f^2 to obtain a first integral[3] of this ODE system, i.e., a preserved quantity. Recognize this quantity as $\mathrm{g}\big(\dot{\gamma}, \frac{\partial}{\partial\theta}\big)$ and conclude that it is constant along γ. (This is called the *Clairaut relation.*)

e) Show that the equation $\ddot{r}(t) + \cdots = 0$ is equivalent to γ having constant speed, i.e., $\mathrm{g}(\dot{\gamma}, \dot{\gamma})$ being constant along γ, provided $\dot{r} \not\equiv 0$.

f) Show that if a unit speed geodesic γ is not a meridian, then we can globally invert $\theta(t)$ and write $\gamma(t) = (r(t), \theta(t))$ as $\gamma(\theta) = (r(\theta), \theta)$. What ODE does $r(\theta)$ satisfy?

g) Show that if a unit speed geodesic γ is not a parallel, then we can locally invert $r(t)$ and write $\gamma(t) = (r(t), \theta(t))$ as $\gamma(r) = (r, \theta(r))$. What ODE does $\theta(r)$ satisfy?

a) The Christoffel symbols Γ_{ij}^k, $1 \leq i, j, k \leq 2$, of the Levi-Civita connection are

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{\ell} \mathrm{g}^{k\ell} \big(\tfrac{\partial}{\partial x_i} \mathrm{g}_{\ell j} + \tfrac{\partial}{\partial x_j} \mathrm{g}_{i\ell} - \tfrac{\partial}{\partial x_\ell} \mathrm{g}_{ij} \big).$$

As $\mathrm{g}_{11} = 1$, $\mathrm{g}_{12} = 0$, $\mathrm{g}_{22} = f(r)^2$, the only nonvanishing Christoffel symbols are

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{f'(r)}{f(r)} \qquad \text{and} \qquad \Gamma_{22}^1 = -f(r) f'(r).$$

b) Using the above, the geodesic equation for $\gamma(t) = (r(t), \theta(t))$ is the following coupled system of 2 second order ODEs, where we denote by $\dot{x}$ the derivative $\frac{\mathrm{d}x}{\mathrm{d}t}$:

$$\ddot{r} - f(r) f'(r) \dot{\theta}^2 = 0$$
$$\ddot{\theta} + 2 \frac{f'(r)}{f(r)} \dot{r} \dot{\theta} = 0$$

c) A meridian has $r(t) = t$ and $\theta(t) \equiv \theta_0$, hence clearly satisfies the above system of ODEs. A parallel has $r(t) \equiv r_0$ and $\theta(t) = t$, so it satisfies the above system of ODEs if and only if $f'(r_0) = 0$, i.e., if and only if $r_0 \in (a, b)$ is a critical point of f.

d) Multiplying $\ddot{\theta} + 2 \frac{f'(r)}{f(r)} \dot{r} \dot{\theta} = 0$ by $f(r)^2$, we find

$$0 = f(r)^2 \ddot{\theta} + 2 f(r) f'(r) \dot{r} \dot{\theta} = \tfrac{\mathrm{d}}{\mathrm{d}t} \big(f(r)^2 \dot{\theta} \big),$$

so the quantity $f(r(t))^2 \dot{\theta}(t) \equiv C$ is constant[4] along the geodesic $\gamma = (r, \theta)$. This quantity is precisely $\mathrm{g}\big(\dot{\gamma}, \frac{\partial}{\partial\theta}\big) = f(r)^2 \dot{\theta}$.

[3] Such a first integral for this system of 2 second order ODEs allows us to reduce it to a single first order ODE, see f) and g). If $|f'(t)| \leq 1$, then γ describes the trajectory of a particle moving along a surface of revolution in $\mathbb{R}^3$ without external forces, and this preserved quantity is the angular momentum of γ.

[4] Note that the values of the constant C may be different for different geodesics.

e) We have $\mathrm{g}(\dot\gamma,\dot\gamma) = \dot r^2 + f(r)^2\dot\theta^2$. This quantity is constant along γ if and only if

$$\ddot r\dot r + f(r)f'(r)\dot r\dot\theta^2 + f(r)^2\ddot\theta\dot\theta = 0.$$

Using the equation for $\ddot\theta$, i.e., the Clairaut relation, the above can be rewritten as

$$0 = \ddot r\dot r + f(r)f'(r)\dot r\dot\theta^2 + f(r)^2\dot\theta\big(-2\tfrac{f'(r)}{f(r)}\dot r\dot\theta\big) = \dot r\big(\ddot r - f(r)f'(r)\dot\theta^2\big).$$

f) If γ is not a meridian, then $\dot\theta \not\equiv 0$, i.e., the constant C such that $f(r)^2\dot\theta \equiv C$ is nonzero. Thus, $\theta(t)$ has nowhere vanishing derivative, hence is monotonic and admits a global inverse $t = t(\theta)$, so we can write $\gamma(\theta) = (r(t(\theta)), \theta)$. Since γ has unit speed geodesic, $\dot r^2 + f(r)^2\dot\theta^2 = 1$. Solving for $\dot r$ we find

$$\dot r = \pm\sqrt{1 - f(r)^2\dot\theta^2} = \pm\sqrt{1 - \frac{C^2}{f(r)^2}}.$$

By the Chain Rule,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\dot r}{\dot\theta} = \frac{\dot r\, f(r)^2}{C},$$

so we conclude that $r(\theta)$ satisfies the first order ODE

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \pm\frac{f(r)\sqrt{f(r)^2 - C^2}}{C}.$$

g) If γ is not a parallel, then $\dot r \not\equiv 0$ but it may have zeros, so $r(t)$ need not be monotonic. Restricting to an interval around t_0 such that $\dot r(t_0) \neq 0$, we may locally invert $t = t(r)$ and write $\gamma(r) = (r, \theta(t(r)))$. Since γ has unit speed geodesic and $f(r)^2\dot\theta = C$, as in the previous item (except that, here, C may be zero), we have

$$\dot r = \pm\sqrt{1 - \frac{C^2}{f(r)^2}}.$$

By the Chain Rule,

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\dot\theta}{\dot r} = \frac{C}{\dot r\, f(r)^2},$$

so we conclude that $\theta(r)$ satisfies the first order ODE

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \pm\frac{C}{f(r)\sqrt{f(r)^2 - C^2}}.$$

Note that the right-hand side of the above is a function of r, so we may integrate:

$$\theta(r) = \pm\int \frac{C}{f(r)\sqrt{f(r)^2 - C^2}}\,\mathrm{d}r.$$

X. (Will not be graded) Use Problem 6 to find all unit speed geodesics of $\mathbb{S}^2$, $\mathbb{R}^2$, and $\mathbb{H}^2$.

For $\mathbb{S}^2$, use $f(r) = \sin r$ in the above and conclude that the only parallel which is a geodesic is the equator $r = \pi/2$; all other geodesics are either meridians or given by $\gamma(r) = (r, \theta(r))$ with

$$\theta(r) = \pm \int \frac{C}{\sin r \sqrt{\sin r^2 - C^2}} \, \mathrm{d}r = \pm \arctan\left(\frac{\sqrt{2} C \cos r}{\sqrt{1 - 2C^2 - \cos 2r}} \right) + \theta_0.$$

Writing $\gamma(r)$ in Euclidean coordinates $(\cos\theta(r)\sin r, \sin\theta(r)\sin r, \cos r) \in \mathbb{S}^2 \subset \mathbb{R}^3$ it becomes clear that $\gamma(r)$ is contained in a linear subspace of $\mathbb{R}^3$, i.e., is a great circle.

For $\mathbb{R}^2$, use $f(r) = r$ in the above and conclude that no parallels are geodesics; geodesics are either meridians (lines through the origin) or given by $\gamma(r) = (r, \theta(r))$ with

$$\theta(r) = \pm \int \frac{C}{r\sqrt{r^2 - C^2}} \, \mathrm{d}r = \pm \arctan\left(\frac{\sqrt{r^2 - C^2}}{C} \right) + \theta_0.$$

Writing $\gamma(r)$ in polar coordinates $(r\cos\theta(r), r\sin\theta(r)) \in \mathbb{R}^2$ it becomes clear that $\gamma(r)$ is a straight line.

For $\mathbb{H}^2$, use $f(r) = \sinh r$ in the above and conclude that no parallels are geodesics; geodesics are either meridians or given by $\gamma(r) = (r, \theta(r))$ with

$$\theta(r) = \pm \int \frac{C}{\sinh r \sqrt{\sinh r^2 - C^2}} \, \mathrm{d}r = \pm \arctan\left(\frac{\sqrt{2} C \cosh r}{\sqrt{-1 - 2C^2 + \cosh 2r}} \right) + \theta_0.$$

Writing $\gamma(r)$ in Lorentzian coordinates $(\cos\theta(r)\sinh r, \sin\theta(r)\sinh r, \cosh r) \in \mathbb{H}^2 \subset \mathbb{R}^{2,1}$ it becomes clear that $\gamma(r)$ is contained in a linear subspace of Minkowski space $\mathbb{R}^{2,1}$, i.e., is a great hyperbola.