Lecture 6 2/14/2024 ← Happy Valentines Day!

Def: The torsion of a connection ∇ on TM is the $(1,2)$-tensor

$$T_X Y = \nabla_X Y - \nabla_Y X - [X,Y]$$

Def: The connection ∇ is compatible with a Riemannian metric g if $\nabla g \equiv 0$, i.e., $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, $\forall X,Y,Z \in \mathfrak{X}(M)$

(also say ∇ is a "metric connection")

Thm. Given a Riemannian metric g on M, there exists a unique torsion-free connection on TM compatible with g, given by the "Koszul formula":

$$g(\nabla_Y X, Z) = \frac{1}{2}\Big(X\,g(Y,Z) + Y\,g(Z,X) - Z\,g(X,Y) - g([X,Z],Y) - g([Y,Z],X) - g([X,Y],Z)\Big).$$

or, equivalently, whose Christoffel symbols are:

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{\ell} g^{k\ell}\left(\frac{\partial}{\partial x_i} g_{\ell j} + \frac{\partial}{\partial x_j} g_{i\ell} - \frac{\partial}{\partial x_\ell} g_{ij}\right)$$

where $(g^{k\ell})$ is the inverse matrix to $(g_{k\ell})$.

Def: This connection is called the Levi-Civita connection of the metric g.

Pf: Suppose such a connection ∇ exists, and compute:

- $X\,g(Y,Z) = g(\nabla_X Y, Z) + \underline{g(Y, \nabla_X Z)}$
- $Y\,g(Z,X) = \underline{g(\nabla_Y Z, X)} + g(Z, \nabla_Y X)$
- $Z\,g(X,Y) = \underline{g(\nabla_Z X, Y)} + \underline{g(X, \nabla_Z Y)}$

Note: Can replace the underlined terms with brackets, i.e., terms independent of ∇ if we subtract the last line from the sum of first two...

so $$X\,g(Y,Z) + Y\,g(Z,X) - Z\,g(X,Y) = \underline{g([X,Z],Y)} + \underline{g([Y,Z],X)} + g(\nabla_X Y + \nabla_Y X, Z).$$

Use $\nabla_X Y = \nabla_Y X + [X,Y]$ to replace last term with $g([X,Y],Z) + 2g(\nabla_Y X, Z)$.

Solving for $g(\nabla_Y X, Z)$, one obtains the Koszul formula. This proves uniqueness of ∇, and, for existence, simply define it by the Koszul formula.

To compute Christoffel symbols, set $Y = \frac{\partial}{\partial x_i}$, $X = \frac{\partial}{\partial x_j}$, $Z = \frac{\partial}{\partial x_\ell}$, so, as $[X,Y]=[X,Z]=[Y,Z]=0$,

$$g\left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_\ell}\right) = g(\nabla_Y X, Z) = \frac{1}{2}\left(X g(Y,Z) + Y g(Z,X) - Z g(X,Y)\right)$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial x_j} g_{i\ell} + \frac{\partial}{\partial x_i} g_{\ell j} - \frac{\partial}{\partial x_\ell} g_{ij}\right)$$

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k} \Rightarrow g\left(\nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_\ell}\right) = \sum_k \Gamma_{ij}^k \underbrace{g\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell}\right)}_{g_{k\ell}}$$

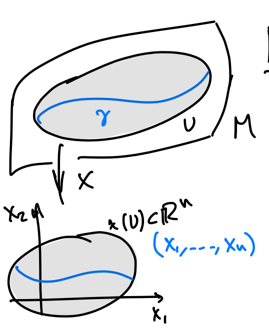
Careful: $\left\{\frac{\partial}{\partial x_i}\right\}$ need not be orthonormal...

$$\Rightarrow \sum_\ell g^{\ell m} \frac{1}{2}\left(\frac{\partial}{\partial x_j} g_{i\ell} + \frac{\partial}{\partial x_i} g_{\ell j} - \frac{\partial}{\partial x_\ell} g_{ij}\right) = \sum_{k,\ell} \Gamma_{ij}^k \underbrace{g_{k\ell}\, g^{\ell m}}_{\delta_{km}} = \sum_k \Gamma_{ij}^k \delta_{km} = \Gamma_{ij}^m$$

so $$\Gamma_{ij}^{\cancel{m}\,k} = \sum_\ell g^{\ell \cancel{m}\,k} \frac{1}{2}\left(\frac{\partial}{\partial x_j} g_{i\ell} + \frac{\partial}{\partial x_i} g_{\ell j} - \frac{\partial}{\partial x_\ell} g_{ij}\right).$$

Recall $g_{ij} = g_{ji}$ and $g^{\ell k} = g^{k\ell}$ because inverse of a symm. matrix is symmetric too. $\square$

__Def__: A curve $\gamma : (a,b) \to (M,g)$ in a Riem. mfld. is a geodesic if it is a geodesics for the Levi-Civita connection ∇ of g, i.e., $\nabla_{\dot\gamma}\dot\gamma = 0$.



__Note__: The geodesic equation in a chart $X = (x_1, \dots, x_n)$ is given by

$$\ddot{x}_k + \sum_{i,j} \dot{x}_i \dot{x}_j \Gamma_{ij}^k = 0, \quad k = 1, \dots, n$$

and Γ_{ij}^k for the Levi-Civita connection can be written as functions of g_{ij} and $\frac{\partial}{\partial x_\ell} g_{ij}$, so geodesics are determined by g.

Prop: If γ is a geodesic in (M,g), then its speed $g(\dot\gamma,\dot\gamma)^{1/2}$ is constant.

Pf: $\frac{d}{dt} g(\dot\gamma,\dot\gamma) \overset{\text{metric compatibility}}{=} 2g(\nabla_{\dot\gamma}\dot\gamma,\dot\gamma) \overset{\text{Geodesic equation}}{=} 0 \Rightarrow g(\dot\gamma,\dot\gamma)$ is constant along γ. $\square$

Def: A vector field $X \in \mathfrak{X}(M)$ is Killing on (M,g) if its flow $\phi_t : M \to M$ is a 1-par. subgroup of $\mathrm{Isom}(M,g)$.

Prop: If $X \in \mathfrak{X}(M)$ is a Killing field of (M,g), i.e., $\mathcal{L}_X g = 0$, then $g(X,\dot\gamma)$ is constant along any geodesic γ.

Pf: Recall $\mathcal{L}_X g = 0 \iff \nabla X$ is skew, i.e., $g(\nabla_Y X, Z) = -g(\nabla_Z X, Y)$

So $\frac{d}{dt} g(X,\dot\gamma) \overset{\text{metric compatibility}}{=} g(\nabla_{\dot\gamma} X,\dot\gamma) + g(X, \underbrace{\nabla_{\dot\gamma}\dot\gamma}_{=0}) = 0$ b/c ∇X is skew. $\square$

Careful: $|X|$ need not be constant along γ, so the angle between a Killing field and a geodesic need not be constant. Moreover, if X is Killing but $|X|$ is not constant, then $\frac{X}{|X|}$ need not be Killing; in general, $f\cdot X$ satisfies

$$\nabla_Y(fX) = Y(f)X + f\nabla_Y X, \text{ so } Y(f)\,g(X,Z) + f\cdot g(\nabla_Y X,Z) = -Z(f)g(X,Y) - f\cdot g(\nabla_Z X,Y)$$

need not be skew

$\iff Y(f)Z + Z(f)Y \equiv 0$, but generically Y and Z are lin. indep.!

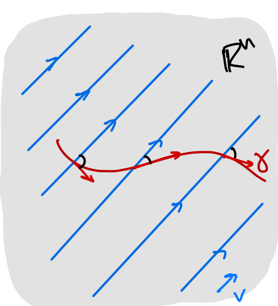
Ex: On $\left((a,b)\times S^1,\ dr^2 + f(r)^2 d\theta^2\right)$, the vector field $X = \frac{\partial}{\partial\theta}$ is Killing, but $|X| = f(r)$.

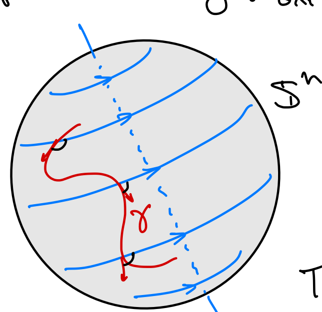
HW2: $g(X,\dot\gamma) = f^2\cdot\dot\theta$ is constant along $\gamma(t) = (r(t),\theta(t))$.

Can prove this in lots of ways, e.g., computing explicitly. (But I find this proof more elegant...)

Cor. Geodesics in $\mathbb{R}^n$ are straight lines; geodesics in S^n are great circles.

Pf: Every constant vector field in $\mathbb{R}^n$ is Killing, since if $v \in \mathbb{R}^n$, $\phi_t(p) = p + tv$ are isometries. In particular, the coordinate vector fields $\{\frac{\partial}{\partial x_i}\}$ are Killing, and form an orthonormal basis at all points. So given a geodesic $\gamma : (a,b) \to \mathbb{R}^n$, it follows that $g(\dot\gamma, \frac{\partial}{\partial x_i}) \equiv c_i$ is constant, thus $\dot\gamma \equiv \sum_i c_i \frac{\partial}{\partial x_i}\big|_{\gamma(t)}$ is constant, i.e., γ is a straight line.





Similarly, if $\gamma : (a,b) \to S^n$ is a geodesic, there are $(n-1)$ lin. indep. Killing vector fields (rotations!) $\{X_i\}$ s.t. $\dot\gamma(t_0)^\perp = \mathrm{span}\{X_i(\gamma(t_0))\}$, and a Killing field Y with $\dot\gamma(t_0) = Y(\gamma(t_0))$.

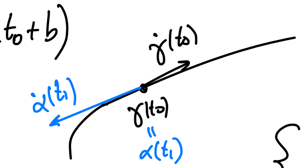
Thus, $g(\dot\gamma(t), X_i(\gamma(t))) \equiv 0$, and $\dot\gamma(t) = Y(\gamma(t))$, so γ is a flow line of the rotation field Y, i.e., a great circle. $\square$

← affine reparametrization.

Note: If $\gamma(t)$ is a geodesic, then so is $\alpha(t) := \gamma(at+b)$ for any $a \neq 0$, $b \in \mathbb{R}$.

