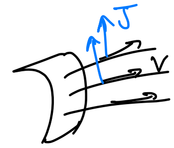
Lecture 20 4/12/2024

From last time: $J'' + R_V J = 0 \iff \begin{cases} J' = SJ \\ S' + S^2 + R_V = 0 \end{cases}$ $(S = \nabla V)$



Thm. Let $R_1, R_2 : \mathbb{R} \to \mathrm{Sym}^2 E$ be smooth curves with $R_1(t) \geq R_2(t)$, $\forall t$. Let $S_i : [t_0, t_i) \to \mathrm{Sym}^2 E$ be the maximal solutions to $S_i' + S_i^2 + R_i = 0$. If $S_1(t_0) \leq S_2(t_0)$, then $t_1 \leq t_2$ and $S_1(t) \leq S_2(t)$ for all $t \in [t_0, t_1)$.

Next, we apply the above to get a comparison of lengths of Jacobi fields:

Thm. Let $S_1, S_2 : (t_0, t') \to \mathrm{Sym}^2 E$ be smooth curves with $S_1(t) \leq S_2(t)$. Let $J_i : (t_0, t') \to E$ be nonzero sol. to $J_i' = S_i J_i$. Then $t \mapsto \dfrac{\|J_1(t)\|}{\|J_2(t)\|}$ is nonincreasing. Moreover, if $\lim\limits_{t \searrow t_0} \dfrac{\|J_1(t)\|}{\|J_2(t)\|} = 1$, then $\|J_1(t)\| \leq \|J_2(t)\|$ for all $t \in (t_0, t')$. Equality holds for some $t_* \in (t_0, t')$ if and only if $J_i = j \cdot v_i$ on $[t_0, t']$ for some $v_i \in E$ with $S_i v_i = \lambda v_i$, $j' = \lambda j$, and $S_1 \leq \lambda \,\mathrm{Id} \leq S_2$.

Pf: Since $\|J_i(t)\|$ is smooth, we can differentiate:

$$\frac{\|J_i\|'}{\|J_i\|} = \frac{1}{\|J_i\|} \frac{1}{2\sqrt{\langle J_i, J_i \rangle}} 2 \langle J_i', J_i \rangle = \frac{\langle J_i', J_i \rangle}{\|J_i\|^2} = \frac{\langle S_i J_i, J_i \rangle}{\|J_i\|^2}$$

$$\in [\lambda_{min}(S_i), \lambda_{max}(S_i)]$$

min. and max eigenvalues of $S_i \in \mathrm{Sym}^2 E$.

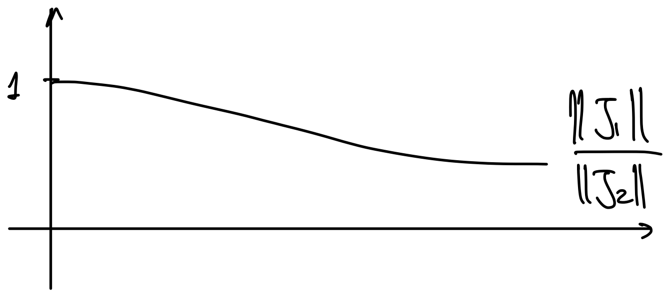
Thus $$(\log \|J_1\|)' = \frac{\|J_1\|'}{\|J_1\|} \leq \lambda_{max}(S_1) \underset{S_1 \leq S_2}{\leq} \lambda_{min}(S_2) \leq \frac{\|J_2\|'}{\|J_2\|} = (\log \|J_2\|)'$$

i.e. $\left(\log \dfrac{\|J_1\|}{\|J_2\|}\right)' \leq 0$ so $\dfrac{\|J_1\|}{\|J_2\|}$ is non-increasing.

By monotonicity, if $\|J_1\| = \|J_2\|$ at $t = t_0$, and $t = t_*$, then

from = in the inequalities.

$\|J_1\| = \|J_2\|$, $\forall t \in (t_0, t_*)$ and hence $J_i' = S_i J_i = \lambda J_i$, from which the stated conclusions follow. $\square$



The following corollaries are originally due to Berger and Rauch:

Thm (Rauch I). Suppose J_i are sol to $J_i'' + R_i J_i = 0$ with $R_1 \geqslant R_2$ and $J_i(0) = 0$, $\|J_1'(0)\| = \|J_2'(0)\|$. Then $\|J_1\| \leq \|J_2\|$ up to the first zero of J_1.

Thm (Rauch II). Suppose J_i are sol to $J_i'' + R_i J_i = 0$ with $R_1 \geqslant R_2$ and $J_i'(0) = 0$, $\|J_1(0)\| = \|J_2(0)\|$. Then $\|J_1\| \leq \|J_2\|$ up to the first zero of J_1.

Both Rauch I and II follow from comparison theorems above; namely $R_1(t) \geqslant R_2(t)$ and $S_1(0) = S_2(0)$ give $S_1(t) \leqslant S_2(t)$ for all $t \in (0, t_1)$. Then:

Rauch I: use singular initial condition "$S_i(0) = \infty$", i.e., $S_i(t) \sim \frac{1}{t}\,\mathrm{Id}$ as $t \downarrow 0$

$$J_i' = S_i J_i \Rightarrow t J_i' \sim J_i \quad \text{as } t \downarrow 0 \rightarrow J_i(0) = 0$$

$$\|J_1'(0)\| = \|J_2'(0)\| \Rightarrow \lim_{t \downarrow 0} \frac{\|J_1(t)\|}{\|J_2(t)\|} = \lim_{t \downarrow 0} \frac{t\|J_1'(t)\|}{t\|J_2'(t)\|} = 1. \quad \text{Apply Thm.}$$

