MATH71000, Spring 2024 **Renato Ghini Bettiol**

Homework #3

DUE: MAR 20, 2024

1. Let $\gamma\colon [a,b] \to M$ be a geodesic and K be a Killing field on M. Show that the restriction of K to γ is a Jacobi field along γ.

   Since K is a Killing field, its flow ϕ_s is a 1-parameter subgroup of isometries of (M, g), with $\phi_s = \mathrm{Id}$. Since the image of a geodesic by an isometry is a geodesic, it follows that $\gamma_s := \phi_s(\gamma)$ are geodesics for all $s \in (-\varepsilon, \varepsilon)$. Thus, the restriction of K to γ is the variational field $K(t) = \frac{\mathrm{d}}{\mathrm{d}s}\gamma_s(t)\big|_{s=0}$ of a variation of γ by geodesics, hence it is a Jacobi field along γ.

2. Show that if $J(t)$ is a Jacobi field along a geodesic $\gamma\colon [0,1] \to M$, then $\mathrm{g}(J(t), \dot\gamma(t)) = a\,t + b$, where $a = \mathrm{g}(J'(0), \dot\gamma(0))$ and $b = \mathrm{g}(J(0), \dot\gamma(0))$. In particular, if $J(0)$ and $J'(0)$ are both orthogonal to $\dot\gamma(0)$, then $J(t)$ is orthogonal to $\dot\gamma(t)$ for all $t \in [0,1]$.

   Since γ is a geodesic, $\frac{D}{\mathrm{d}t}\dot\gamma = 0$ and hence $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{g}(J(t), \dot\gamma(t)) = \mathrm{g}(J'(t), \dot\gamma(t))$ for all $t \in [0,1]$. Differentiating again and using the Jacobi equation, we have

   $$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{g}(J'(t), \dot\gamma(t)) = \mathrm{g}(J''(t), \dot\gamma(t)) = -\mathrm{g}\big(R(J(t), \dot\gamma(t))\dot\gamma(t), \dot\gamma(t)\big) = 0$$

   by the symmetries of R. Thus, the function $[0,1] \ni t \mapsto \mathrm{g}(J'(t), \dot\gamma(t))$ is constant, so $\mathrm{g}(J'(t), \dot\gamma(t)) = \mathrm{g}(J'(0), \dot\gamma(0)) = a$. Integrating in t, it follows that $\mathrm{g}(J(t), \dot\gamma(t)) = a\,t + \mathrm{g}(J(0), \dot\gamma(0)) = a\,t + b$. In particular, if $J(0)$ and $J'(0)$ are both orthogonal to $\dot\gamma(0)$, then $a = b = 0$, so $J(t)$ is orthogonal to $\dot\gamma(t)$ for all $t \in [0,1]$.

3. Let $f\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ be a smooth function with $f(0) = 1$ and $f'(0) = 0$. Consider the submanifolds P and Q of $\mathbb{R}^2$ given by neighborhoods of $(0,0)$ in the y-axis and of $(1,0)$ in the graph of $x = f(y)$, i.e.,

   $$P = \big\{(0,s) \in \mathbb{R}^2 : s \in (-\varepsilon, \varepsilon)\big\}, \qquad \text{and} \qquad Q = \big\{(f(s), s) \in \mathbb{R}^2 : s \in (-\varepsilon, \varepsilon)\big\}.$$

   Consider the energy functional $E_{\mathrm{g}}\colon \Omega_{P,Q} \to \mathbb{R}$ on the set $\Omega_{P,Q}$ of curves joining P to Q, where g is the Euclidean metric. Let $\gamma_0 \in \Omega_{P,Q}$ be the geodesic $\gamma_0(t) = (t, 0)$, $t \in [0,1]$, which is a critical point of $E_{\mathrm{g}}\colon \Omega_{P,Q} \to \mathbb{R}$.

   a) Show that the second variation of $E_{\mathrm{g}}\colon \Omega_{P,Q} \to \mathbb{R}$ at γ_0 is positive-semidefinite if and only if $f''(0) \geq 0$. Conclude that if $f''(0) > 0$, then $\gamma_0 \in \Omega_{P,Q}$ is a local minimum of energy.

   b) If $f''(0) \leq 0$, is the second variation of $E_{\mathrm{g}}\colon \Omega_{P,Q} \to \mathbb{R}$ at γ_0 is negative-semidefinite? What about the second variation of E_{g} among the subset of *geodesics* in $\Omega_{P,Q}$?