Pf: $\dot\alpha(t) = a\,\dot\gamma(at+b)$ so $\nabla_{\dot\alpha}\dot\alpha = \nabla_{a\dot\gamma(at+b)} a\,\dot\gamma(at+b) = a^2 \nabla_{\dot\gamma(at+b)} \dot\gamma(at+b) = 0$ b/c $\nabla_{\dot\gamma}\dot\gamma = 0$.

Note: $\begin{cases} \gamma(t_0) = p \\ \dot\gamma(t_0) = v \end{cases} \overset{(*)}{\iff} \begin{cases} \alpha(t_1) = p \\ \dot\alpha(t_1) = a v \end{cases}$ $(t_1 = a t_0 + b)$



Initial conditions are the same, up to rescaling the initial velocity! Thus,

$$\left\{\begin{matrix}\text{Geometrically}\\ \text{distinct pointed}\\ \text{geodesics in } M\end{matrix}\right\} \cong P_{\mathbb{R}}(T_1 M)$$

projectivized unit tangent bundle. $\mathbb{RP}^{n-1} \to P_{\mathbb{R}}(T_1M) \to M$

Def: $T_1 M := \{(p,v) \in TM : g_p(v,v) = 1\}$ is the "unit tangent bundle" of (M,g). ← $(n-1)$-sphere bundle over (M,g).

Careful: If a reparametrization $f(t)$ is not affine, then $\alpha(t) = \gamma(f(t))$ need not be geod:

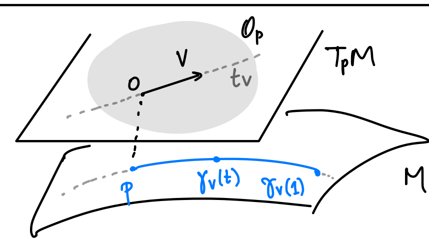
$\dot\alpha(t) = f'(t)\,\dot\gamma(f(t))$ so $\nabla_{\dot\alpha}\dot\alpha = \nabla_{f'\dot\gamma(f)} f'\dot\gamma(f) = f' \nabla_{\dot\gamma(f)}(f'\dot\gamma(f)) = f'\Big(f''\,\dot\gamma(f) + f' \underbrace{\nabla_{\dot\gamma(f)}\dot\gamma(f)}_{=0}\Big)$.

Using that $\mathrm{Isom}(M,g)$ acts transitively on T_1M if $M = \mathbb{R}^n$ or $M = S^n$

Alternative pf: Show that at least one straight line γ_0 in $\mathbb{R}^n$ and one great circle γ_0 in S^n are geodesics. Given any initial conditions (p,v), up to affinely reparam. γ_0, we have a geodesic with the prescribed initial conditions, so by uniqueness all geodesics are (possibly reparametrized) images of γ_0 via an isometry. $\square$

Def: The exponential map of (M,g) at $p \in M$ is

$$\exp_p : \mathcal{O}_p \subset T_pM \longrightarrow M$$
$$\exp_p(v) = \gamma_v(1)$$



where $\gamma_v(t)$ is the (unique) geodesic in (M,g) with $\begin{cases}\gamma_v(0) = p \\ \dot\gamma_v(0) = v\end{cases}$ and $\mathcal{O}_p \subset T_pM$ is the open subset of $v \in T_pM$ s.t. $\gamma_v(t)$ is defined at least up to $t = 1$.

By $(*)$ above, $\gamma_{sv}(t) = \gamma_v(st)$ provided $|t|, |s|$ are sufficiently small. Thus, $\forall v \in T_pM$ ($\cong T_0 T_pM$),

$$d(\exp_p)_0 v = \frac{d}{dt}\exp_p(tv)\Big|_{t=0} = \frac{d}{dt}\gamma_{tv}(1)\Big|_{t=0} = \frac{d}{dt}\gamma_v(t)\Big|_{t=0} = \dot\gamma_v(0) = v.$$

i.e., $d(\exp_p)_0 = \mathrm{id}$. Thus, by the Inverse Function Theorem, there exist open neighborhoods $U \ni 0$ in T_pM and $V \ni p$ in M s.t. $(\exp_p)|_U : U \to V$ is a diffeo.

This defines a local chart around $p \in M$, whose coord. are called "normal coordinates".

← identify $T_pM \cong \mathbb{R}^n$ by choosing a g-orthonormal basis.

Lecture 7 2/16/2024

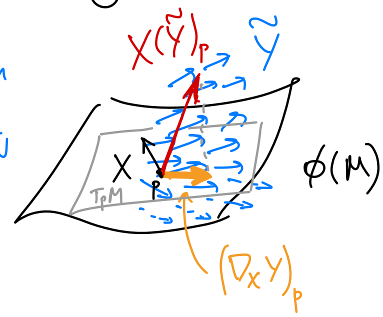
Recall: Levi-Civita connection of g is the unique torsion-free connection compatible with g.

EX: Let $\phi: M \hookrightarrow \mathbb{R}^N$ be an isom. embedding, i.e., $g = \phi^*(g_{\text{Eucl}})$. Then

$$(\nabla_X Y)_p := \text{proj}_{T_pM}\left(X(\tilde{Y})_p\right)$$

locally extend Y to a vector field on $U \subset \mathbb{R}^N$, then use connection from $\mathbb{R}^N$

orthogonal projection to $T_pM \subset \mathbb{R}^N$.



is torsion-free and compatible with g, hence it is the Levi-Civita connection on (M,g).

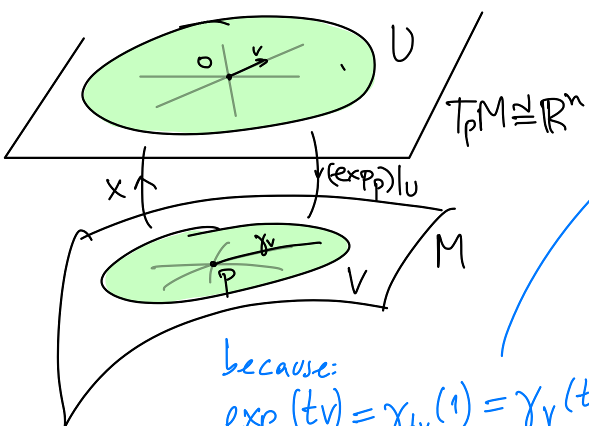
Recall: $\exp_p: \mathcal{O}_p \subset T_pM \to M$ satisfies $d(\exp_p)_0 = id$, hence

$\exp_p(v) = \gamma_v(1)$

Inverse Fund. Thm.

$\exists\, U \ni 0$ in T_pM and $\exists\, V \ni p$ in M s.t. $(\exp_p)|_U : U \to V$ is a diffeo

Properties of Normal Coordinates. $x = (x_1, \dots, x_n): V_{\subset M} \longrightarrow U_{\subset T_pM}$ s.t. $x^{-1} = (\exp_p)|_U$



because: $\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$

- $x(\gamma_v(t)) = tv \quad \forall v \in T_pM, \quad |t|$ small
- $x(p) = 0$
- $g_{ij}(p) = \delta_{ij}, \quad \left(\frac{\partial}{\partial x_k} g_{ij}\right)(p) = 0$
- $\Gamma_{ij}^k(p) = 0$

All of these follow from the above.

Questions of "naturality":

Levi-Civita connections

$$\boxed{\textbf{Prop: } \text{If } \phi: (M,g) \to (N,h) \text{ is an isometry, i.e., } g = \phi^* h, \text{ then } \nabla^g = \phi^* \nabla^h.}$$

Pf: $\phi^* \nabla^h$ is torsion-free and compatible with g, hence equal to ∇^g by uniqueness of LC connection. $\square$

Checking this is a good exercise, see e.g. [Lee] Prop 5.8, 5.9 for solution.

Cor: If $\gamma:(a,b)\to(M,g)$ is a geodesic, and $\phi:(M,g)\to(N,h)$ an isometry, then $\phi\circ\gamma:(a,b)\to(N,h)$ is a geodesic.

Pf. Let $\alpha=\phi\circ\gamma$, so $\dot\alpha(t)=d\phi_{\gamma(t)}(\dot\gamma(t))$ and compute:

$$\nabla^h_{\dot\alpha(t)}\dot\alpha(t)=\nabla^h_{d\phi_{\gamma(t)}\dot\gamma(t)}\,d\phi_{\gamma(t)}(\dot\gamma(t))=\left(\phi^*\nabla^h\right)_{\dot\gamma(t)}\dot\gamma(t)=\nabla^g_{\dot\gamma(t)}\dot\gamma(t)=0.$$

$\square$

Cor: If $\phi:(M,g)\to(N,h)$ is an isometry, then $\phi(\exp^g_p v)=\exp^h_{\phi(p)}(d\phi_p v)$, i.e., the following diagram commutes:

$$\begin{array}{ccc} T_pM & \xrightarrow{d\phi_p} & T_{\phi(p)}N \\ \exp^g_p\downarrow & & \downarrow\exp^h_{\phi(p)} \\ M & \xrightarrow{\phi} & N \end{array}$$

Pf: By definition, $\phi\left(\exp^g_p(v)\right)=\phi\left(\gamma^g_v(1)\right)$ and $\exp^h_{\phi(p)}\left(d\phi_p v\right)=\gamma^h_{d\phi_p v}(1)$

g-geod. on M s.t. $\begin{cases}\gamma_v(0)=p\\ \dot\gamma_v(0)=v\end{cases}$

h-geod on N s.t. $\begin{cases}\gamma_{d\phi_p v}(0)=\phi(p)\\ \dot\gamma_{d\phi_p v}(0)=d\phi_p v\end{cases}$

By Prop., $\phi\circ\gamma^g_v$ is a geod in (N,h), with same initial conditions as $\gamma^h_{d\phi_p v}$, so $\phi\circ\gamma^g_v=\gamma^h_{d\phi_p v}$. $\square$

Cor: If $\phi,\psi:(M,g)\to(N,h)$ are local isometries and $\exists p\in M$ such that $\begin{cases}\phi(p)=\psi(p)\\ d\phi_p=d\psi_p\end{cases}$, then $\phi\equiv\psi$ on the connected component of $p\in M$.