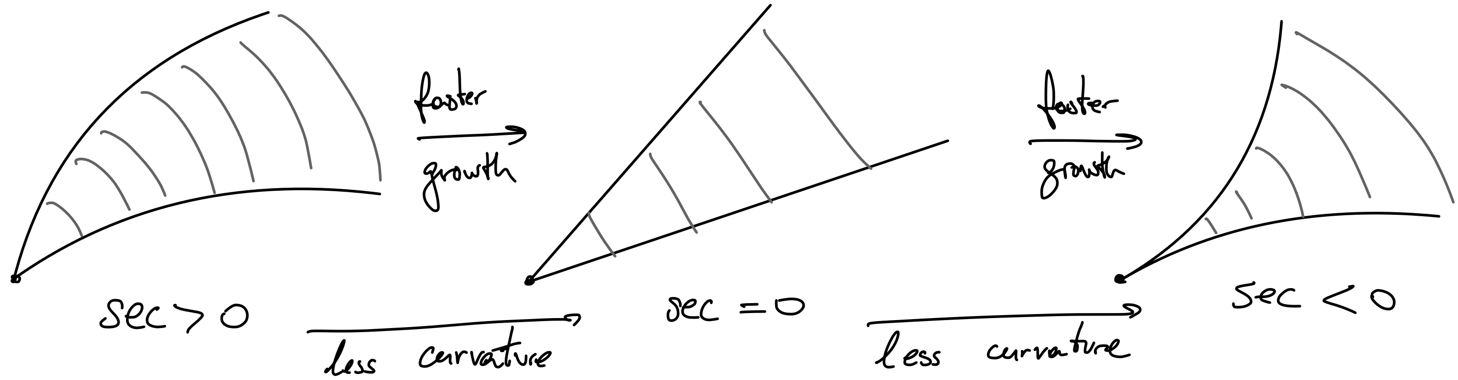
$\square$

Rauch II: use initial condition $S_i(0) = 0$

$$J_i' = S_i \cdot J_i \Rightarrow J_i'(0) = 0.$$

$$\|J_1(0)\| = \|J_2(0)\| \Rightarrow \lim_{t \downarrow 0} \frac{\|J_1(t)\|}{\|J_2(t)\|} = 1. \quad \text{Apply Thm.} \quad \square$$

Picture to have in mind from Rauch I:

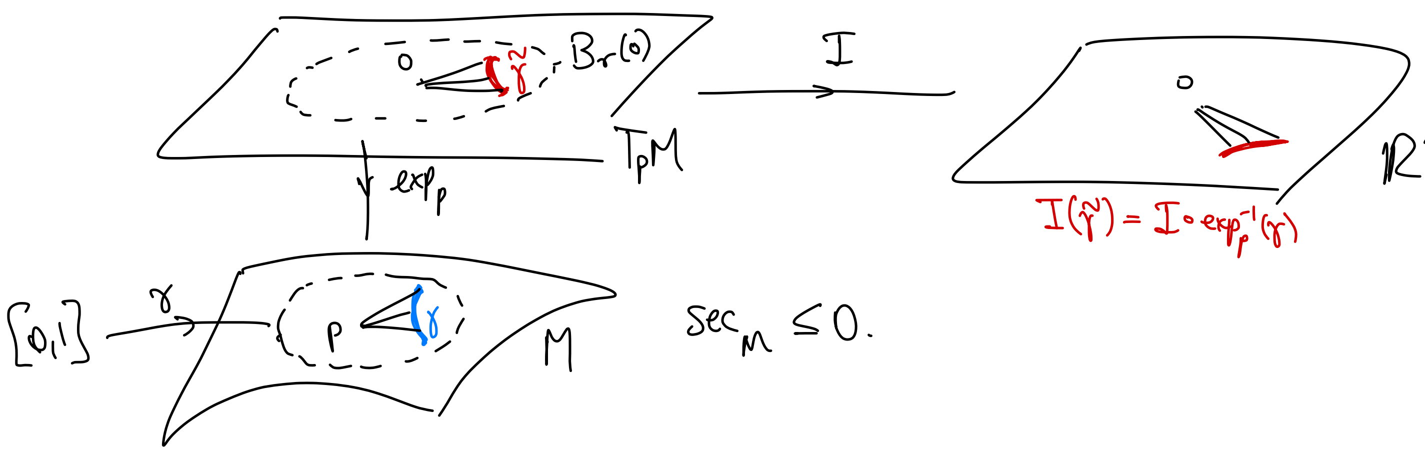


(We knew this for $t \approx 0$ from Taylor Series expansion of $\|J(t)\|^2$ at $t=0$, now this is known for $0 \leq t \leq t_1$ where t_1 is the first conjugate time.)

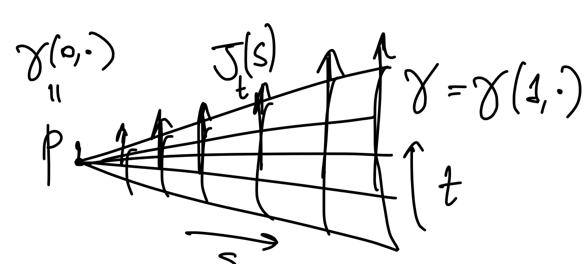
Application of Rauch I:

Cor: Let (M^n, g) be a complete Riem. mfld with sec ≤ 0, and $r > 0$ s.t. $\exp_p : B_r(0) \to M$ is a diffeom. onto its image. Fix a linear isometry $I : T_pM \to \mathbb{R}^n$. Given $\gamma : [0,1] \to \exp_p(B_r(0))$, we have

$$\text{length}_g(\gamma) \geq \text{length}_{\mathbb{R}^n}\left(I \circ \exp_p^{-1}(\gamma)\right).$$



Pf: Let $\tilde{\gamma} = \exp_p^{-1} \gamma$, and consider the "rectangle" $\gamma(s,t) = \exp_p s\tilde{\gamma}(t)$



For fixed t, $s \mapsto \gamma(s,t)$ is a geodesic, and $J_t(s) = \frac{\partial}{\partial t}\gamma(s,t)$ is a Jacobi field along $s \mapsto \gamma(s,t)$; with $J_t(0) = 0$ and $J_t(1) = \dot\gamma(t)$.