   Recall the second variation of energy along a geodesic $\gamma\colon [a,b] \to M$ with variational field $V(s,t) = \frac{\mathrm{d}}{\mathrm{d}s}\gamma_s(t)$ is

   $$\tfrac{\mathrm{d}^2}{\mathrm{d}s^2}E_{\mathrm{g}}(\gamma_s)\big|_{s=0} = \mathrm{g}\big(\tfrac{DV}{\mathrm{d}s}, \dot\gamma\big)\Big|_a^b + \int_a^b \mathrm{g}\left(\tfrac{DV}{\mathrm{d}t}, \tfrac{DV}{\mathrm{d}t}\right) + \mathrm{g}(R(V, \dot\gamma)V, \dot\gamma)\,\mathrm{d}t.$$

a) Let $\gamma_s(t)$, $s \in (-\varepsilon, \varepsilon)$, $t \in [0,1]$, be an arbitrary variation of γ_0 in $\Omega_{P,Q}$, i.e., a 1-parameter family of curves in $\mathbb{R}^2$ joining P to Q, so with endpoints

$$\gamma_s(0) = \big(0, \phi(s)\big), \quad \text{and} \quad \gamma_s(1) = \big(f(\psi(s)), \psi(s)\big),$$

for some smooth functions $\phi, \psi \colon (-\varepsilon, \varepsilon) \to \mathbb{R}$, with $\phi(0) = \psi(0) = 0$.
The corresponding variational field $V(s,t) = \frac{\mathrm{d}}{\mathrm{d}s}\gamma_s(t)$ satisfies

$$V(s,0) = \big(0, \phi'(s)\big), \quad \text{and} \quad V(s,1) = \big(f'(\psi(s))\psi'(s), \psi'(s)\big),$$

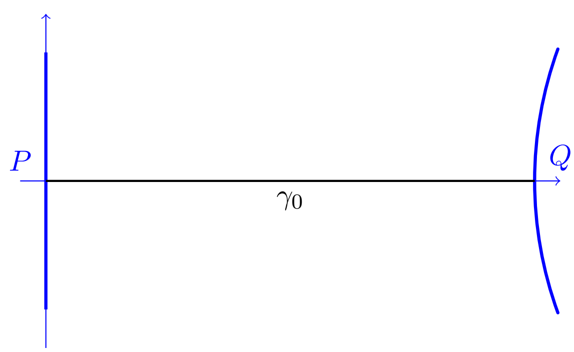
$$\frac{DV}{\mathrm{d}s}(s,0) = \big(0, \phi''(s)\big), \quad \text{and} \quad \frac{DV}{\mathrm{d}s}(s,1) = \big(f''(\psi(s))\psi'(s)^2 + f'(\psi(s))\psi''(s), \psi''(s)\big),$$

$$\frac{DV}{\mathrm{d}s}(0,0) = \big(0, \phi''(0)\big), \quad \text{and} \quad \frac{DV}{\mathrm{d}s}(0,1) = \big(f''(0)\psi'(0)^2, \psi''(0)\big),$$