Pf: Let $U\ni p$ be a neighborhood of $p\in M$ and let $\delta=\phi\circ\psi^{-1}$, so $\delta|_U:U\to\delta(U)$ is an isometry. By Prop., since $\begin{cases}\delta(p)=p\\ d\delta_p=id\end{cases}$, we have:

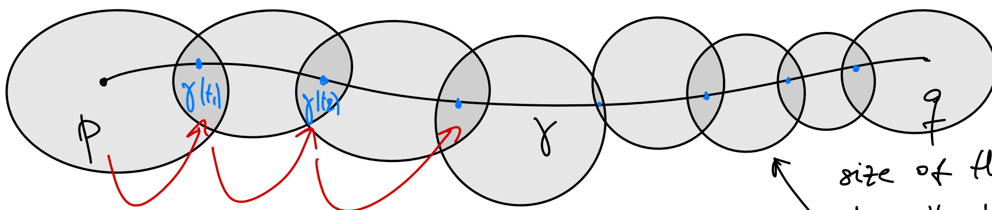
$$\delta\left(\exp_p v\right)=\exp_{\delta(p)} d\delta(p)v=\exp_p v,\qquad \forall v\in\mathcal{O}_p\subset T_pM$$

$\leftarrow$ domain of $\exp_p$.

So $\delta(x)=x$ for all $x \in \underbrace{U \cap (\exp_p \mathcal{O}_p)}$; i.e., $\phi \equiv \psi$ near $p\in M$.

still a neighborhood of $p \in M$

Propagate this to the connected component of $p \in M$ "as usual": let $\gamma:[0,1]\to M$ be a curve with $\gamma(0)=p$, $\gamma(1)=q$ and note $\exists\, \varepsilon>0$ s.t. $B_\varepsilon(0)\subset \mathcal{O}_{\gamma(t)} \subset T_{\gamma(t)}M$ for all $t\in[0,1]$, by continuity (of the "injectivity radius"). Then apply argument above at $\gamma(t_i)$, where $0=t_0<t_1<\dots<t_k=1$ is a sufficiently fine partition so that $\gamma(t_{i+1})\in \exp_{\gamma(t_i)} \mathcal{O}_{\gamma(t_i)}$ to get from p to q.



"Geometric induction"

If hypotheses hold at $\gamma(t_i)$ get conclusion to hold at $\gamma(t_{i+1})$, which is also the hypotheses there...

size of the neighborhoods doesn't shrink to zero by compactness of $\gamma([0,1])$! This will be justified better later, using "uniformly normal" coordinates.

$\square$

Cor: $\mathrm{Isom}(\mathbb{R}^n) = O(n)\ltimes \mathbb{R}^n = \{x\mapsto Ax+b,\ A\in O(n),\ b\in\mathbb{R}^n\}$.

Pf: Clearly $x\mapsto Ax+b$, $A\in O(n)$, $b\in\mathbb{R}^n$ are isometries of $(\mathbb{R}^n, g_{\text{Eucl}})$.

Conversely, if $\phi:\mathbb{R}^n\to\mathbb{R}^n$ is an isom., let $\psi(x)=\phi(x)-\phi(0)$ and note that $\psi:\mathbb{R}^n\to\mathbb{R}^n$ is also an isometry, with $\psi(0)=0$. Then, $d\psi_0: T_0\mathbb{R}^n\to T_0\mathbb{R}^n$ is a (linear) isometry of $T_0\mathbb{R}^n\cong\mathbb{R}^n$. Since the isometries ψ and $d\psi_0$ satisfy $\begin{cases}\psi(0)=0=d\psi_0(0)\\ d\psi_0 = d(d\psi_0)_0\end{cases}$, and $\mathbb{R}^n$ is connected,

it follows that $\psi = d\psi_0$ is linear, hence acts as an element of $O(n)$.

Thus $\psi(x)=Ax$, $A\in O(n)$ and setting $\underset{\mathbb{R}^n}{b=\phi(0)}$, we have $\phi(x)=Ax+b$

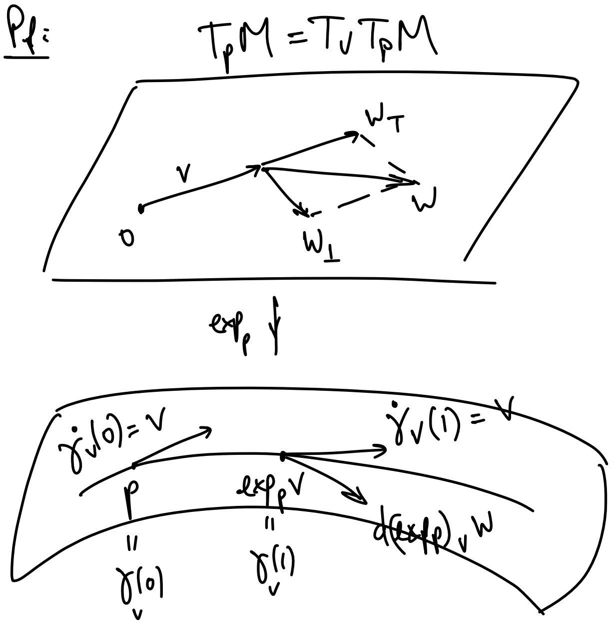
$\square$

Gauss Lemma: $\exp_p$ is a radial isometry, i.e.,

$$\left\langle d(\exp_p)_v v,\ d(\exp_p)_v w\right\rangle = \langle v, w\rangle, \quad \forall v, w \in T_pM = T_vT_pM$$

Here we use $\langle \cdot,\cdot\rangle$ instead of g to simplify notation...

Pf:



Write $w = w_T + w_\perp$, where $\begin{cases} w_T = \alpha v. \\ \langle w_\perp, v\rangle = 0 \end{cases}$

Clearly,

$$d(\exp_p)_v\, v = \frac{d}{dt}(\exp_p)((t+1)v)\Big|_{t=0}$$

$$= \frac{d}{dt}(\exp_p)(tv)\Big|_{t=1}$$

$$= \frac{d}{dt}\gamma_v(t)\Big|_{t=1} = \dot\gamma_v(1) = \underbrace{P_p^{\gamma_v(1)}(v)}.$$

parallel transport of $v \in T_pM$ along γ_v to $\gamma_v(1)$.

$P_p^{\gamma_v(1)} : T_pM \to T_{\gamma_v(1)}M$

Thus

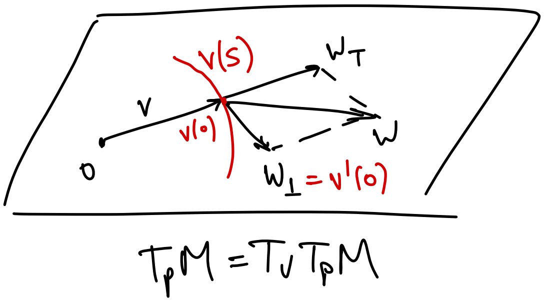
$$\langle d(\exp_p)_v v, d(\exp_p)_v w\rangle = \langle d(\exp_p)_v v, d(\exp_p)_v(\alpha v)\rangle + \langle d(\exp_p)_v v, d(\exp_p)_v w_\perp\rangle$$

$$= \alpha\langle P_p^{\gamma_v(1)} v, P_p^{\gamma_v(1)} v\rangle + \langle d(\exp_p)_v v, d(\exp_p)_v w_\perp\rangle$$

$$= \langle v, \underbrace{\alpha v}_{w_T}\rangle + \langle d(\exp_p)_v v, d(\exp_p)_v w_\perp\rangle$$

$$= \langle v, w\rangle + \langle d(\exp_p)_v v, d(\exp_p)_v w_\perp\rangle$$

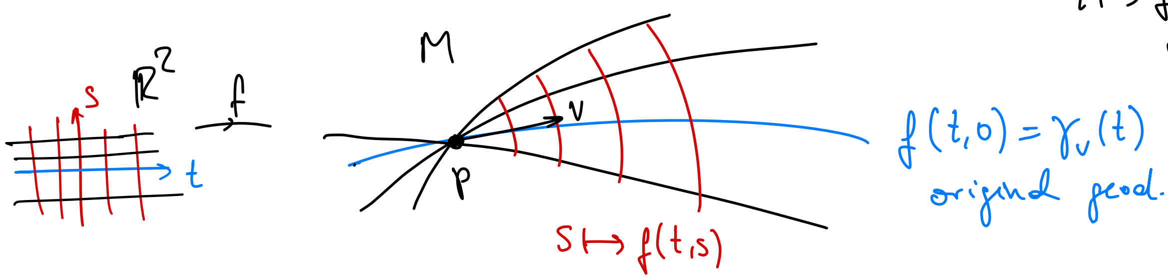
So we must show $\langle d(\exp_p)_v v,\ d(\exp_p)_v w_\perp \rangle = 0$.



Let $v(s) = (\cos s)v + (\sin s)w_\perp$ so $\begin{cases} v(0) = v \\ v'(0) = w_\perp \\ \|v(s)\| = \text{const.} \end{cases}$

and $f(t,s) = \exp_p(t v(s)) = \gamma_{v(s)}(t)$

$t \mapsto f(t,s)$ are geodesics $\gamma_{v(s)}(t)$



$$\left.\begin{array}{l} d(\exp_p)_v v = \dfrac{\partial}{\partial t} \exp_p(t v(s))\Big|_{\substack{t=1\\ s=0}} = \dfrac{\partial f}{\partial t}(1,0) \\[2ex] d(\exp_p)_v w_\perp = \dfrac{\partial}{\partial s} \exp_p(t v(s))\Big|_{\substack{t=1\\ s=0}} = \dfrac{\partial f}{\partial s}(1,0) \end{array}\right\} \Rightarrow \langle d(\exp_p)_v v,\ d(\exp_p)_v w_\perp \rangle = \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle (1,0).$$

Compute:

$$\frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle = \langle \underbrace{\nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial t}}_{=0}, \frac{\partial f}{\partial s} \rangle + \left\langle \frac{\partial f}{\partial t}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial s} \right\rangle = \left\langle \frac{\partial f}{\partial t}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial f}{\partial t} \right\rangle$$

metric compatibility of ∇

b/c $t \mapsto f(t,s) = \gamma_{v(s)}(t)$ are geodesics.

$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = 0$

and

$$= \frac{1}{2} \frac{\partial}{\partial s} \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle = 0,$$

b/c $\left\| \frac{\partial f}{\partial t}(t) \right\| = \|\dot\gamma_{v(s)}(t)\| = \|\dot\gamma_{v(s)}(0)\| = \|v(s)\| = \text{const.}$

Therefore $t \mapsto \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle (t,0)$ is constant, and, computing at $t=0$:

$$\frac{\partial f}{\partial s}(t,0) = \frac{\partial}{\partial s}(\exp_p)(t v(s))\Big|_{s=0} = d(\exp_p)(t \underbrace{v(0)}_{v})(t \underbrace{v'(0)}_{w_\perp}) = d(\exp_p)_{tv}\, t w_\perp$$

$$\lim_{t\to 0} \frac{\partial f}{\partial s}(t,0) = \lim_{t\to 0} d(\exp_p)_{tv}\, t w_\perp = 0; \quad \text{so } \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle (1,0) = 0. \ \square$$

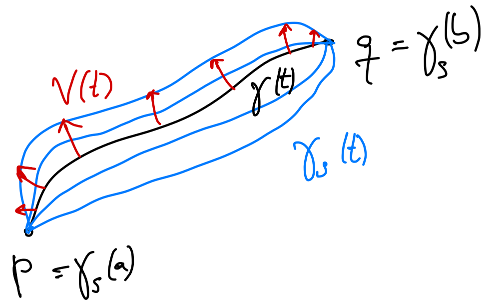
Lecture 8 (by Dan Lee) 2/23/2024

Def. A curve γ from p to q is a minimizing curve (or minimal) if $\mathrm{dist}(p,q) = L_g(\gamma)$; i.e. if it realizes the inf in $\mathrm{dist}(p,q)$.

Prop: A unit speed minimizing curve is a geodesic.

Pf. (First variation of length). Let $\gamma_s(t)$, $|s|<\varepsilon$, be a smooth family of curves s.t. $\gamma = \gamma_0$ is minimizing from $p=\gamma(a)$ to $q=\gamma(b)$, and $\gamma_s(a) = p$, $\gamma_s(b) = q$. Then, letting $V = \frac{d}{ds}\gamma_s\big|_{s=0}$, we compute

$$\frac{d}{ds} L_g(\gamma_s)\Big|_{s=0} = \int_a^b \frac{d}{ds}\, g(\dot\gamma_s, \dot\gamma_s)^{1/2}\Big|_{s=0}\, dt$$



$$\overset{*}{=} \frac{1}{2}\int_a^b \frac{1}{|\dot\gamma_s|}\left(g\left(\frac{D}{dt}\dot\gamma_s, \dot\gamma_s\right) + g\left(\dot\gamma_s, \frac{D}{dt}\dot\gamma_s\right)\right)\Big|_{s=0} dt$$

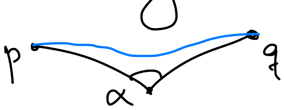
$$= \int_a^b \frac{1}{\underbrace{\cancel{|\dot\gamma|}}_{1 \text{ b/c } \gamma \text{ is unit speed}}}\, g\left(\frac{DV}{dt}, \dot\gamma\right) dt \overset{\text{parts}}{=} \underbrace{g(V, \dot\gamma)\Big|_a^b}_{=0 \text{ b/c } V(a)=0,\ V(b)=0} - \int_a^b g\Big(V, \underbrace{\frac{D\dot\gamma}{dt}}_{= D_{\dot\gamma}\dot\gamma}\Big)\, dt.$$

If the above vanishes for all smooth families of curves with fixed endpoints at p and q, then $\frac{D\dot\gamma}{dt} = \nabla_{\dot\gamma}\dot\gamma = 0$, i.e., γ is a geodesic. $\square$

(*) **Lemma.** $\frac{DV}{dt} = \frac{D}{dt}\frac{d}{ds}\gamma_s(t)\Big|_{s=0} = \frac{D}{ds}\frac{d}{dt}\gamma_s(t)\Big|_{s=0} = \frac{D}{ds}\dot\gamma_s\Big|_{s=0}$.

Pf. Compute both sides of $=$ in local coordinates, use that $\Gamma^k_{ij} = \Gamma^k_{ji}$. $\square$

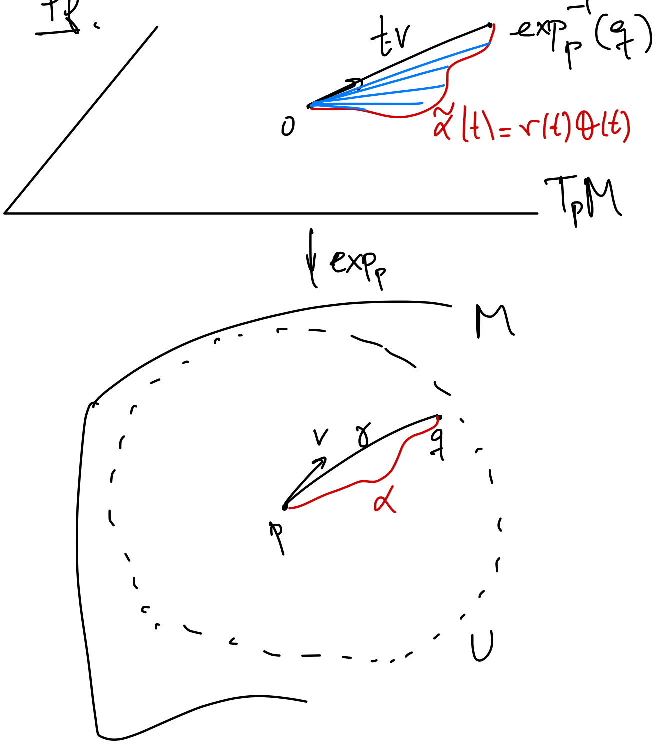
Note: The above shows that smooth curves with fixed endpoints that minimize, or, more generally, are stationary/critical points for L_g are geodesics. Smoothness can be assumed by a "cutting corners" argument:



If $\alpha < \pi$, then moving inwards, i.e., replacing with blue curve, would decrease distances, see, e.g. [Lee], p.156

Prop. Up to reparametrization, radial geodesics $t \mapsto \exp_p(tv)$, where $|t| < \varepsilon$ is small enough so that curve stays in a normal neighbhd. of p, are the only minimizing geodesics from p to $\exp_p(tv)$.

Pf.



$\gamma : [0, t_*] \to U, \quad \gamma(t) = \exp_p tv$

$\alpha : [a, b] \to U, \quad \alpha(a) = p, \quad \alpha(b) = \exp_p(t_* v)$

has the same endpoints as γ.

Let $\alpha(t)$ be another curve joining p to $q = \exp_p(t_* v)$ in U, and write in T_pM

$$\tilde{\alpha}(t) := \exp_p^{-1}(\alpha(t)) = r(t)\,\theta(t)$$

where r is a (positive) real-valued function and $\theta(t) \in T_pM$ is a unit vector for each t, i.e. $\|\theta(t)\| \equiv 1$. Then,

$$\alpha(t) = \exp_p \tilde{\alpha}(t) = \exp_p(r(t)\theta(t))$$

$$\dot{\alpha}(t) = d(\exp_p)_{\tilde{\alpha}(t)}\, \tilde{\alpha}'(t)$$

$$= d(\exp_p)_{\tilde{\alpha}(t)}\big(r'(t)\theta(t) + r(t)\theta'(t)\big).$$

$$= r'(t)\, d(\exp_p)_{\tilde{\alpha}(t)}\, \theta(t) + r(t)\, d(\exp_p)_{\tilde{\alpha}(t)}\, \theta'(t)$$

$$= \frac{r'(t)}{r(t)}\, d(\exp_p)_{\tilde{\alpha}(t)}\, \tilde{\alpha}(t) + r(t)\, d(\exp_p)_{\tilde{\alpha}(t)}\, \theta'(t)$$

By the Gauss Lemma, $\langle d(\exp_p)_{\tilde{\alpha}(t)} \tilde{\alpha}(t), d(\exp_p)_{\tilde{\alpha}(t)} w \rangle = \langle \tilde{\alpha}(t), w \rangle$ for any $w \in T_pM$, so

$$\|\dot{\alpha}(t)\|^2 = \frac{r'(t)^2}{r(t)^2} \|d(\exp_p)_{\tilde{\alpha}(t)}\, \tilde{\alpha}(t)\|^2 + r(t)^2 \|d(\exp_p)_{\tilde{\alpha}(t)}\, \theta'(t)\|^2$$

$$+ 2 r'(t) \langle d(\exp_p)_{\tilde{\alpha}(t)} \tilde{\alpha}(t), d(\exp_p)_{\tilde{\alpha}(t)}\, \theta'(t) \rangle$$

Gauss Lemma

$$\geqslant r'(t)^2 \|d(\exp_p)_{\tilde{\alpha}(t)}\, \theta(t)\|^2 + 2 r'(t) \underbrace{\langle \tilde{\alpha}(t), \theta'(t) \rangle}_{= r(t)\langle \theta(t), \theta'(t)\rangle = 0}$$

b/c $\|\theta(t)\|^2 \equiv 1$.

$$\overset{\downarrow}{=} r'(t)^2 \|\theta(t)\|^2 = r'(t)^2.$$

Thus, as $\tilde{\alpha} : [a,b] \longrightarrow T_pM$ joins $r(a) = 0$ to $r(b) = t_* v$, we have

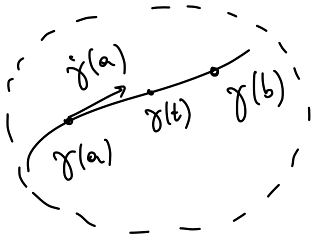
$$L_g(\alpha) = \int_a^b \|\dot{\alpha}(t)\| \, dt \geq \int_a^b r'(t)\, dt = r(b) - r(a) = \|t_* v\| = L_g(\gamma)$$

i.e., the length of α is at least as large as the length of the radial geodesic $\gamma : [0, t_*] \to U$, $\gamma(t) = \exp_p(tv)$. $\square$

<u>Corollary</u>. Geodesic balls are metric balls, i.e., if $\exp_p$ is well-defined on $B_r(0) \subset T_pM$, $B_r(0) = \{v \in T_pM : \|v\| < r\}$,
then $B_r(p) = \{x \in M : \operatorname{dist}(p,x) < r\} = \exp_p(B_r(0))$.

<u>Corollary</u>. Geodesics are locally distance-minimizing.

<u>Pf.</u> Let $\gamma(t)$ be a geodesic, and a, b s.t. $\gamma(a)$ and $\gamma(b)$ are sufficiently close, in the sense that $\gamma(t)$ is in a normal neighborhood of $\gamma(a)$ for all $t \in [a,b]$. Then, γ agrees with the (only) radial geodesic from $\gamma(a)$ to $\gamma(b)$, up to reparametrization, so by the Prop. above, γ is distance-minimizing from $\gamma(a)$ to $\gamma(b)$.



<u>Prop.</u> For all $p \in M$, there exists $r > 0$ sufficiently small so that $B_r(p)$ is <u>convex</u>, i.e., $\forall x, y \in B_r(p)$, there is a unique minimizing geodesic from x to y, and this geodesic is entirely contained in $B_r(p)$.

<u>Lecture 9</u> 3/1/2024

<u>Energy v. Length</u>

$E_g(\gamma) = \frac{1}{2} \int_a^b \|\dot{\gamma}(t)\|^2 \, dt$ is <u>not</u> invariant under reparametrizations (fixed gauge)

$L_g(\gamma) = \int_a^b \|\dot{\gamma}(t)\| \, dt$ is invariant under reparametrizations (gauge-invariant).

By Cauchy-Schwartz, $L_g(\gamma)^2 = \left(\int_a^b \|\dot{\gamma}\| \, dt \right)^2 \leq \int_a^b \|\dot{\gamma}\|^2 dt \cdot \int_a^b 1 \, dt = 2(b-a) E_g(\gamma)$;

and equality holds iff $\|\dot{\gamma}\| = \text{const}$, i.e., iff γ has constant speed.

Prop. Let $p,q\in M$ and $\gamma:[a,b]\to M$ a curve joining p to q. Then γ is a minimizer for E_g iff it is a minimizer for L_g and has constant speed.

Pf. If γ has constant speed and minimizes L_g, then any other curve $\alpha:[a,b]\to M$ with $\alpha(a)=p$, $\alpha(b)=q$ has $L_g(\alpha)\geq L_g(\gamma)$; so $E_g(\alpha)\geq \frac{1}{2(b-a)}L_g(\alpha)^2 \geq \frac{1}{2(b-a)}L_g(\gamma)^2 \underset{\uparrow\, \|\dot\gamma\|=\text{const.}}{=} E_g(\gamma)$, i.e., α minimizes E_g. Converse will follow from first variation of energy (below). $\square$

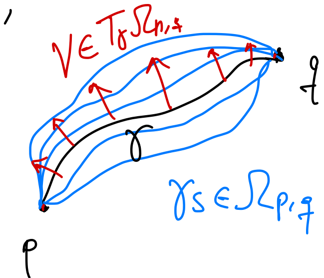
Analytically, E_g is easier to handle than L_g. We can consider the Hilbert mfld $W^{1,2}([a,b],M)$ of paths in M, whose tangent space at γ is

$$T_\gamma W^{1,2}([a,b],M) \cong W^{1,2}([a,b],\gamma^*TM) = \left\{V:[a,b]\to TM,\ W^{1,2}\text{-vector field along } \gamma\right\}$$

and submanifolds, such as, given fixed endpoints $p,q\in M$,

$$\Omega_{p,q} = \left\{\gamma\in W^{1,2}([a,b],M) : \gamma(a)=p,\ \gamma(b)=q\right\}$$

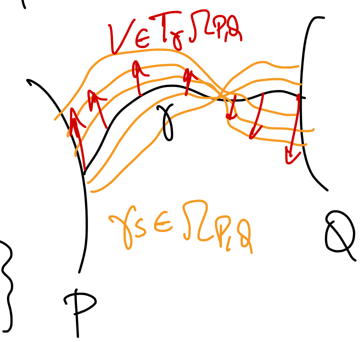
$$T_\gamma\Omega_{p,q} = \left\{V\in T_\gamma W^{1,2}([a,b],M) : V(a)=0,\ V(b)=0\right\}$$



or, given submanifolds $P,Q\subset M$,

$$\Omega_{P,Q} = \left\{\gamma\in W^{1,2}([a,b],M) : \gamma(a)\in P,\ \gamma(b)\in Q\right\}$$

$$T_\gamma\Omega_{P,Q} = \left\{V\in T_\gamma W^{1,2}([a,b],M) : V(a)\in T_{\gamma(a)}P,\ V(b)\in T_{\gamma(b)}Q\right\}$$



or $\Omega_{\text{closed}} = \left\{\gamma\in W^{1,2}([a,b],M) : \gamma(a)=\gamma(b)\right\}$, etc.

Still an open problem to establish existence of infinitely many geometrically distinct closed geodesics on all closed Riemannian manifolds!

1st variation of Energy.

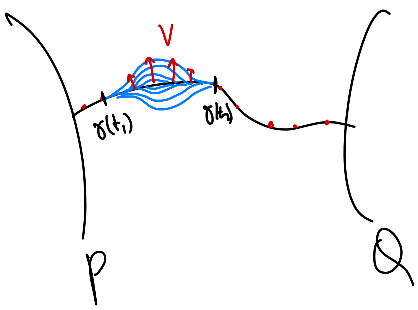
Let $\gamma_s\in W^{1,2}([a,b],M)$, $|s|<\varepsilon$, and set $V=\frac{d}{ds}\gamma_s\big|_{s=0}$. Note that $\frac{DV}{dt}=\frac{d}{ds}\dot\gamma_s\big|_{s=0}$ by the Lemma of previous Lecture.

$$dE_g(\gamma)V = \frac{d}{ds}E_g(\gamma_s)\Big|_{s=0} = \frac{1}{2}\int_a^b \frac{d}{ds}g(\dot\gamma_s,\dot\gamma_s)\Big|_{s=0}dt = \int_a^b g\left(\frac{DV}{dt},\dot\gamma\right)dt \overset{\substack{\text{Int. by}\\ \text{parts}}}{=} g(V,\dot\gamma)\Big|_a^b - \int_a^b g\left(V,\frac{D\dot\gamma}{dt}\right)dt.$$

- Thus, if $\gamma = \gamma_0$ is a critical point of $E_g : \Omega_{p,q} \to \mathbb{R}$, then $dE_g(\gamma)V = 0$ for all $V \in T_\gamma \Omega_{p,q}$, so it follows from the Fundamental Lemma of Calculus of Variations that $\frac{D\dot\gamma}{dt} = 0$, i.e., $\nabla_{\dot\gamma}\dot\gamma = 0$, i.e., γ is a geodesic curve (hence constant speed) joining p to q.

- Similarly, if γ is a critical point of $E_g : \Omega_{P,Q} \to \mathbb{R}$, then $dE_g(\gamma)V = 0$ for all $V \in T_\gamma \Omega_{P,Q}$ so γ is a geodesic joining P to Q and meeting them <u>orthogonally</u>.

<u>Note</u>: First, use variational fields supported in the interior of $[a,b]$ to see that γ is a geodesic, i.e., $\nabla_{\dot\gamma}\dot\gamma = 0$:

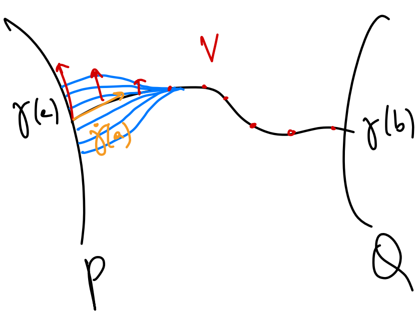


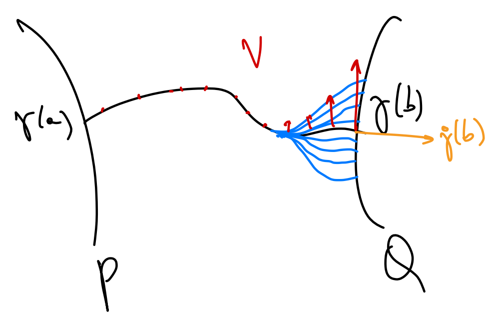
$$dE_g(\gamma)V = 0, \quad \forall V \in C_c^\infty([a,b], \gamma^*TM)$$

$$\iff \int_a^b g\left(V, \frac{D\dot\gamma}{dt}\right) dt = 0, \quad \forall V \in C_c^\infty([a,b], \gamma^*TM)$$

$$\iff \frac{D\dot\gamma}{dt} = 0 \text{ on } (a,b).$$

Then, to see that $g(V(t), \dot\gamma(t)) = 0$ for $t = a$ and $t = b$ individually, use variational fields supported in a neighborhood of $t = a$ and $t = b$.





$$dE_g(\gamma)V = 0, \quad \forall V \in C^\infty([a,\varepsilon], \gamma^*TM)$$

$$\iff g(V(a), \dot\gamma(a)) = 0, \quad \forall V \in C^\infty([a,\varepsilon], \gamma^*TM)$$

$$\iff \dot\gamma(a) \in T_{\gamma(a)}P^\perp.$$

similarly at $\gamma(b)$, get $\dot\gamma(b) \in T_{\gamma(b)}Q^\perp$.

2^{nd} Variation of Energy

Lemma. Given a vector field $W(s,t)$ along $\gamma_s(t)$, we have

$$\frac{D}{ds}\frac{D}{dt}W - \frac{D}{dt}\frac{D}{ds}W = R\left(\frac{d}{ds}\gamma_s, \frac{d}{dt}\gamma_s\right)W,$$

where R is the $(1,3)$-tensor given by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

"Curvature tensor"

Pf. Compute in coordinates, using $X = (\gamma_s)_* \frac{\partial}{\partial s}$, $Y = (\gamma_s)_* \frac{\partial}{\partial t}$ so that $[X,Y]=0$ and $\frac{D}{ds} = \nabla_X$, $\frac{D}{dt} = \nabla_Y$, so result follows. $\square$

Suppose $\gamma = \gamma_0$ is a geodesic. Then,

$$d^2E_g(\gamma)(V,V) = \frac{d^2}{ds^2}E_g(\gamma_s)\Big|_{s=0} = \int_a^b \frac{d}{ds}\, g\left(\frac{DV}{dt}, \dot\gamma_s\right)\Big|_{s=0} dt$$

$$= \int_a^b g\left(\frac{D}{ds}\frac{D}{dt}V, \dot\gamma\right) + g\left(\frac{DV}{dt}, \frac{DV}{dt}\right) dt$$

$$\overset{\text{Lemma}}{=} \int_a^b g\left(\frac{DV}{dt}, \frac{DV}{dt}\right) + g\left(\frac{D}{dt}\frac{D}{ds}V, \dot\gamma\right) + g\left(R(V,\dot\gamma)V, \dot\gamma\right) dt$$

$$\overset{\text{Int. by parts}}{=} g\left(\frac{DV}{ds}, \dot\gamma\right)\Big|_a^b - \int_a^b g\Big(\frac{DV}{ds}, \underbrace{\cancel{\frac{D\dot\gamma}{dt}}}_{\to 0}\Big)$$

b/c γ is a geodesic.

$$+ \int_a^b g\left(\frac{DV}{dt}, \frac{DV}{dt}\right) + g\left(R(V,\dot\gamma)V, \dot\gamma\right) dt$$

$$\overset{\substack{\text{Int. by parts} \\ \text{+ symmetry of } R}}{=} g\left(\frac{DV}{ds}, \dot\gamma\right)\Big|_a^b + g\left(\frac{DV}{dt}, V\right)\Big|_a^b - \int_a^b g\Big(\underbrace{\frac{D^2V}{dt^2} + R(V,\dot\gamma)\dot\gamma}_{\text{"Jacobi operator"}}, V\Big) dt.$$

often write V''

Note: Using polarization, can easily compute $d^2E_g(\gamma)(V,W)$ for any V, W.

Def. A vector field $J:[a,b]\to TM$ along a geodesic $\gamma:[a,b]\to M$ is a Jacobi field if it solves the Jacobi equation $J''+R(J,\dot\gamma)\dot\gamma=0$.

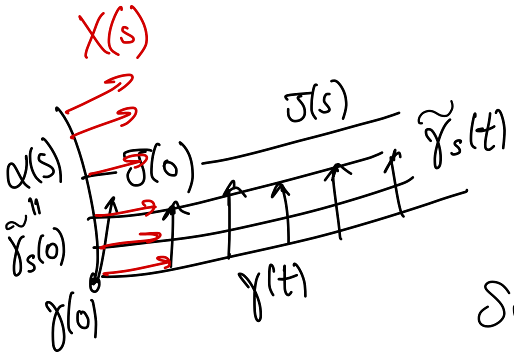
Prop. The variational field $J(t)=\frac{d}{ds}\gamma_s\big|_{s=0}$ is a Jacobi field along the geodesic $\gamma=\gamma_0$ iff the curves $t\longmapsto\gamma_s(t)$ are geodesics for $|s|<\varepsilon$.

Proof. If $J(t)=\frac{d}{ds}\gamma_s(t)\big|_{s=0}$ where $\gamma_s(t)$ is a variation by geodesics, then

$$J''(t)=\frac{D}{dt}\frac{D}{dt}\frac{d}{ds}\gamma_s(t)=\frac{D}{dt}\frac{D}{ds}\underbrace{\frac{d}{dt}\gamma_s(t)}_{\dot\gamma_s(t)}=\frac{D}{ds}\underbrace{\frac{D}{dt}\dot\gamma_s(t)}_{=0 \text{ b/c } \gamma_s(t) \text{ is geod.}}-R(J,\dot\gamma)\dot\gamma$$

so J is a Jacobi field. Conversely, if J is a Jacobi field, then let

$$\alpha(s)=\exp_{\gamma(0)}sJ(0) \quad \text{and let } X(s) \text{ be a vector field}$$



along $\alpha(s)$ with $X(0)=\dot\gamma(0)$, $X'(0)=J'(0)$.

Set $\tilde\gamma_s(t)=\exp_{\alpha(s)}tX(s)$.

Since $t\mapsto\tilde\gamma_s(t)$ are geodesics, by the above, the vector field $\tilde J(t)=\frac{d}{ds}\tilde\gamma_s(t)\big|_{s=0}$ satisfies $\tilde J''+R(\tilde J,\tilde\gamma')\tilde\gamma'=0$.

Moreover, $\tilde J(0)=\frac{d}{ds}\tilde\gamma_s(0)\big|_{s=0}=\alpha'(0)=J(0)$ and

$$\tilde J'(0)=\frac{D}{dt}\frac{d}{ds}\tilde\gamma_s(t)\Big|_{\substack{s=0\\t=0}}=\frac{D}{ds}\frac{d}{dt}\tilde\gamma_s(t)\Big|_{\substack{s=0\\t=0}}=\frac{D}{ds}X(s)\Big|_{s=0}=X'(0)=J'(0).$$

So $J(t)=\tilde J(t)=\frac{d}{ds}\tilde\gamma_s(t)\big|_{s=0}$ for all t, by uniqueness of sol. to ODE w/ same initial conditions; hence J is the variational field of the family of geodesics $\tilde\gamma_s(t)$. $\square$

See HW3.

Rmk: The Jacobi field along $\gamma_v(t) = \exp_p tv$ with $J(0)=0$ and $J'(0)=w$ is given by $J(t) = d(\exp_p)_{tv}\, tw$, cf. end of pf. of Gauss Lemma.
Similarly, can also write the unique Jacobi field along $\gamma_v(t)$ with arbitrary initial conditions $J(0)$ and $J'(0)$ using $d(\exp_p)$.

Symmetries of the Curvature Tensor

Let $R(X,Y,Z,W) = g(R(X,Y)Z,W)$, so $R: TM\otimes TM\otimes TM\otimes TM \to \mathbb{R}$ is a $(0,4)$-tensor. Then, it satisfies:

$$R(\overbrace{\underbrace{X,Y}_{\text{skew}}}\,,\,\overbrace{\underbrace{Z,W}_{\text{skew}}})\quad\text{symm.}$$

1) $R(X,Y,Z,W) = R(Z,W,X,Y)$

2) $R(X,Y,Z,W) = -R(Y,X,Z,W)$
$= R(Y,X,W,Z)$

3) 1^{st} Bianchi identity: $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

Together 1) and 2) correspond to the fact that R defines a symmetric endomorphism $R: \Lambda^2 TM \to \Lambda^2 TM$ called the "curvature operator":

$$g\big(R(X\wedge Y), Z\wedge W\big) = g(R(X,Y)W,Z)$$

⚠ Careful with the flip here!

for all $X,Y,Z,W \in TM$ and extended by linearity to $\Lambda^2 TM$.

Def. (Sectional Curvature) The sectional curvature of the plane σ spanned by X, Y is

$$\sec(X\wedge Y) = \frac{g(R(X\wedge Y), X\wedge Y)}{g(X\wedge Y, X\wedge Y)} = \frac{g(R(X,Y)Y,X)}{\|X\|^2\|Y\|^2 - g(X,Y)^2}.$$

Note. If X', Y' are s.t. $\operatorname{span}\{X',Y'\} = \operatorname{span}\{X,Y\}$, then $\sec(X'\wedge Y') = \sec(X\wedge Y)$, so we write $\sec: Gr_2^+ T_pM \to \mathbb{R}$, where $Gr_2^+T_pM \subset \Lambda^2 T_pM$ is the (oriented) Grassmannian of 2-planes in T_pM, given by $Gr_2^+T_pM = \{\sigma \in \Lambda^2 T_pM : \|\sigma\|^2 = 1,\ \underline{\sigma\wedge\sigma = 0}\}$, as $\sec(\sigma) = \langle R\sigma, \sigma\rangle$.

curvature operator $R: \Lambda^2 T_pM \to \Lambda^2 T_pM$

"Plücker relations" characterize the elements $\sigma \in \Lambda^2 T_pM$ of the form $\sigma = X\wedge Y$ for some $X, Y \in T_pM$, i.e., "rank 1 tensors".

Pf: Any other basis is obtained by performing finitely many of the following operations:

a) $\{X,Y\} \to \{Y,X\}$

b) $\{X,Y\} \to \{\lambda X, Y\}$ $\lambda \in \mathbb{R}$

c) $\{X,Y\} \to \{X+\lambda Y, Y\}$ $\lambda \in \mathbb{R}$.

All the above clearly preserve $\sec(X,Y)$; e.g., (c):

$$\langle R(X+\lambda Y, Y)Y, X+\lambda Y\rangle = \langle R(X,Y)Y, X\rangle \text{ b/c } R(Y,Y)=0 \quad \langle R(\cdot,\cdot)Y,Y\rangle = 0.$$

$$\|X+\lambda Y\|^2\|Y\|^2 - \langle X+\lambda Y, Y\rangle^2 = \left(\|X\|^2 + 2\lambda\langle X,Y\rangle + \lambda^2\|Y\|^2\right)\|Y\|^2 - \left(\langle X,Y\rangle + \lambda\|Y\|^2\right)^2$$
$$= \|X\|^2\|Y\|^2 - \langle X,Y\rangle^2.$$

(or, more elegantly, note: $\|(X+\lambda Y)\wedge Y\|^2 = \|X\wedge Y + \lambda \underbrace{Y\wedge Y}_{=0}\|^2 = \|X\wedge Y\|^2$.) $\square$

Rmk: Given $\sigma \subset T_pM$, let $\Sigma = \exp_p(\sigma)$. Then $\sec(\sigma) = K_\Sigma$.

Gaussian curvature of Σ with induced metric from $\Sigma \hookrightarrow M$.

Lecture 10 3/6/2024

From the 2^{nd} variation of energy, we were led to the curvature tensor

$R: TM \otimes TM \to \mathrm{End}(TM)$ $\quad R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$

(or $R: TM \otimes TM \otimes TM \to TM$)

Due to its symmetries, one may equivalently write R as a symmetric endomorphism $R: \Lambda^2 TM \to \Lambda^2 TM$, called the curvature operator,
$\langle R(X\wedge Y), Z\wedge W\rangle := \langle R(X,Y)W, Z\rangle$.

Def. Sectional curvature: $\sec(X\wedge Y) = \dfrac{\langle R(X\wedge Y), X\wedge Y\rangle}{\|X\wedge Y\|^2} = \dfrac{\langle R(X,Y)Y, X\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}$

Prop. Curvature operator $R: \Lambda^2 TM \to \Lambda^2 TM$, curvature tensor $R: TM\otimes TM \to \mathrm{End}(TM)$, and sectional curvature $\sec: Gr_2 TM \to \mathbb{R}$ are uniquely determined by one another.

Pf: Curvature operator and curvature tensor uniquely determine each other by basic Linear Algebra. Only left to show sec determines R. Use "polarization" and symmetries:

Suppose R' is s.t. $\dfrac{\langle R'(X,Y)Y,X\rangle}{\|X\wedge Y\|^2} = \dfrac{\langle R(X,Y)Y,X\rangle}{\|X\wedge Y\|^2} = \sec(X\wedge Y)$

for all X,Y; want to show $R'=R$.

By hypothesis, $\langle R'(X+Z,Y)Y,X+Z\rangle = \langle R(X+Z,Y)Y,X+Z\rangle$

So $$\underline{\langle R'(X,Y)Y,X\rangle} + 2\langle R'(X,Y)Y,Z\rangle + \langle R'(Z,Y)Y,Z\rangle$$
$$= \underline{\langle R(X,Y)Y,X\rangle} + 2\langle R(X,Y)Y,Z\rangle + \langle R(Z,Y)Y,Z\rangle$$

so $\langle R'(X,Y)Y,Z\rangle = \langle R(X,Y)Y,Z\rangle$. $\forall X,Y,Z$

Thus, $\langle R'(X,Y+W)(Y+W),Z\rangle = \langle R(X,Y+W)(Y+W),Z\rangle$

So $\underline{\langle R'(X,Y)Y,Z\rangle} + \langle R'(X,Y)W,Z\rangle + \langle R'(X,W)Y,Z\rangle + \langle R'(X,W)W,Z\rangle =$

$= \underline{\langle R(X,Y)Y,Z\rangle} + \langle R(X,Y)W,Z\rangle + \langle R(X,W)Y,Z\rangle + \langle R(X,W)W,Z\rangle$

so $\langle R'(X,Y)W,Z\rangle + \langle R'(X,W)Y,Z\rangle = \langle R(X,Y)W,Z\rangle + \langle R(X,W)Y,Z\rangle$

i.e. $\langle R'(X,Y)W,Z\rangle - \langle R(X,Y)W,Z\rangle = \langle R(X,W)Y,Z\rangle - \langle R'(X,W)Y,Z\rangle$
$$= \langle R'(W,X)Y,Z\rangle - \langle R(W,X)Y,Z\rangle \quad \forall X,Y,Z,W$$

Therefore $R'(X,Y)W - R(X,Y)W$ is invariant under cyclic perm. of (X,Y,W) and hence, by the 1st Bianchi identity,

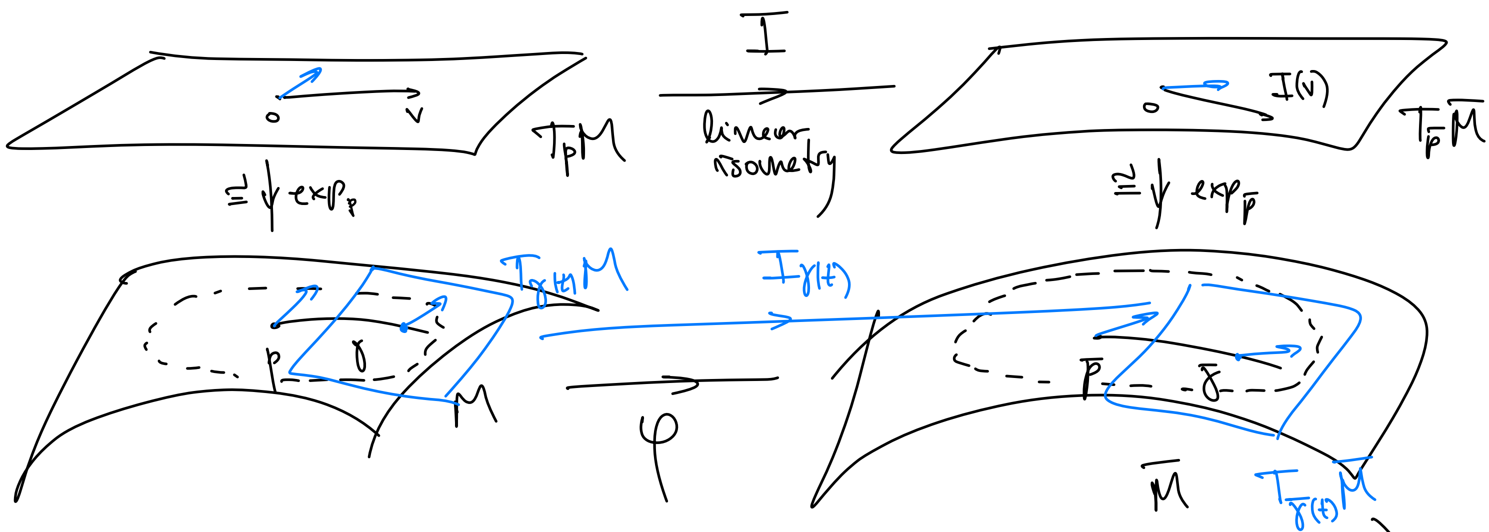
$3\big(R'(X,Y)W - R(X,Y)W\big) = 0$, $\forall X,Y,W$ so $R=R'$. $\square$

Cor. If $R: \Lambda^2 TM \to \Lambda^2 TM$ is s.t. $\sec(\sigma) = K$ for all σ, then $R = K \cdot \mathrm{Id}$, i.e.

$$\langle R(X,Y)W, Z\rangle = \langle R(X\wedge Y), Z\wedge W\rangle = K\langle X\wedge Y, Z\wedge W\rangle$$
$$= K\big(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle\big).$$

curvature operator of a space form w/ $\sec \equiv K$

$$\boxed{\text{Cartan: Curvature is the only local invariant of a Riem mfld.}}$$



$\varphi = \exp_{\bar p} \circ I \circ \exp_p^{-1}$ is a diffeom. (on geod. normal coord.)

Let $\bar\gamma = \varphi\circ\gamma$, $I_{\gamma(t)} : T_{\gamma(t)}M \to T_{\bar\gamma(t)}\bar M$ (Note: $I_{\gamma(t)}$ are linear isometries!)

$$I_{\gamma(t)} := P_{\bar p}^{\bar\gamma(t)} \circ I \circ P_{\gamma(t)}^{p}$$

parallel transport

So $I_{\gamma(t)}$ is linear isometry

Preserving curvature is the "Integrability condition" to become a local isometry:

Thm (Cartan). If for all geodesics $\gamma(t)$ starting at $p\in M$,

$$I_{\gamma(t)}(R(X,Y)Z) = \bar R(I_{\gamma(t)}X, I_{\gamma(t)}Y)I_{\gamma(t)}Z \quad \forall |t| \text{ small}$$

then φ is a local isometry, and $d\varphi_{\gamma(t)} = I_{\gamma(t)}$.

Pf. Given q near p, and $X \in T_q M$, let $\gamma:[0,L]\to M$ be minimizing geodesic w/ $\gamma(0)=p$, $\gamma(L)=q$ and let $J:[0,L]\to TM$ be the Jacobi field along γ with $J(0)=0$ and $J(L)=X$. See Lemma later.

Let $\bar{J}(t) = I_{\gamma(t)}(J(t))$. By hypothesis, $\bar{J}(t)$ is a Jacobi field along $\bar{\gamma}$, since:

$$\bar{J}''(t) + \bar{R}(\bar{J}(t), \bar{\gamma}'(t))\bar{\gamma}'(t) = I_{\gamma(t)}\Big(J''(t) + R(J(t),\gamma'(t))\gamma'(t)\Big) = 0.$$

Note: $I_{\gamma(t)} J''(t) = \bar{J}''(t)$ b/c $I_{\gamma(t)}$ is defined using parallel transport. Say $J(t)=\sum_i a_i(t)e_i(t)$, w/ $e_i(t)$ parallel frame along $\gamma(t)$. Then $\bar{e}_i(t) = I_{\gamma(t)}e_i(t)$ is a parallel frame along $\bar{\gamma}(t)$, and $\bar{J}(t) = \sum_i a_i(t)\bar{e}_i(t)$.
Thus, $I_{\gamma(t)} J''(t) = \sum_i a_i''(t)\, I_{\gamma(t)}e_i(t) = \sum_i a_i''(t)\bar{e}_i(t) = \bar{J}''(t)$.

Moreover, See HW3

$$\begin{cases} J(t) = d(\exp_p)_{t\gamma'(0)}\, t J'(0) \\ \bar{J}(t) = d(\exp_{\bar{p}})_{t\bar{\gamma}'(0)}\, t\bar{J}'(0) \end{cases}$$

$$tJ'(0) = d(\exp_p)^{-1}_{t\gamma'(0)} J(t) \underset{\text{Inverse Fct Thm}}{=} d(\exp_p^{-1})_{\underbrace{\exp_p(t\gamma'(0))}_{\gamma(t)}} J(t)$$

so
$$\begin{aligned}
\bar{J}(t) &= d(\exp_{\bar{p}})_{t\bar{\gamma}'(0)}\, t\bar{J}'(0) \\
&= d(\exp_{\bar{p}})_{t\bar{\gamma}'(0)}\, t\, I(J'(0)) \\
&= d(\exp_{\bar{p}})_{tI\gamma'(0)} \circ I \circ d(\exp_p^{-1})_{\gamma(t)} J(t) \\
&= d\big(\underbrace{\exp_{\bar{p}} \circ I \circ \exp_p^{-1}}_{\varphi}\big)_{\gamma(t)} J(t) = d\varphi_{\gamma(t)} J(t)
\end{aligned}$$

Computing at $t=L$, we have $\bar{J}(L) = d\varphi_{\gamma(L)} J(L) = d\varphi_q X$ and

$$\|d\varphi_q X\| = \|\bar{J}(L)\| \underset{(I_q \text{ is a linear isometry})}{=} \|J(L)\| = \|X\|$$ so $d\varphi_q$ is an isometry. $\square$

<u>Lemma</u>. Let $\gamma:[0,L]\to M$ be a geodesic, $v\in T_{\gamma(0)}M$, $w\in T_{\gamma(L)}M$. If $L>0$ is suff. small, there exists a unique Jacobi field J along γ with $J(0)=v$, $J(L)=w$.

<u>Pf:</u> Let $\mathcal{J}=\{J \text{ is a Jacobi field along } \gamma,\ J(0)=0\}$;

$$\overset{\text{HW3}}{=}\left\{J(t)=d(\exp_{\gamma(0)})_{t\gamma'(0)}\, tJ'(0)\right\}$$

this is a vector space

$\dim \mathcal{J}=\dim T_pM$

Consider $ev_L: \mathcal{J}\longrightarrow T_{\gamma(L)}M$

$\qquad\qquad J\longmapsto J(L)$ (Linear map)

If $L>0$ is small, then ev_L is injective: otherwise $J_1, J_2\in\mathcal{J}$, $J_1(L)=J_2(L)$ but $J_1\neq J_2$. Then $J_1-J_2\in\mathcal{J}$ satisfies $0=(J_1-J_2)(L)=d(\exp_{\gamma(0)})_{L\gamma'(0)}L\cdot(J_1-J_2)'(0)$ and for L small $d(\exp)_{L\gamma'(0)}$ is invertible, so $(J_1-J_2)'(0)=0$, hence $(J_1-J_2)(0)=0$ and $(J_1-J_2)'(0)=0$ so $J_1\equiv J_2$ (contradiction)

Since $ev_L: \mathcal{J}\to T_{\gamma(L)}M$ is linear and $\dim\mathcal{J}=\dim T_{\gamma(L)}M$, ev_L is bijective. So $\exists\, J_1\in\mathcal{J}$ with $J_1(L)=w$.

By the same argument starting from $\gamma(L)$, $\exists\, J_2$ a Jacobi field along γ with $J_2(0)=v$ and $J_2(L)=0$.

Thus, $J:=J_1+J_2$ satisfies $J(0)=v$ and $J(L)=w$. $\square$

<u>Rmk:</u> The above holds for any $L>0$ s.t. $\gamma(L)$ is <u>not</u> <u>conjugate</u> to $\gamma(0)$ along γ. (We define conjugate points later).

Lecture 11 3/8/2024

Completeness:

- (M,g) is geodesically complete if every geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$ can be extended to $\bar{\gamma} : \mathbb{R} \to M$.
- (M,g) is metrically complete if the metric space (M, dist_g) is complete, i.e., every Cauchy sequence converges.

Thm (Hopf–Rinow, 1931). Let (M,g) be a connected Riem. mfld, and $p \in M$. The following are equivalent:

a) $\exp_p : T_pM \to M$ is defined on all of T_pM,

b) Closed and bounded subsets of M are compact

c) (M,g) is metrically complete

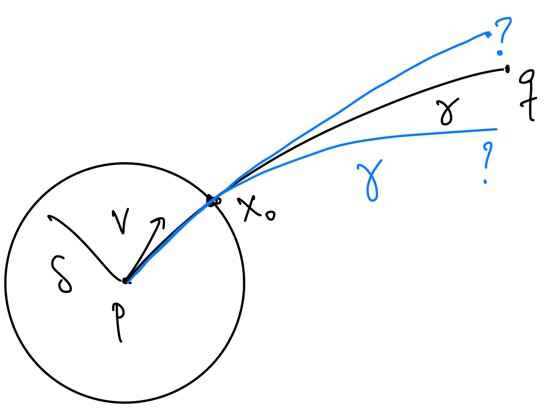
d) (M,g) is geodesically complete

e) $\exists K_n \subset M$ nested sequence of compact subsets ($K_n \subset \operatorname{int} K_{n+1}$) s.t. $M = \bigcup_n K_n$, and if $q_n \notin K_n$ $\forall n$, then $\mathrm{dist}(p, q_n) \to +\infty$.

If any (hence all) of the above hold, then:

f) For all $q \in M$, there exists a minimizing geodesic from p to q, i.e., $\gamma : [0,L] \to M$ with $\gamma(0) = p$, $\gamma(L) = q$, and $\mathrm{dist}(p,q) = L_g(\gamma)$.

Pf. a) $\Rightarrow$ f) Let $r = \mathrm{dist}(p,q)$ and $B_\delta(p)$ be a normal neighbd of p.



The function $f : \partial B_\delta(p) \to \mathbb{R}$, $f(x) = \mathrm{dist}(x,q)$ is continuous hence has a minimum $x_0 \in \partial B_\delta(p)$.
Let $v \in T_pM$ be s.t. $\exp_p \delta v = x_0$ and $\|v\| = 1$; let $\gamma(t) = \exp_p tv$, which is defined $\forall t \in \mathbb{R}$ by a)

<u>Claim.</u> $\gamma(r) = q$.

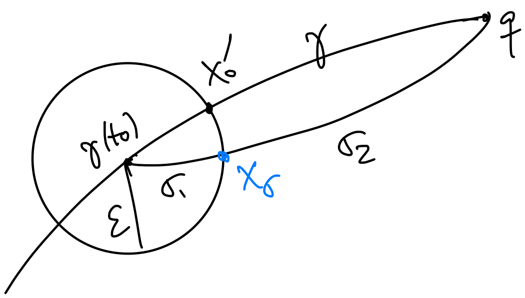
Pf of Claim: (Continuity method) Consider the subset

$$A = \{ t \in [0,r] : \operatorname{dist}(\gamma(t), q) = r - t \}$$

again by continuity of $\operatorname{dist}(\cdot, q)$

and note $A \neq \emptyset$ because $0 \in A$, and $A \subset [0,r]$ is closed.
It suffices to show that if $t_0 \in A$, then $t_0 + \varepsilon \in A$ for suff. small $\varepsilon > 0$, since then $A = [0,r]$, and $r \in A$ is the desired claim.

Let $t_0 \in A$ and $\varepsilon > 0$ small; We want to show that $t_0 + \varepsilon \in A$.
By making $\varepsilon > 0$ suff. small, we may assume $B_\varepsilon(\gamma(t_0))$ is a normal neighbhd of $\gamma(t_0)$. Let σ be a curve from $\gamma(t_0)$ to q and $x_\sigma \in \partial B_\varepsilon(\gamma(t_0))$ be the first time it intersects $\partial B_\varepsilon(\gamma(t_0))$; write $\sigma = \sigma_1 \cup \sigma_2$, where σ_1 joins $\gamma(t_0)$ to x_σ, as in the picture.



Every point in $\partial B_\varepsilon(\gamma(t_0))$ is at distance ε from $\gamma(t_0)$, so $L_g(\sigma_1) \geq \operatorname{dist}(\gamma(t_0), x_\sigma) = \varepsilon = \operatorname{dist}(\gamma(t_0), x_0')$
and $L_g(\sigma_2) \geq \operatorname{dist}(x_\sigma, q) \geq \operatorname{dist}(x_0', q)$ where $x_0' \in \partial B_\varepsilon(\gamma(t_0))$ is a minimum for $\operatorname{dist}(x, q)$, $x \in \partial B_\varepsilon(\gamma(t_0))$. Thus,

$$L_g(\sigma) = L_g(\sigma_1) + L_g(\sigma_2) \geq \operatorname{dist}(\gamma(t_0), x_0') + \operatorname{dist}(x_0', q)$$

Taking the infimum over all such σ, since $\operatorname{dist}(\gamma(t_0), q) = \inf L_g(\sigma)$,

$$r - t_0 \overset{t_0 \in A}{=} \operatorname{dist}(\gamma(t_0), q) \geq \underbrace{\operatorname{dist}(\gamma(t_0), x_0')}_{\varepsilon} + \operatorname{dist}(x_0', q)$$

which, together with the triangle inequality, implies that

$$r - t_0 = \varepsilon + \operatorname{dist}(x_0', q)$$

i.e. $\operatorname{dist}(x_0', q) = r - t_0 - \varepsilon$. Thus, it suffices to show $x_0' = \gamma(t_0 + \varepsilon)$; for that will imply $t_0 + \varepsilon \in A$. By the triangle inequality,

$$\operatorname{dist}(p, x_0') \geq \operatorname{dist}(p, q) - \operatorname{dist}(q, x_0') = r - (r - t_0 - \varepsilon) = t_0 + \varepsilon$$

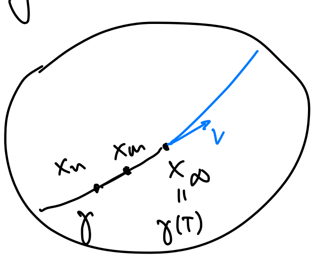
Moreover, the curve $\underset{\text{from } p \text{ to } \gamma(t_0)}{\gamma([0,t_0])} \cup \underset{\text{from } \gamma(t_0) \text{ to } x_0'}{\alpha}$ where α is a radial geod.

from $\gamma(t_0)$ to x_0' has length $t_0+\varepsilon$, and therefore is minimizing. Minimizing geodesics are smooth, so α must be a piece of γ, namely $\alpha = \gamma([t_0, t_0+\varepsilon])$, so $x_0' = \gamma(t_0+\varepsilon)$ as desired. $\square$

a) $\Rightarrow$ b) Let $K \subset M$ be closed and bounded. Boundedness gives $R>0$ s.t. $K \subset B_R(p)$, so $K \subset \underset{\text{defined on all } T_pM \text{ by a)}}{\exp_p} \overline{B_R(0)}$, where $\overline{B_R(0)} \subset T_pM$ is compact, and as $\exp_p$ is continuous, also $\exp_p \overline{B_R(0)}$ is compact. Since $K \subset \exp_p \overline{B_R(0)}$ is closed in a compact, it is also compact. $\square$

b) $\Rightarrow$ c) Let $\{x_n\}$ be a Cauchy sequence and $K = \overline{\{x_n : n \in \mathbb{N}\}}$. Since K is closed and bounded, it is compact by b), so $\{x_n\}$ has a convergent subsequence, hence (as it is Cauchy) it converges. $\square$

c) $\Rightarrow$ d) Suppose $\gamma : [0,T) \to M$ is a unit speed geodesic that we wish to extend to T and beyond. Let $t_n = T - 1/n$ and $x_n = \gamma(t_n)$.
Since $\mathrm{dist}(x_n, x_m) = |t_n - t_m| = \left|\frac{1}{n} - \frac{1}{m}\right|$, the sequence x_n is Cauchy, hence converges to $x_\infty \in M$ by c). Let $B_\varepsilon(x_\infty)$ be a normal neighborhood at x_∞.
For n, m suff. large, $x_n, x_m \in B_\varepsilon(x_\infty)$ so there exists a (unique) minimizing geodesic α_{nm} from x_n to x_m, which hence coincides with $\gamma([t_n, t_m])$.



Since $\exp_{x_\infty}$ is a diffeo onto $B_\varepsilon(x_\infty)$, the geodesic γ can be extended to $\gamma(T) = x_\infty$ and beyond, as $\gamma(t) = \exp_{x_\infty}(t-T)v$ for $t \geq T$ where $v = \lim_{n\to\infty} \underset{\substack{\cap \\ T_{x_n}M}}{\dot\gamma(t_n)} \in T_{x_\infty}M$

(limit in the unit tangent bundle)

d) $\Rightarrow$ a) is trivial; b) $\Leftrightarrow$ e) follows from general topology. $\square$

<u>Rmk.</u> f) $\not\Rightarrow$ a), b), c), d), e). E.g., let $M = \{x \in \mathbb{R}^n : \|x\| < 1\}$ be an open ball.

<u>Cor:</u> Compact manifolds are complete. Closed submanifolds of a complete manifold are complete.