Since $\sec_M \leq 0$, by Rauch I,

$$\|J_t(s)\| \geq \underbrace{s\,\|J_t'(0)\|}_{\circledast\ \text{length of comparison Jacobi field in } \mathbb{R}^n} \quad \text{so} \quad \text{length}_g(\gamma) = \int_0^1 \|\dot\gamma(t)\|\,dt = \int_0^1 \|J_t(1)\|\,dt$$

$$\overset{s=1}{\geq} \int_0^1 \|J_t'(0)\|\,dt = \text{length}_{\mathbb{R}^n}\left(I \circ \exp_p^{-1}\gamma\right)$$

Indeed, $$J_t'(0) = \frac{D}{ds} J_t(s)\Big|_{s=0} = \frac{D}{ds}\frac{\partial}{\partial t} \exp_p s\tilde\gamma(t)\Big|_{s=0}$$

$$= \frac{D}{dt}\frac{\partial}{\partial s} \exp_p s\tilde\gamma(t)\Big|_{s=0} = \frac{D}{dt}\underbrace{d(\exp_p)_0}_{id}\tilde\gamma(t) = \tilde\gamma'(t)$$

and so $$\text{length}_{\mathbb{R}^n}\left(I \circ \exp_p^{-1}\gamma\right) = \int_0^1 \Big\| \frac{\partial}{\partial t} \underbrace{I \circ \exp_p^{-1}(\gamma)}_{\tilde\gamma} \Big\|\,dt = \int_0^1 \|J_t'(0)\|\,dt. \qquad \square$$

$\circledast$ In $\mathbb{R}^n$, the Jacobi equation $J'' = 0$ has solutions $J(s) = J(0) + sJ'(0)$, so Jacobi fields with $J(0) = 0$ are given by $J(s) = s\,J'(0)$.

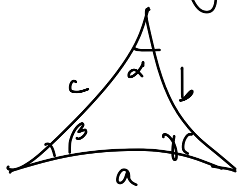
Rmk: Reasoning as above, Rauch I gives a more refined estimate

$$\|J(t)\| \geq t\,\|J'(0)\| > 0$$

for Jacobi fields with $J(0) = 0$ on manifolds with $\sec \leq 0$, compared to our earlier observation (a crucial step in the proof of Cartan–Hadamard Thm) that $J(t) \neq 0$, $\forall t > 0$; cf. Remark in p.2 of Lectures3.pdf.

Def. A geodesic triangle is a triple of minimizing geodesics with endpoints that are match pairwise (as in a triangle).

Cor: A geodesic triangle on a complete manifold with $\sec \le 0$ satisfies

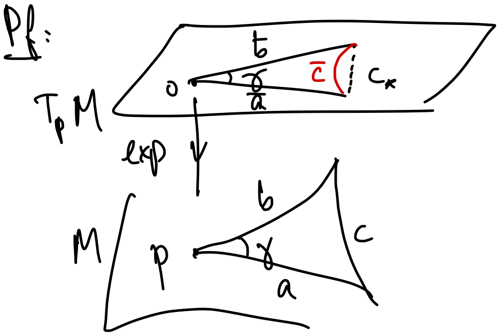


(i) $\ell(c)^2 \geq \ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma$ $\quad (\ell = \text{length})$

(ii) $\alpha + \beta + \gamma \le \pi$

If $\sec < 0$, then get strict inequalities.

Pf:



Let $\bar{a}, \bar{b}, \bar{c}$ in T_pM be such that

$$a = \exp_p \bar{a}, \qquad b = \exp_p \bar{b}, \qquad c = \exp_p \bar{c}$$

Note that $\bar{a}$ and $\bar{b}$ are straight line segments ($\exp_p$ is radial isometry); with $\ell(\bar{a}) = \ell(a)$ and $\ell(\bar{b}) = \ell(b)$. Let c_* be the straight line segment with same endpoints as $\bar{c}$, so $\ell(\bar{c}) \geq \ell(c_*)$. By the Application of Rauch I, $\ell(c) \geq \ell(\bar{c}) \geq \ell(c_*)$. Thus, altogether:

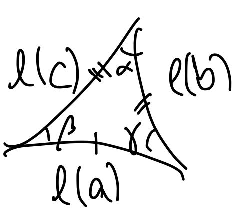
$$\ell(c)^2 \geq \ell(c_*)^2 \overset{\text{Law of cosines in } T_pM \cong \mathbb{R}^n}{=} \ell(\bar{a})^2 + \ell(\bar{b})^2 - 2\ell(\bar{a})\ell(\bar{b})\cos\gamma$$

$$\overset{\text{Gauss Lemma}}{=} \ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma.$$

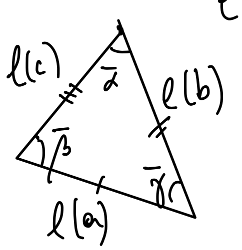
To compare angles, since $\ell(a), \ell(b), \ell(c)$ satisfy the triangle inequalities (b/c every geodesic is minimizing in $\sec \le 0$, i.e., $\ell(a), \ell(b), \ell(c)$ achieve distances) we can build a comparison triangle in $\mathbb{R}^2$, with same side angles, but possibly different angles, say $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$. Then, from the above:

$$\ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma \le \ell(c)^2$$

$$= \ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\bar{\gamma}$$



in M

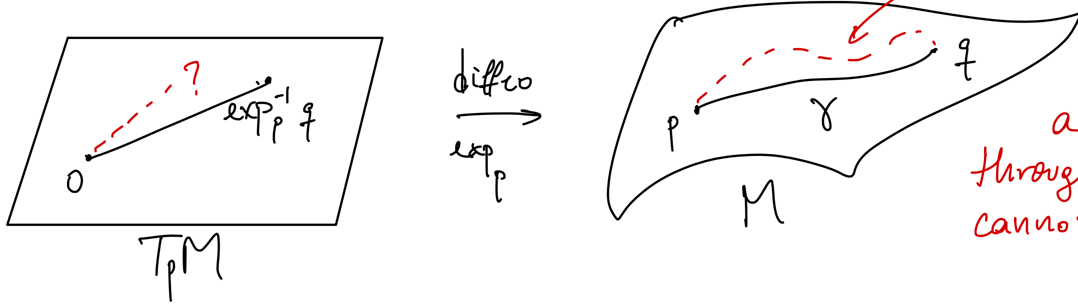


in $\mathbb{R}^2$

$$\Rightarrow \cos\gamma \geq \cos\bar{\gamma} \Rightarrow \gamma \le \bar{\gamma}$$

Same for α, β and get $\alpha + \beta + \gamma \le \bar{\alpha} + \bar{\beta} + \bar{\gamma} = \pi$. $\square$

Rmk: If (M^n, g) is a complete Riem. mfld with $\pi_1 M = \{1\}$ and $\sec \leq 0$, then by Cartan-Hadamard $\exp_p : T_pM \to M$ is a diffeo, so given any $q \in M$ there is a unique geodesic joining p and q, which is hence minimizing (b/c there exists some minimizing geodesic by Hopf–Rinow).



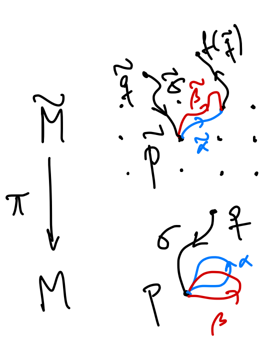
any other geodesic from p to q would lift to a straight line through $0 \in T_pM$, so cannot end at q.

Thus, if M is complete, $\pi_1 M = \{1\}$, and $\sec \leq 0$, then the above facts about geodesic triangles hold for any triangles with geodesic sides (b/c the sides are automatically minimizing.)

Lecture 21 4/17/2024

Def: (M, g) closed Riem. mfld, $(\tilde{M}, \tilde{g})$ universal covering. A deck transformation $f : \tilde{M} \to \tilde{M}$ is a translation along the geodesic $\tilde{\gamma}$ in $\tilde{M}$ if $f(\tilde{\gamma}) = \tilde{\gamma}$. Note: If $f \neq id$, then $f(\tilde{\gamma}(t)) = \tilde{\gamma}(t + a)$.

From basic topology: $\pi_1(M) \cong Aut(\tilde{M}) = \{ f : \tilde{M} \to \tilde{M} : \text{deck transformation} \}$



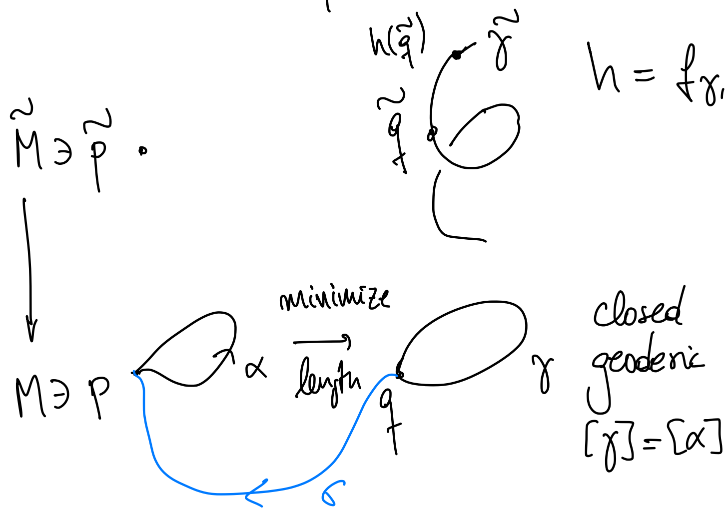
$$[\alpha] \in \pi_1(M, p) \longmapsto \left(f_{\alpha,p} : \tilde{M} \longrightarrow \tilde{M} \right) \in Aut(\tilde{M})$$

$f_{\alpha,p}(\tilde{q})$ = endpoint of lift of $\sigma^{-1} \alpha \sigma$ to $\tilde{M}$, starting at $\tilde{q}$.

Recall: curves in M are homotopic $\iff$ lifts to $\tilde{M}$ have same endpoint so the above is well-defined.

Prop. Given a deck transformation $f : \tilde{M} \to \tilde{M}$, there exists a geodesic $\tilde{\gamma}$ in $\tilde{M}$ s.t. f is a translation along $\tilde{\gamma}$.

Pf. $f = f_{\alpha,p}$ for some $\alpha \in \pi_1(M,p)$. Let $\gamma \sim \alpha$ (free homotopy) be a closed geodesic. Then



$h = f_{\gamma,q} \in Aut(\tilde{M})$ is s.t. $h(\tilde{\gamma}) = \tilde{\gamma}$; by construction.

Claim: $f = h$.

Since h, f are deck transformations, suffices to show $h(\tilde{q}) = f(\tilde{q})$.

As α, γ are freely homotopic, it follows $\sigma^{-1} \alpha \sigma$ is homotopic to γ rel. q. (as elements of $\pi_1(M,q)$) So the endpoints of their lifts are the same, i.e., $h(\tilde{q}) = f(\tilde{q})$ so $h = f$ hence $f(\tilde{\gamma}) = \tilde{\gamma}$. $\square$

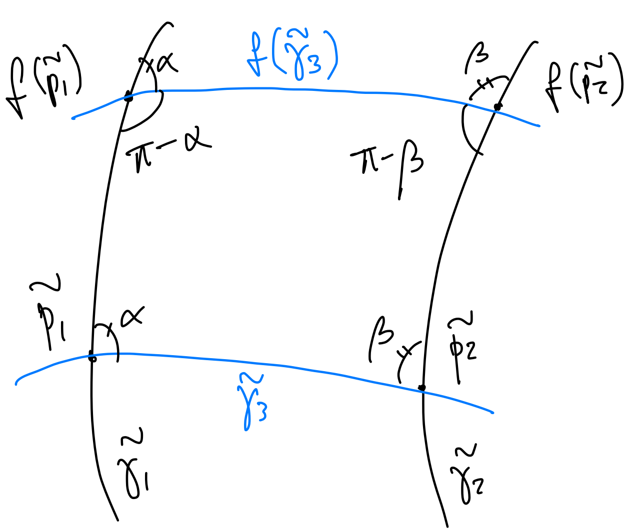
Lemma. If (M,g) is a closed mfld with $\sec < 0$, then a deck transformation $f : \tilde{M} \to \tilde{M}$, $f \neq id$ is a translation along a unique geodesic $\tilde{\gamma}$ in $\tilde{M}$.

Pf. Suppose $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are geodesics in $\tilde{M}$ s.t. f is a translation along $\tilde{\gamma}_i$. Then, if $\tilde{p} \in \tilde{\gamma}_1 \cap \tilde{\gamma}_2$, we have $\tilde{p} \neq f(\tilde{p}) \in \tilde{\gamma}_1 \cap \tilde{\gamma}_2$, but this contradicts injectivity of $\exp_{\tilde{p}}$.

($f \neq id \Rightarrow f$ has no fixed pts.)

by Cartan-Hadamard, $\exp_{\tilde{p}} : T_{\tilde{p}}\tilde{M} \to \tilde{M}$ is a diffeo

Thus $\tilde{\gamma}_1 \cap \tilde{\gamma}_2 = \phi$.



Let $\tilde{p}_i \in \tilde{\gamma}_i$ and $\tilde{\gamma}_3$ be a minimizing geodesic from $\tilde{p}_1$ to $\tilde{p}_2$. As f is an isometry of $\tilde{M}$, the angles α, β in the diagram are the same. Subdividing this quadrangle into two triangles Δ_1, Δ_2 it follows that

$$\sum_{\text{int. angles}} \Delta_1 + \sum_{\text{int. angles}} \Delta_2 \geq 2\pi$$

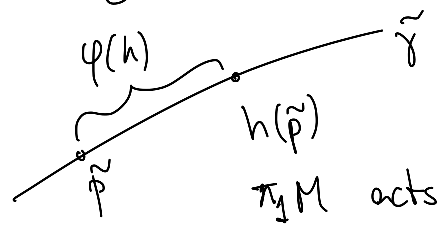
So $\sum_{\text{int angles}} \Delta_i \geq \pi$ for $i=1$ or 2, contradicting Cor. from last Lecture that $\sum_{\text{int. angles}} \Delta < \pi$ if $\sec < 0$. $\square$

<u>Lemma</u>. If (M,g) is a closed mfld with $\sec<0$, then commuting deck transformations are translations along the same geodesic.

<u>Pf</u>. If $f_1, f_2 : \tilde{M} \to \tilde{M}$ are as above, with $f_i(\tilde{\gamma}_i) = \tilde{\gamma}_i$, then $f_2(f_1(\tilde{\gamma}_2)) = f_1(f_2(\tilde{\gamma}_2)) = f_1(\tilde{\gamma}_2)$ so f_2 preserves $f_1(\tilde{\gamma}_2)$ hence $f_1(\tilde{\gamma}_2) = \tilde{\gamma}_2$. By uniqueness proved above, $\tilde{\gamma}_1 = \tilde{\gamma}_2$. $\square$

<u>Thm</u> (Preissmann, 1943). If (M^n, g) is a closed Riem. mfld with $\sec<0$ and $H < \pi_1 M$ is Abelian, $H \neq \{1\}$, then $H \cong \mathbb{Z}$.

<u>Pf</u>. Let $H < \pi_1 M$ be Abelian, and $\tilde{\gamma}$ be the geodesic in $\tilde{M}$ along which every $h \in H$ is a translation. Recall (see Remark at end of last lecture) that given two points in $\tilde{M}$ there is a unique geodesic (hence minimizing) joining them. Fix $\tilde{p} \in \tilde{\gamma}$ and define $\varphi : H \to \mathbb{R}$, $\varphi(h) = \pm \operatorname{dist}(\tilde{p}, h(\tilde{p}))$, according to $h(\tilde{p})$ being before/after $\tilde{p}$ along $\tilde{\gamma}$. Then φ is a group homomorphism and injective, so the subgroup $\varphi(H) < \mathbb{R}$ is either dense or isomorphic to $\mathbb{Z}$. It cannot be dense because $\pi_1 M$ acts properly discontinuously on $\tilde{M}$. $\square$



<u>Cor.</u> If M_1, M_2 are closed manifolds, then $M_1 \times M_2$ does not admit any metric with $\sec < 0$.

<u>Pf.</u> Suppose $(M_1 \times M_2, g)$ has $\sec < 0$; in particular, by Cartan–Hadamard, $\widetilde{M_1 \times M_2} \cong \widetilde{M}_1 \times \widetilde{M}_2 \cong \mathbb{R}^n$ so $\pi_1 M_i \neq \{1\}$ for $i = 1, 2$. Indeed, if, say $\pi_1 M_1 = \{1\}$, then $\widetilde{M}_1 \times \widetilde{M}_2 \cong M_1 \times \widetilde{M}_2 \not\cong \mathbb{R}^n$ because M_1 is closed. Let $h_i \in \pi_1 M_i$ be nontrivial elements and $\langle h_i \rangle$ the corresponding cyclic subgroups. Then $H = \langle h_1 \rangle \oplus \langle h_2 \rangle$ is an Abelian subgroup of $\pi_1 M$ that is not isomorphic to $\mathbb{Z}$. $\square$

E.g., T^n does not have any metric with $\sec < 0$

$\Sigma^2 \times S^1$ does not have any metric with $\sec < 0$

But they have metrics with $\sec \leq 0$!

e.g., closed surface of genus ≥ 2

<u>Rmk.</u> Byers showed that if a closed manifold (M, g) with $\sec < 0$ has $H < \pi_1 M$ a nontrivial <u>solvable</u> subgroup, then $H \cong \mathbb{Z}$. Moreover, $\pi_1 M$ does not admit finite-index cyclic subgroups.

<u>Rmk.</u> It is unknown if any $M_1 \times M_2$ where M_i are closed, $\pi_1 M_i = \{1\}$, can admit metrics with $\sec > 0$. The case $M_1 = M_2 = S^2$ is known as the Hopf Question.

<u>Rmk.</u> The analogous question for $\sec > 0$ was proposed by Chern in 1965: If (M, g) is a closed Riem. mfld w/ $\sec > 0$, and $H < \pi_1 M$ is Abelian, then is H cyclic? By Synge, the answer is affirmative in even dimensions. Ravi Shankar (1998) found infinitely many counter-examples in dimension 7, as there are homogeneous spaces M^7 with $\sec > 0$ and a free action by $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ which is isometric. Thus, $M^7 / \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is a closed manifold with $\sec > 0$ and fundamental group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Lecture 22 4/19/2024

Comparison theorems for sec: Rauch I, II $\rightsquigarrow$ Preissmann, Cartan–Hadamard, Myers, Synge, ...

also with second variation of length/energy.

maybe later? Toponogov $\rightsquigarrow$ Gromov's bounds on generators of π_1, total Betti #, ... Alexandrov geometry

Comparison theorem for Ric: Bishop Volume Comparison (today) $\rightsquigarrow$ Milnor (π_1 has polynomial growth)

Rigidity in Myers, (today)

Gromov's compactness theorem, ...

Recall Riccati equation:

$$S' + S^2 + R_V = 0 \quad \text{(in } \mathrm{Sym}^2 E) \quad \overset{\mathrm{tr}}{\rightsquigarrow} \quad \mathrm{tr}\, S' + \underbrace{\mathrm{tr}\,(S^2)} + \mathrm{Ric}(V) \overset{\circledast}{=} 0 \quad (\text{in } \mathbb{R})$$

this is not a function of $\mathrm{tr}\, S$...

like the actual shape operator...

Since $S(V) = \nabla_V V = 0$, can restrict S to $V^\perp$, $S : V^\perp \to V^\perp$, and $S \in \mathrm{Sym}^2 V^\perp$. Let $a = \frac{\mathrm{tr}\, S}{n-1}$, and note that

$$S = a\,\mathrm{Id} + S_0, \quad \text{where} \quad \mathrm{tr}\, S_0 = 0.$$

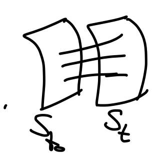
"trace-free part"

So $\langle S_0, I \rangle = 0$. recall $\langle A, B \rangle = \mathrm{tr}\, AB$

Then $\mathrm{tr}(S^2) = \|S\|^2 = a^2 \|\mathrm{Id}\|^2 + \|S_0\|^2 = (n-1)a^2 + \|S_0\|^2$ so $\circledast$

gives $a' + a^2 + r = 0$, where $r = \frac{1}{n-1}\left(\|S_0\|^2 + \mathrm{Ric}(V)\right) \geq \frac{\mathrm{Ric}(V)}{n-1}$

Rmk: Geometrically, $a(t) = \frac{H}{n-1}$ where $H = \mathrm{tr}\, S$ is the mean curvature of S_t.



Thm. Suppose $S : [t_0, t_1) \to \mathrm{Sym}^2 V^\perp$ is the maximal solution to $S' + S^2 + R = 0$, where $R : \mathbb{R} \to \mathrm{Sym}^2 V^\perp$ is given. Suppose $\exists\, K \in \mathbb{R}$ s.t.

(i) $\mathrm{tr}\, R \geq (n-1)K$

(ii) $\mathrm{tr}\, S(t_0) \leq (n-1)\bar{a}(t_0)$

where $\bar{a} : [t_0, t_2) \to \mathbb{R}$ is the maximal solution to $\bar{a}' + \bar{a}^2 + K = 0$. Let $a = \frac{\mathrm{tr}\, S}{n-1}$

Then $t_1 \leq t_2$ and $a(t) \leq \bar{a}(t)$ for all $t \in [t_0, t_1)$.

Pf: Apply ODE comparison from Lectures 19-20:

> **Thm.** Let $R_1, R_2 : \mathbb{R} \to \mathrm{Sym}^2 E$ be smooth curves with $R_1(t) \geq R_2(t)$, $\forall t$.
> Let $S_i : [t_0, t_i) \to \mathrm{Sym}^2 E$ be the maximal solutions to $S_i' + S_i^2 + R_i = 0$
> If $S_1(t_0) \leq S_2(t_0)$, then $t_1 \leq t_2$ and $S_1(t) \leq S_2(t)$ for all $t \in [t_0, t_1)$.

setting $E = \mathbb{R}$, $R_1 = r$, $R_2 = K$, so (i) $\Rightarrow r \geq K \Rightarrow R_1 \geq R_2$

$$S_1' + S_1^2 + R_1 = 0 \iff a' + a^2 + r = 0$$
$$S_2' + S_2^2 + R_2 = 0 \iff \bar{a}' + \bar{a}^2 + K = 0.$$

$\square$

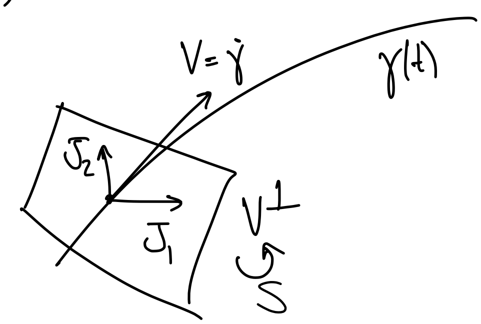
Rmk: Above result remains true if $\bar{a}$ has a pole at t_0; namely

$$S(t) \sim \frac{1}{t - t_0} \mathrm{Id}, \qquad \bar{a} = \frac{sn_K'}{sn_K} \quad \text{where} \begin{cases} sn_K'' + K\, sn_K = 0 \\ sn_K(t_0) = 0 \\ sn_K'(t_0) = 1. \end{cases}$$

Let $J_1, \ldots, J_{n-1}$ be Jacobi fields along γ that form a basis of solutions to

$$J' = SJ \qquad (S : V^\perp \to V^\perp)$$

all identified via parallel transport



and set $j = \det(J_1, J_2, \ldots, J_{n-1})$.

$$j' = \det(J_1', J_2, \ldots, J_{n-1}) + \det(J_1, J_2', J_3, \ldots, J_{n-1}) + \cdots + \det(J_1, \ldots, J_{n-1}')$$
$$= \det(SJ_1, J_2, \ldots, J_{n-1}) + \det(J_1, SJ_2, J_3, \ldots, J_{n-1}) + \cdots + \det(J_1, \ldots, SJ_{n-1})$$
$$= \mathrm{tr}\, S \cdot \det(J_1, \ldots, J_{n-1}) = \mathrm{tr}\, S \cdot j$$

or: $d(\det)_I X = \mathrm{tr}\, X$; more generally, if A is invertible, $d(\det)_A X = (\det A)\,\mathrm{tr}(A^{-1} X)$

Let $j(t) = \det A(t)$, where $A(t) = (J_1(t), \ldots, J_{n-1}(t))$.

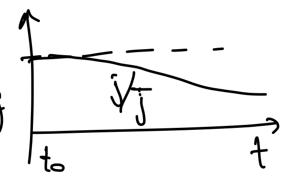
$$j'(t) = d(\det)_{A(t)} A'(t) = (\det A(t))\, \mathrm{tr}\left(A(t)^{-1} A'(t)\right)$$
$$= j(t) \cdot \mathrm{tr}\left(A^{-1}(t) \cdot S(t) \cdot A(t)\right) = (\mathrm{tr}\, S) \cdot j$$

so we have $j' = (n-1)\, a\, j$

Thm. Let $S:[t_0,t_1)\to \mathrm{Sym}^2 V^\perp$ and $a=\frac{1}{n-1}\operatorname{tr} S$ be s.t. $a\le \overline{a}$, and $j'=(n-1)aj$. Choose $\overline{j}$ s.t. $\overline{j}'=(n-1)\overline{a}\,\overline{j}$. Then $j/\overline{j}$ is nonincreasing.

Pf: Once again, apply ODE comparison from before!

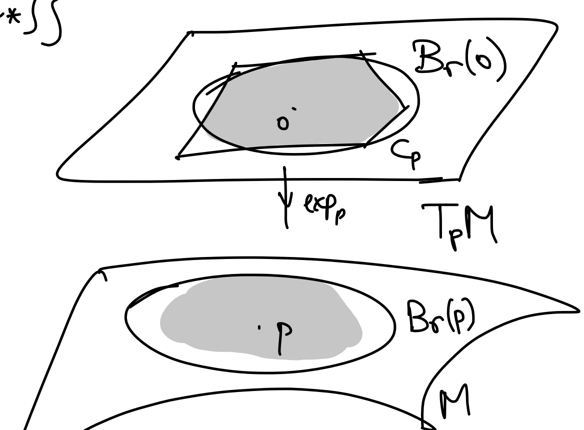
$$(n-1)a\le (n-1)\overline{a} \implies \left(\log\frac{j}{\overline{j}}\right)'\le 0 \implies j/\overline{j} \text{ nonincreasing}$$



Thm (Bishop Volume Comparison). Let (M^n,g) be a Riem. mfld with $\mathrm{Ric}\ge(n-1)K$ and $\overline{M}$ be the simply-connected Riem. mfld with $\sec_{\overline{M}}\equiv K$. Then $\forall p\in M$, $\mathrm{Vol}(B_r(p))\le \mathrm{Vol}(\overline{B_r})$, where $B_r(p)\subset M$ and $\overline{B_r}\subset\overline{M}$ are balls of radius r. Moreover, equality holds if and only if $B_r(p)\underset{\text{isom}}{\cong}\overline{B_r}$.

Pf: We will show that $r\mapsto \dfrac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(\overline{B_r})}$ is nonincreasing; the conclusion follows since $\displaystyle\lim_{r\downarrow 0}\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(\overline{B_r})}=1$ because both approach Euclidean balls as $r\downarrow 0$.

Let $\mathrm{cut}(v)=\max\{t_*>0:\ \exp_p tv \text{ is min. geod. on } [0,t_*]\}$ and $C_p=\{tv:\ t\le \mathrm{cut}(v),\ \|v\|=1\}\subset T_pM$. Then $\exp_p: C_p\to M$ is a diffeom. onto its image, so:



$\exp_p C_p=\mathrm{Cut}(p)$ is the cut locus of p

$$\mathrm{Vol}(B_r(p))=\int_{B_r(p)}1\,d\mathrm{vol}=\int_{\exp_p(B_r(0)\cap C_p)}1\,d\mathrm{vol}$$

$$\overset{\substack{\text{Change of}\\ \text{variables}\\ \text{formula}}}{=}\int_{B_r(0)\cap C_p}\det\left(d(\exp_p)_u\right)du$$

$$\overset{\substack{\text{Polar}\\ \text{coord.}}}{=}\int_{S^{n-1}(1)}\int_0^{r(v)}\det\left(d(\exp_p)_{tv}\right)t^{n-1}\,dt\,dv$$

Recall: $B_r(p)=\exp_p(B_r(0))=\exp_p(B_r(0)\cap C_p)$

where $r(v)=\min\{r,\mathrm{cut}(v)\}$ for $v\in T_pM$, $\|v\|=1$, i.e. $v\in S^{n-1}(1)\subset T_pM$.

Since $d(\exp_p)_{tv}\, e_i = \frac{1}{t}\left(d(\exp_p)_{tv}\, te_i\right) = \frac{1}{t} J_i(t)$ is the Jacobi field along $t \mapsto \exp_p tv$ with $J_i(0)=0$ and $J_i'(0) = e_i$, it follows that $\det\left(d(\exp_p)_{tv}\right) = \frac{1}{t^{n-1}} \det\left(J_1(t), \dots, J_{n-1}(t)\right)$ and hence:

$$\mathrm{Vol}(B_r(p)) = \int_{S^{n-1}(1)} \int_0^{r(v)} \underbrace{\det(J_1(t), \dots, J_{n-1}(t))}_{j_v(t)}\, dt\, dv$$

if needed, extend $j_v(t)$ as $j_v(t) = 0$ for $t > \mathrm{cut}(v)$.

By previous result, $j_v(t)/\bar{j}(t)$ is non increasing on $[0,r]$, where $\bar{j}(t) = \det(\bar{J}_1, \dots, \bar{J}_{n-1})$, for corresponding Jacobi fields $\bar{J}_i$ on $\bar{M}$.

Set $q(t) = \frac{1}{\mathrm{Vol}(S^{n-1}(1))} \int_{S^{n-1}(1)} \frac{j_v(t)}{\bar{j}(t)}\, dv$, which is also non-increasing (because it is an average of nonincreasing quantities). As before,

$$\mathrm{Vol}(\bar{B}_r) = \int_{S^{n-1}(1)} \int_0^r \bar{j}(t)\, dt\, dv \overset{\substack{\text{(space w/ sec} \equiv K\\ \text{is isotropic)}}}{=} \mathrm{Vol}(S^{n-1}) \int_0^r \bar{j}(t)\, dt$$

Thus,

$$\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(\bar{B}_r)} = \frac{\int_{S^{n-1}(1)} \int_0^r j_v(t)\, dt\, dv}{\mathrm{Vol}(S^{n-1}(1)) \cdot \int_0^r \bar{j}(t)\, dt} \overset{\text{Fubini}}{=} \frac{\int_0^r q(t) \cdot \bar{j}(t)\, dt}{\int_0^r \bar{j}(t)\, dt}$$

is non increasing, because RHS is the ($\bar{j}$ – weighted) average(*) of the non increasing function $q(t)$ over growing intervals.

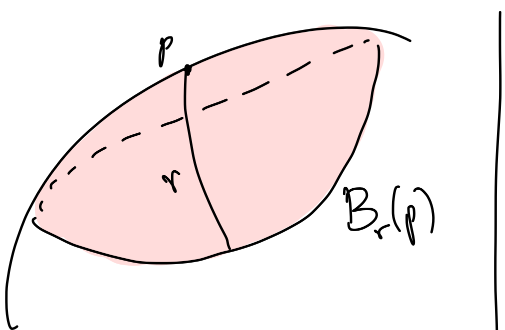
(*) More explicitly: if $\phi, \psi > 0$, and $t \mapsto \dfrac{\phi(t)}{\psi(t)}$ is non increasing, then

$$r \mapsto \frac{\int_0^r \phi(t)\,dt}{\int_0^r \psi(t)\,dt} = \frac{\int_0^{\bar r} \frac{\phi(s)}{\psi(s)}\,ds}{\int_0^{\bar r} ds}$$ is non increasing, where $\begin{cases} ds = \psi(t)\,dt \\ \bar r = s(r). \end{cases}$

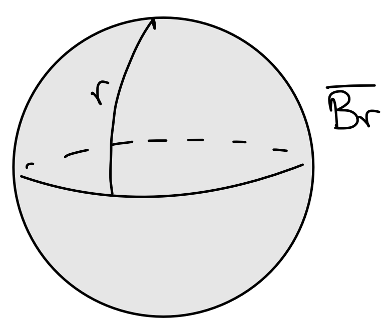
Rigidity statement follows from rigidity statements in ODE comparison: if $\forall v \in S^{n-1}(1)$, $\forall\, 0 \le t \le r$, $j_v(t) = \bar J(t)$, then $a(t) = \bar a(t)$, for all $0 \le t \le r$, so $R(t) = \bar R(t) = \kappa\,\mathrm{Id}$. Thus $B_r(p)$ has constant curvature $\sec \equiv \kappa$ and is hence isometric to $\overline{B_r}$. $\square$

<u>Remark:</u> Similarly, one can prove $r \mapsto \dfrac{\mathrm{Vol}(\partial B_r(p))}{\mathrm{Vol}(\partial \overline{B_r})}$ is non increasing.

Geometrically:



$\mathrm{Ric} \geq (n-1)\kappa$

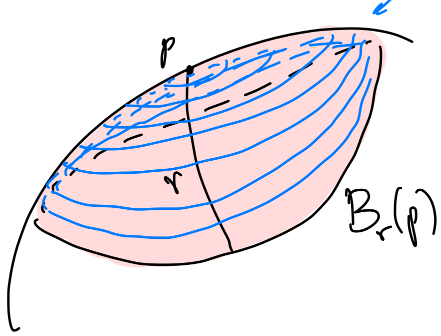


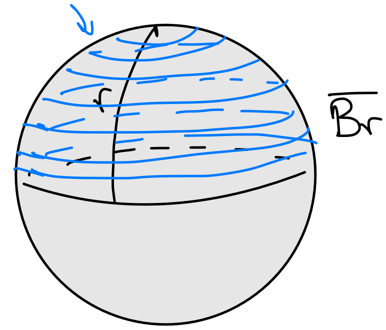
$\sec \equiv \kappa$
(so $\mathrm{Ric} = (n-1)\kappa$)

$$\mathrm{Vol}(B_r(p)) \le \mathrm{Vol}(\overline{B_r})$$
$$=$$
$$\Updownarrow$$
$$B_r(p) \underset{\text{isom}}{\cong} \overline{B_r}$$

With <u>stronger</u> control on curvature $\sec \geq \kappa$ we know that:

longer / shorter





so "integrating" get the above.

BUT

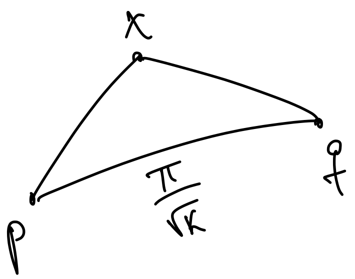
$\mathrm{Ric} \geq \kappa(n-1)$ is enough for this "integral" control.

Rigidity in Myers Theorem

(Originally by Shiu-Yuen Cheng, with different proof)
↳ student of S.S. Chern

$$\boxed{\text{Thm. Let } (M^n,g) \text{ be a complete Riem. mfld with } Ric \geq K\cdot(n-1) > 0 \text{ and } \operatorname{diam}(M^n,g) = \operatorname{diam}\left(S^n\left(\tfrac{1}{\sqrt{K}}\right)\right) = \tfrac{\pi}{\sqrt{K}}. \text{ Then } (M^n,g) \underset{\text{isom.}}{\cong} S^n\left(\tfrac{1}{\sqrt{K}}\right).}$$

Pf: Let $p,q \in M$ be points at maximal distance, i.e. $\operatorname{dist}(p,q) = \frac{\pi}{\sqrt{K}}$. Then, for all $r>0$, the balls $B_r(p)$ and $B_{\frac{\pi}{\sqrt{K}}-r}(q)$ are disjoint: if $d(p,x)<r$ and $d(x,q) < \frac{\pi}{\sqrt{K}} - r$, then



$$\frac{\pi}{\sqrt{K}} = d(p,q) \leq d(p,x) + d(x,q) < \frac{\pi}{\sqrt{K}}$$

so no such x can exist. