Homework #4

Due: Apr 19, 2024

1. Prove that if (M^n, g) is a complete connected Riemannian manifold with $\sec > 0$, then any two totally geodesic closed submanifolds $N_1, N_2 \subset M$ with $\dim N_1 + \dim N_2 \geq n$ must intersect. (This is known as *Frankel's Theorem*.)

   Hint: If $N_1 \cap N_2 = \emptyset$, adapt the proof of Myers' Theorem (p.7 of `Lectures3.pdf`).

   Suppose $N_1 \cap N_2 = \emptyset$. Since N_1 and N_2 are compact, there exists a unit speed minimizing geodesic $\gamma\colon [0, L] \to M$ such that $\gamma(0) \in N_1$, $\gamma(L) \in N_2$, and, for all $p_i \in N_i$,

$$\operatorname{dist}(p_1, p_2) \geq \operatorname{dist}(\gamma(0), \gamma(L)) = L > 0.$$

   By the first variation formula, γ meets N_i orthogonally at its endpoints. Thus, the parallel transport of $T_{\gamma(L)}N_2$ along γ from $\gamma(L)$ to $\gamma(0)$ is a linear subspace of $T_{\gamma(0)}M$ whose intersection with $T_{\gamma(0)}N_1$ has dimension ≥ 1, since both are linear subspaces orthogonal to $\dot\gamma(0)$ and the sum of their dimensions is at least n. Let $v \in T_{\gamma(0)}N_1$ be a vector in this intersection, so that its parallel transport $V(t)$ along $\gamma(t)$ satisfies $V(0) \in T_{\gamma(0)}N_1$ and $V(L) \in T_{\gamma(L)}N_2$. The variational field of $\gamma_s(t) = \exp_{\gamma(t)} sV(t)$, $s \in (-\varepsilon, \varepsilon)$, is clearly the parallel vector field $V(t)$, and since N_i are totally geodesic, $\gamma_s(0) \in N_1$ and $\gamma_s(L) \in N_2$ for all $s \in (-\varepsilon, \varepsilon)$. Thus, by the second variation formula,

$$\begin{aligned}\tfrac{\mathrm{d}^2}{\mathrm{d}s^2} E_{\mathrm{g}}(\gamma_s)\big|_{s=0} &= \mathrm{g}\big(\tfrac{DV}{\mathrm{d}s}, \dot\gamma\big)\Big|_0^L + \int_0^L \mathrm{g}\left(\tfrac{DV}{\mathrm{d}t}, \tfrac{DV}{\mathrm{d}t}\right) + \mathrm{g}(R(V,\dot\gamma)V, \dot\gamma)\,\mathrm{d}t \\ &= -\int_0^L \mathrm{g}(R(V,\dot\gamma)\dot\gamma, V)\,\mathrm{d}t \\ &< 0,\end{aligned}$$

   so for sufficiently small $0 < s < \varepsilon$, the curve γ_s, is shorter than $\gamma_0 = \gamma$ and joins N_1 to N_2, contradicting the choice of γ as minimizing geodesic between N_1 and N_2.

2. Prove that a closed hypersurface $M^n \subset \mathbb{R}^{n+1}$ with $\sec > 0$ is diffeomorphic to $\mathbb{S}^n$.

   Hint: If $\vec{n}$ is a unit normal to M, show that $M \ni p \mapsto \vec{n}_p \in \mathbb{S}^n$ is a covering map.

   Choose a unit normal $\vec{n}$ to the hypersurface $M^n \subset \mathbb{R}^{n+1}$, which is possible as embedded submanifolds of codimension 1 in $\mathbb{R}^{n+1}$ are two-sided. We write the second fundamental form of M^n as $\mathrm{II}(X,Y) = h(X,Y)\,\vec{n}$, where $h(X,Y) = \langle S_{\vec{n}}X, Y\rangle$ and $S_{\vec{n}}X = -(\nabla_X \vec{n})^T$ is the shape operator. By the Gauss Equation, for all $X, Y \in T_pM$, we have

$$0 < \sec(X \wedge Y) = h(X,X)\,h(Y,Y) - h(X,Y)^2 = \langle S_{\vec{n}}X, X\rangle\langle S_{\vec{n}}Y, Y\rangle - \langle S_{\vec{n}}X, Y\rangle^2.$$

   Since $S_{\vec{n}}\colon T_pM \to T_pM$ is symmetric, we can diagonalize it with an orthonormal basis $\{e_i\}$ of eigenvectors and corresponding eigenvalues κ_i, say $S_{\vec{n}}e_i = \kappa_i e_i$. Setting $X = e_i$ and $Y = e_j$ in the above, we find that $\kappa_i\kappa_j > 0$ for all $i \neq j$. In particular, $\kappa_i \neq 0$ for all $1 \leq i \leq n$, which means that the linear map $S_{\vec{n}}x = -(\nabla_x \vec{n})^T$ is invertible at all points, so the map $M \ni p \mapsto \vec{n}_p \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a local diffeomorphism, hence a covering map. Since $\mathbb{S}^n$ is simply-connected, it follows that this map is a diffeomorphism.

3. Let (M^n, g) be a complete Riemannian manifold, and $f\colon M \to \mathbb{R}$ a smooth function. Prove that f is convex, i.e., $\mathrm{Hess} f \succeq 0$, if and only if for all geodesics $\gamma\colon \mathbb{R} \to M$, the function $(f \circ \gamma)\colon \mathbb{R} \to \mathbb{R}$ is convex. What can you say about the topology of (M^n, g) if it admits a *strictly* convex function, i.e., with $\mathrm{Hess} f \succ 0$?

If $f\colon M \to \mathbb{R}$ is smooth and $\gamma(t)$ is a geodesic, then

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(\gamma(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{g}\big(\nabla f(\gamma(t)), \dot\gamma(t)\big) \\ &= \mathrm{g}\big(\nabla_{\dot\gamma}\nabla f(\gamma(t)), \dot\gamma(t)\big) + \mathrm{g}\big(\nabla f(\gamma(t)), \nabla_{\dot\gamma(t)}\dot\gamma(t)\big) \\ &= (\mathrm{Hess} f)_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)).\end{aligned}$$

Thus, $\mathrm{Hess} f \succeq 0$ implies $f \circ \gamma$ is convex. Conversely, suppose $f \circ \gamma$ is convex for all geodesics γ. For all $v \in T_pM$ and all $p \in M$, there is a geodesic γ with $\gamma(0) = p$ and $\dot\gamma(0) = v$, hence $(\mathrm{Hess} f)_p(v, v) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} f(\gamma(t)) \geq 0$ for all $p \in M$ and $v \in T_pM$.

Quite a lot can be said about a complete manifold (M^n, g) that admits a strictly convex function. Since $\mathrm{Hess} f \succ 0$ cannot hold at a maximum, the function $f\colon M \to \mathbb{R}$ can only have critical points which are nondegenerate local minima. In particular, M is noncompact, for otherwise $f\colon M \to \mathbb{R}$ would have a maximum, besides a minimum.

If $f\colon M \to \mathbb{R}$ does not have a minimum, then M is diffeomorphic to $N \times \mathbb{R}$ where $N = f^{-1}(c)$ is the preimage of any (regular) value $c \in \mathbb{R}$, as can be seen using the flow of the nowhere vanishing unit vector field $\nabla f / \|\nabla f\|$. Moreover, if $f\colon M \to \mathbb{R}$ has a minimum, then it is unique. Indeed, if p, q are distinct local minima, then let $\gamma\colon [0, 1] \to M$ be a geodesic with $\gamma(0) = p$ and $\gamma(1) = q$. But as $f \circ \gamma$ is strictly convex, we have $f(\gamma(t)) < \min\{f(p), f(q)\}$ for all $0 < t < 1$, contradicting the fact that p, q are local minima. Let $p \in M$ be the unique minimum of f. Since the vector field $\nabla f / \|\nabla f\|$ is bounded and nowhere vanishing on $M \setminus \{p\}$, it follows that M is contractible. If, in addition, $f\colon M \to \mathbb{R}$ is *proper*, i.e., $f^{-1}(K)$ is compact in M for all compact $K \subset \mathbb{R}$, then M is diffeomorphic to $\mathbb{R}^n$ as a consequence of the Brown–Stallings Theorem.[1]

4. Let (M^n, g) be a connected closed Riemannian manifold, and consider smooth functions $f, f_1, f_2\colon M \to \mathbb{R}$. Recall that $\Delta f = \operatorname{div} \nabla f = \operatorname{tr} \mathrm{Hess} f$, and a real number λ is an *eigenvalue* of $-\Delta$ if there exists a nonzero function f such that $-\Delta f = \lambda f$, in which case f is called an *eigenfunction* of $-\Delta$ with eigenvalue λ.

   a) Prove the Green's identity $\displaystyle\int_M f_1 \,\Delta f_2 \;\mathrm{vol}_{\mathrm{g}} = -\int_M \mathrm{g}(\nabla f_1, \nabla f_2) \;\mathrm{vol}_{\mathrm{g}}$.

[1] The *Brown–Stallings Theorem* states that if M^n is a smooth manifold such that for all compact subsets $K \subset M$ there exists an open subset O that contains K and is diffeomorphic to an open ball, then M is diffeomorphic to $\mathbb{R}^n$. It can be proved as an application of the Palais–Cerf Disc Theorem, see Palais "Extending diffeomorphisms", on Proc. AMS 1960. In particular, exotic $\mathbb{R}^4$'s have compact subsets not contained in any open subset diffeomorphic to a ball!

b) Show that *harmonic functions* on M, i.e., solutions to $\Delta f = 0$ on M, are constant. Conclude that the smallest eigenvalue of $-\Delta$ on (M^n, g) is

$$\lambda_0(M^n, \mathrm{g}) := \inf_{f \in W^{1,2}(M)} \frac{\displaystyle\int_M \|\nabla f\|^2 \, \mathrm{vol_g}}{\displaystyle\int_M f^2 \, \mathrm{vol_g}} = 0,$$

and the corresponding eigenspace is formed by constant functions.

c) Decompose the symmetric 2-tensor $\mathrm{Hess} f$ as the sum of its traceless part and a multiple of the identity[2] to show that if $f\colon M \to \mathbb{R}$ is an eigenfunction of $-\Delta$ with eigenvalue λ, then $\lambda \displaystyle\int_M \|\nabla f\|^2 \, \mathrm{vol_g} \leq n \int_M \|\mathrm{Hess} f\|^2 \, \mathrm{vol_g}$.

d) Use the Bochner identity $\frac{1}{2}\Delta\|\nabla f\|^2 = \mathrm{g}(\nabla\Delta f, \nabla f) + \|\mathrm{Hess} f\|^2 + \mathrm{Ric}(\nabla f, \nabla f)$ to prove that $\displaystyle\int_M (\Delta f)^2 \, \mathrm{vol_g} = \int_M \|\mathrm{Hess} f\|^2 + \mathrm{Ric}(\nabla f, \nabla f) \, \mathrm{vol_g}$.

e) Using the above, prove that if (M^n, g) has $\mathrm{Ric} \geq (n-1)k\,\mathrm{g}$, where $k > 0$, then the smallest nonzero eigenvalue of $-\Delta$ satisfies the Lichnerowicz estimate

$$\lambda_1(M^n, \mathrm{g}) := \inf_{\substack{f \in W^{1,2}(M) \\ \int_M f \, \mathrm{vol_g} = 0}} \frac{\displaystyle\int_M \|\nabla f\|^2 \, \mathrm{vol_g}}{\displaystyle\int_M f^2 \, \mathrm{vol_g}} \geq n\,k.$$

a) A simple computation gives $\mathrm{div}(f_1 \nabla f_2) = \mathrm{g}(\nabla f_1, \nabla f_2) + f_1 \Delta f_2$. By the Stokes theorem, since M is closed,

$$0 = \int_M \mathrm{div}(f_1 \nabla f_2) \, \mathrm{vol_g} = \int_M \mathrm{g}(\nabla f_1, \nabla f_2) \, \mathrm{vol_g} + \int_M f_1 \Delta f_2 \, \mathrm{vol_g}.$$

b) If $\Delta f = 0$, then by Green's identity with $f_1 = f_2 = f$, we have that $\nabla f \equiv 0$. Thus, since M is connected, it follows that f is constant.

c) The traceless part of $\mathrm{Hess} f$ is $\mathrm{Hess} f - \frac{\mathrm{tr\,Hess} f}{n}\mathrm{Id} = \mathrm{Hess} f - \frac{\Delta f}{n}\mathrm{Id}$, and it is orthogonal to $\frac{\Delta f}{n}\mathrm{Id}$ in the inner product $\langle A, B\rangle = \mathrm{tr}\, AB$. Thus,

$$\|\mathrm{Hess} f\|^2 = \left\|\mathrm{Hess} f - \frac{\Delta f}{n}\mathrm{Id}\right\|^2 + \frac{(\Delta f)^2}{n} \geq \frac{(\Delta f)^2}{n}.$$

Integrating the above, using $-\Delta f = \lambda f$ and Green's identity, we have:

$$n \int_M \|\mathrm{Hess} f\|^2 \, \mathrm{vol_g} \geq \int_M (\Delta f)^2 \, \mathrm{vol_g} = -\int_M \lambda f \Delta f \, \mathrm{vol_g} = \lambda \int_M \|\nabla f\|^2 \, \mathrm{vol_g}.$$

[2] Recall from Linear Algebra that if A is a symmetric $n \times n$ matrix, then its traceless part $A - \frac{\mathrm{tr}\,A}{n}\mathrm{Id}$ is orthogonal to Id, and hence $\|A\|^2 = \|A - (\frac{\mathrm{tr}\,A}{n})\mathrm{Id}\|^2 + \frac{(\mathrm{tr}\,A)^2}{n}$, since $\|\mathrm{Id}\|^2 = n$.

d) By Green's identity applied with $f_1 = -\Delta f$ and $f_2 = f$, we have

$$\int_M \mathrm{g}(\nabla\Delta f, \nabla f)\ \mathrm{vol_g} = -\int_M (\Delta f)^2\ \mathrm{vol_g}$$

Since M is closed, integrating the Bochner identity, we have

$$0 = \int_M \mathrm{g}(\nabla\Delta f, \nabla f) + \|\mathrm{Hess} f\|^2 + \mathrm{Ric}(\nabla f, \nabla f)\ \mathrm{vol_g},$$

so it follows that $\displaystyle\int_M (\Delta f)^2\ \mathrm{vol_g} = \int_M \|\mathrm{Hess} f\|^2 + \mathrm{Ric}(\nabla f, \nabla f)\ \mathrm{vol_g}$.

e) If $-\Delta f = \lambda f$ and $\mathrm{Ric} \geq (n-1)k\,\mathrm{g}$, where $k > 0$, combining c) and d), we have:

$$\lambda^2 \int_M f^2\ \mathrm{vol_g} = \int_M (\Delta f)^2\ \mathrm{vol_g} = \int_M \|\mathrm{Hess} f\|^2 + \mathrm{Ric}(\nabla f, \nabla f)\ \mathrm{vol_g}$$
$$\geq \frac{\lambda}{n}\int_M \|\nabla f\|^2\ \mathrm{vol_g} + (n-1)k\int_M \|\nabla f\|^2\ \mathrm{vol_g} = \frac{\lambda + n(n-1)k}{n}\int_M \|\nabla f\|^2\ \mathrm{vol_g},$$

so, if $f \not\equiv 0$, we obtain:

$$\big(\lambda + n(n-1)k\big)\frac{\displaystyle\int_M \|\nabla f\|^2\ \mathrm{vol_g}}{\displaystyle\int_M f^2\ \mathrm{vol_g}} \leq n\,\lambda^2.$$

Letting $f \in C^\infty(M)$ be a function with $\int_M f\ \mathrm{vol_g} = 0$ that achieves the infimum in the definition of $\lambda_1 := \lambda_1(M, \mathrm{g})$, the above inequality implies

$$\big(\lambda_1 + n(n-1)k\big)\,\lambda_1 \leq n\,\lambda_1^2.$$

So, dividing both sides by $\lambda_1 > 0$, we conclude that $\lambda_1 \geq n\,k$.

Remark: The above bound $\lambda_1(M^n, \mathrm{g}) \geq n\,k$ for manifolds with $\mathrm{Ric} \geq (n-1)k\,\mathrm{g}$, $k > 0$, is *sharp*: equality is achieved by the round sphere $\mathbb{S}^n(1/\sqrt{k}) \subset \mathbb{R}^{n+1}$ of constant curvature $\sec = k$, whose first eigenfunctions are height functions $f(x) = \langle x, v\rangle$, for any fixed $v \in \mathbb{R}^{n+1}$. Moreover, it is *rigid*: if (M^n, g) is a manifold with $\mathrm{Ric} \geq (n-1)k\,\mathrm{g}$, $k > 0$, and $\lambda_1(M^n, \mathrm{g}) = n\,k$, then (M^n, g) is isometric to $\mathbb{S}^n(1/\sqrt{k})$.

5. Let (P, g) be a Riemannian manifold, and $M \subset N \subset P$ be submanifolds of one another, with metrics induced by g. Prove or disprove (with a counter-example) the statements:

   a) If M is totally geodesic in N and N is totally geodesic in P, then M is totally geodesic in P;

   b) If M is minimal in N and N is minimal in P, then M is minimal in P;

c) If M is totally geodesic in N and N is minimal in P, then M is minimal in P;

d) If M is minimal in N and N is totally geodesic in P, then M is minimal in P.

a) True. If M is totally geodesic in N and N is totally geodesic in P, then geodesics in M are geodesics in N and geodesics in N are geodesics in P. Thus, geodesics in M are geodesics in P, so M is totally geodesic in P.

b) False. Let $P = \mathbb{R}^3$, N be a catenoid in $\mathbb{R}^3$, and M be the (unique) closed geodesic in the catenoid N. Then M is minimal (actually, totally geodesic) in N and N is minimal in P, but M is not minimal in P since it is not a straight line.

c) False. Same counter-example as the previous item.

d) True. If N is totally geodesic in P, then the Levi-Civita connection ∇^N of N agrees with the Levi-Civita connection ∇^P of P, i.e., for all $X, Y \in T_pN$, we have $\nabla^N_X Y = \nabla^P_X Y$. Fix an orthonormal basis of T_pP such that the first $\dim M$ vectors are an orthonormal basis of T_pM and the first $\dim N$ vectors are an orthonormal basis of T_pN. Since $\mathrm{I\!I}^P_M(X,Y) = \nabla^P_X Y - \nabla^M_X Y = \nabla^N_X Y - \nabla^M_X Y$, in this basis

$$\mathrm{I\!I}^P_M = \begin{pmatrix} \mathrm{I\!I}^N_M & 0 \\ 0 & 0 \end{pmatrix},$$

so the trace of $\mathrm{I\!I}^P_M$ is equal to the trace of $\mathrm{I\!I}^N_M$, hence zero, i.e., M is minimal in P.

X. (Will not be graded) In Problem 1, prove M^n is the boundary of a convex body in $\mathbb{R}^{n+1}$.

Given a unit vector $v \in \mathbb{R}^{n+1}$, consider the height function $f_v(p) = \langle p, v\rangle$, $p \in M^n$. Clearly, $\nabla f_v(p)$ is the orthogonal projection of v onto T_pM, so $p \in M$ is a critical point of f_v if and only if $v = \pm \vec{n}_p$. Since $M \ni p \mapsto \vec{n}_p \in \mathbb{S}^n$ is a diffeomorphism, it follows that f_v has exactly two critical points, say $p_v^\pm \in M$. At such critical points,

$$(\mathrm{Hess} f_v)(X,Y) = \langle \nabla_X \nabla f_v, Y\rangle = \pm\langle \nabla_X \vec{n}, Y\rangle = \mp\langle S_{\vec{n}} X, Y\rangle.$$

As explained in the solution to Problem 1, the eigenvalues κ_i of $S_{\vec{n}}$ satisfy $\kappa_i\kappa_j > 0$ for all $i \neq j$. Thus, either $\kappa_i > 0$ for all $1 \leq i \leq n$, or $\kappa_i < 0$ for all $1 \leq i \leq n$, so $(\mathrm{Hess} f_v)_{p_v^\pm}$ is either positive-definite or negative-definite. So each of the two critical points $p_v^\pm$ is either a local minimum or local maximum of f_v. On the other hand, by compactness of M, the function $f_v \colon M \to \mathbb{R}$ has a global minimum and a global maximum. Up to relabeling, let p_v^- be the global minimum and p_v^+ be the global maximum, so for $p \in M$,

$$f_v(p_v^-) \leq f_v(p) \leq f_v(p_v^+).$$

This means that $M \subset \mathbb{R}^{n+1}$ is contained in the slab $\mathcal{S}_v$ between two parallel hyperplanes in $\mathbb{R}^{n+1}$ with normal vector v, that are at a bounded distance from each other and each intersects M at a single point $p_v^\pm$. By the Jordan–Brouwer separation theorem, $\mathbb{R}^{n+1} \setminus M$ has two connected components, the (bounded) *interior* of M and the *exterior* of M. If x, y are in the interior of M, then the line segment $\overline{xy}$ joining them is entirely in the interior of the slab $\mathcal{S}_v$. In particular, $p_v^\pm \notin \overline{xy}$, since $p_v^\pm$ is in the boundary of the slab $\mathcal{S}_v$. Our choice of v, and hence of $p_v^\pm \in M$, was arbitrary, so it follows that no line segment joining two points in the interior of M intersects M. Therefore, the interior of M is convex, i.e., M is the boundary of a (strictly) convex body in $\mathbb{R}^{n+1}$.

REMARK: The above is known as Hadamard's convexity theorem, and it was proved (in dimension 3) in: J. Hadamard, *Sur certaines proprietés des trajectoires en dynamique*, J. Math. Pures Appl. 3 (1897) 331–387.