hence the second variation of energy reads

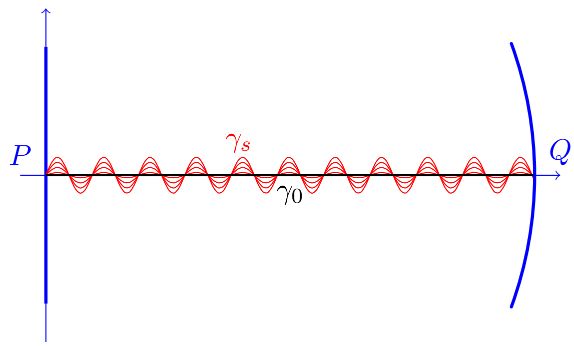
$$\begin{aligned}
\tfrac{\mathrm{d}^2}{\mathrm{d}s^2} E_{\mathrm{g}}(\gamma_s)\big|_{s=0} &= \mathrm{g}\big(\tfrac{DV}{\mathrm{d}s}(0,1), \dot\gamma_0(1)\big) - \mathrm{g}\big(\tfrac{DV}{\mathrm{d}s}(0,0), \dot\gamma_0(0)\big) + \int_0^1 \mathrm{g}\left(\tfrac{DV}{\mathrm{d}t}, \tfrac{DV}{\mathrm{d}t}\right) \,\mathrm{d}t \\
&= f''(0)\psi'(0)^2 + \int_0^1 \mathrm{g}\left(\tfrac{DV}{\mathrm{d}t}, \tfrac{DV}{\mathrm{d}t}\right) \,\mathrm{d}t. \qquad (1)
\end{aligned}$$

Clearly, if $f''(0) \geq 0$, then $\frac{\mathrm{d}^2}{\mathrm{d}s^2} E_{\mathrm{g}}(\gamma_s)\big|_{s=0} \geq 0$ for all variations $s \mapsto \gamma_s \in \Omega_{P,Q}$ of γ_0. Conversely, if $\frac{\mathrm{d}^2}{\mathrm{d}s^2} E_{\mathrm{g}}(\gamma_s)\big|_{s=0} \geq 0$ for an arbitrary variation $s \mapsto \gamma_s \in \Omega_{P,Q}$ of γ_0, then choose a variation with $\frac{DV}{\mathrm{d}t}\big|_{s=0} = 0$ and $\psi'(0) = 1$, e.g., set $\gamma_s(t) = (tf(s), s)$, to conclude that $f''(0) \geq 0$.



Moreover, if $f''(0) > 0$, then $\frac{\mathrm{d}^2}{\mathrm{d}s^2} E_{\mathrm{g}}(\gamma_s)\big|_{s=0} > 0$ for all variations $s \mapsto \gamma_s \in \Omega_{P,Q}$ of γ_0 for which $\gamma_s(1)$ is nonconstant or $\frac{DV}{\mathrm{d}t}$ is not identically zero, so $E_{\mathrm{g}}(\gamma_s) > E_{\mathrm{g}}(\gamma_0)$ for small $s \neq 0$. If a variation $\gamma_s \in \Omega_{P,Q}$ has parallel variational field $\frac{DV}{\mathrm{d}t} \equiv 0$, then it is of the form $\gamma_s(t) = (t + a(s), b(s))$. If, furthermore, it has $\gamma_s(1) \equiv \gamma_0(1)$, then $a \equiv 0$ and $b \equiv 0$ so it is a trivial variation, i.e., $V \equiv 0$.

b) From (1), the second variation of energy cannot be negative-semidefinite even if $f''(0) \leq 0$, since the term $\int_0^1 \mathrm{g}\left(\frac{DV}{\mathrm{d}t}, \frac{DV}{\mathrm{d}t}\right) \mathrm{d}t$ is nonnegative and can be made arbitrarily large, corresponding to variations of γ_0 that oscillate with arbitrarily high frequency, e.g., setting $\gamma_s(t) = (t, s\sin(n\pi t))$ for $n \in \mathbb{N}$ sufficiently large, we have $\mathrm{g}\left(\frac{DV}{\mathrm{d}t}, \frac{DV}{\mathrm{d}t}\right) = n^2\pi^2\cos(n\pi t)$ so $\int_0^1 \mathrm{g}\left(\frac{DV}{\mathrm{d}t}, \frac{DV}{\mathrm{d}t}\right) \mathrm{d}t = n\pi(2n\pi + \sin(2n\pi))/4$ is arbitrarily large. However, such variation $\gamma_s(t)$ is not by other geodesics.



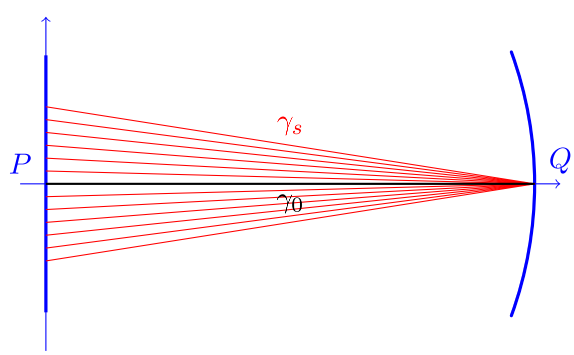
A variation $s \mapsto \gamma_s$ by geodesics in $\mathbb{R}^2$ from P to Q, i.e., by straight line segments from P to Q, can be written as

$$\gamma_s(t) = (c(s)t,\, a(s)\,t + b(s)),$$

where $a, b, c\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ are smooth functions with $c(0) = 1$, $a(0) = b(0) = 0$, and $f(a(s) + b(s)) = c(s)$ for all s because $\gamma_s(1) \in Q$. Thus,

$$V(s,t) = (c'(s)t, a'(s)t + b'(s))$$

so $\frac{DV}{\mathrm{d}s}(s,t) = (c''(s)t, a''(s)t + b''(s))$ and $\frac{DV}{\mathrm{d}t}(s,t) = (c'(s), a'(s))$. Since $f'(0) = 0$, we have $c'(0) = 0$, but $a'(0)$ is not constrained and can be chosen arbitrarily large. Thus, once again, $\int_0^1 \mathrm{g}\left(\frac{DV}{\mathrm{d}t}, \frac{DV}{\mathrm{d}t}\right) \mathrm{d}t = \int_0^1 a'(0)^2\, \mathrm{d}t$ can be made arbitrarily large, e.g., by setting $\gamma_s(t) = (tf(ns), nst)$ for sufficiently large n. This variation has a fixed endpoint at $\gamma_s(0) = (0,0)$. If we let this endpoint vary along P, e.g., setting $\gamma_s(t) = (t, nst - ns)$, then it is geometrically evident (see figure) that γ_0 is a local minimum of $s \mapsto E_{\mathrm{g}}(\gamma_s)$, hence the second variation is not negative-semidefinite.



Altogether, the critical point γ_0 of $E_{\mathrm{g}}\colon \Omega_{P,Q} \to \mathbb{R}$ is a (strict) local minimum if $f''(0) > 0$ and a saddle point if $f''(0) < 0$.

4. Let $\gamma\colon [0,1] \to M$ be a geodesic with initial conditions $\gamma(0) = p$ and $\dot\gamma(0) = v$.

   a) Prove that the Jacobi field J along γ with initial conditions $J(0) = 0$ and $J'(0) = y$ is given by $J(t) = \mathrm{d}(\exp_p)_{tv} ty$.

   b) Prove that the Jacobi field J along γ with initial conditions $J(0) = x$ and $J'(0) = y$ is given by $J(t) = \frac{\partial}{\partial s} \exp_{\alpha(s)} tw(s)\big|_{s=0}$, where $\alpha(s)$ is a curve with $\alpha(0) = p$ and $\dot\alpha(0) = x$, and $w(s)$ is a vector field along $\alpha(s)$ with $w(0) = v$ and $w'(0) = y$.

   a) Define a variation of $\gamma(t) = \exp_p tv$ by geodesics as follows:

   $$\gamma_s(t) = \exp_p(tv + sty).$$

   Its variational field $V(t) = \frac{\mathrm{d}}{\mathrm{d}s}\gamma_s(t)\big|_{s=0} = \mathrm{d}(\exp_p)_{tv} ty$ is a Jacobi field along $\gamma(t)$ with $V(0) = 0$ and $V'(0) = y$, thus, by uniqueness of solutions to ODEs with same initial conditions, $J(t) = V(t)$.

   b) Define a variation of $\gamma(t) = \exp_p tv$ by geodesics as follows:

   $$\gamma_s(t) = \exp_{\alpha(s)}(tw(s)),$$

   where $\alpha(s)$ is a curve with $\alpha(0) = p$ and $\dot\alpha(0) = x$, and $w(s)$ is a vector field along $\alpha(s)$ with $w(0) = v$ and $w'(0) = y$. Note that $\gamma_0(t) = \gamma(t)$.
   Its variational field $V(t) = \frac{\mathrm{d}}{\mathrm{d}s}\gamma_s(t)\big|_{s=0} = \frac{\partial}{\partial s} \exp_{\alpha(s)} tw(s)\big|_{s=0}$ is a Jacobi field along $\gamma(t)$ with $V(0) = \frac{\partial}{\partial s} \exp_{\alpha(s)}(0)\big|_{s=0} = \dot\alpha(0) = x$ and

   $$V'(0) = \frac{D}{\mathrm{d}t}\frac{\partial}{\partial s} \exp_{\alpha(s)} tw(s)\big|_{s=0,t=0} = \frac{D}{\mathrm{d}s}\frac{\partial}{\partial t} \exp_{\alpha(s)} tw(s)\big|_{t=0,s=0} = \frac{Dw}{\mathrm{d}s}\big|_{s=0} = y.$$

   By uniqueness of solutions to ODEs with same initial conditions, $J(t) = V(t)$.

5. Let $R\colon \wedge^2 T_pM \to \wedge^2 T_pM$ be the curvature operator of a Riemannian 3-manifold (M, g) at a point p, and let $\{e_1, e_2, e_3\}$ be an orthonormal basis of T_pM such that R is diagonal in the basis $\{*e_1, *e_2, *e_3\}$ of $\wedge^2 T_pM$, with eigenvalues ν_1, ν_2, ν_3 respectively. (Why does such a basis exist?) Prove that the sectional curvature of a 2-plane $\sigma \subset T_pM$ with unit normal vector $\vec{n} \in T_pM$ is given by:

   $$\sec(\sigma) = \nu_1\, \mathrm{g}(e_1, \vec{n})^2 + \nu_2\, \mathrm{g}(e_2, \vec{n})^2 + \nu_3\, \mathrm{g}(e_3, \vec{n})^2.$$

   Conclude that, on a 3-manifold, the smallest and largest sectional curvatures at $p \in M$ coincide with the smallest and largest eigenvalues of the curvature operator at $p \in M$.

HINT: The Hodge star operator[1] $*\colon \wedge^2 T_pM \cong T_pM$ is a linear isomorphism that maps the oriented Grassmannian $\mathrm{Gr}_2^+(T_pM) \subset \wedge^2 T_pM$ to the unit sphere $\mathbb{S}^2 \subset T_pM$; an oriented 2-plane $\sigma \in \mathrm{Gr}_2^+(T_pM)$ is mapped to its positively oriented unit normal.

Since $R\colon \wedge^2 T_pM \to \wedge^2 T_pM$ is symmetric, there exists an orthonormal basis $\{\alpha_1, \alpha_2, \alpha_3\}$ of $\wedge^2 T_pM$ on which it is diagonal, i.e., $\langle R(\alpha_i), \alpha_j\rangle = \nu_i \delta_{ij}$. Let $e_i = *\alpha_i \in T_pM$, and note that $\{e_1, e_2, e_3\}$ is an orthonormal basis because $*\colon \wedge^2 T_pM \to T_pM$ is a linear isometry. Since $\dim M = 3$, we have that $\sigma \wedge \sigma = 0$ for all $\sigma \in \wedge^2 T_pM$, so the oriented Grassmannian of 2-planes in T_pM is a sphere $\mathrm{Gr}_2^+(T_pM) = \{\sigma \in \wedge^2 T_pM : \|\sigma\| = 1\}$. The image of a 2-plane $\sigma \in \mathrm{Gr}_2^+(T_pM)$ by $*$ is the (positively oriented) unit normal $\vec{n} = *\sigma \in T_pM$ to σ. If $\sigma = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 \in \mathrm{Gr}_2^+(T_pM)$, then $*\sigma = c_1 e_1 + c_2 e_2 + c_3 e_3$, and hence $c_i = \mathrm{g}(e_i, \vec{n})$. Thus, the sectional curvature of such a 2-plane σ is given by

$$\begin{aligned}\sec(\sigma) = \langle R(\sigma), \sigma\rangle = \sum_{i,j} \langle R(c_i\alpha_i), c_j\alpha_j\rangle = \sum_{i,j} c_i c_j \langle R(\alpha_i), \alpha_j\rangle = \\ = \sum_{i,j} c_i c_j \nu_i \delta_{ij} = \sum_i c_i^2 \nu_i = \nu_1\, \mathrm{g}(e_1, \vec{n})^2 + \nu_2\, \mathrm{g}(e_2, \vec{n})^2 + \nu_3\, \mathrm{g}(e_3, \vec{n})^2.\end{aligned}$$

Note that the above is invariant under the involution $\vec{n} \mapsto -\vec{n}$ which reverses orientation of σ, hence descends to the sectional curvature function $\sec\colon \mathrm{Gr}_2(T_pM) \to \mathbb{R}$ on the (unoriented) Grassmannian of 2-planes in T_pM, given by the same expression.

Since $\sec\colon \mathrm{Gr}_2(T_pM) \to \mathbb{R}$ coincides with the restriction of the quadratic form

$$Q\colon \mathbb{R}^3 \to \mathbb{R}, \quad Q(c_1, c_2, c_3) = \nu_1 c_1^2 + \nu_2 c_2^2 + \nu_3 c_3^2$$

to the sphere $\mathbb{S}^2 = \{(c_1, c_2, c_3) \in \mathbb{R}^3 : c_1^2 + c_2^2 + c_3^2 = 1\}$, it follows that its extremal values are $\min_{\mathrm{Gr}_2(T_pM)} \sec = \min_{\mathbb{S}^2} Q = \min \nu_i$ and $\max_{\mathrm{Gr}_2(T_pM)} \sec = \max_{\mathbb{S}^2} Q = \max \nu_i$.

X. (Will not be graded) Give an example of a Jacobi field along a geodesic which is *not* the restriction of an ambient (local) Killing field, cf. Problem 2.

The reasoning here is that Jacobi fields infinitesimally preserve geodesics, while Killing fields infinitesimally preserve the whole Riemannian metric. Thus, to find an example of an ambient vector field which is not Killing but restricts to a Jacobi field along a geodesic, it suffices to find a vector field that infinitesimally preserves geodesics but not the metric. For instance, in Euclidean $\mathbb{R}^2$, any family of linear transformations preserves the geodesics of $\mathbb{R}^2$ since it maps straight lines to straight lines, but only orthogonal linear transformations preserve the metric. Take, e.g., the radial vector field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, which is clearly not Killing since its flow $\phi_t = (t+1)\mathrm{Id}$ is not by isometries of $\mathbb{R}^2$, but $\phi_t\colon \mathbb{R}^2 \to \mathbb{R}^2$ maps straight lines to straight lines. The restriction of X to any straight line in $\mathbb{R}^2$ is a Jacobi field along that geodesic which is not the restriction of an ambient Killing vector field.

[1] https://en.wikipedia.org/wiki/Hodge_star_operator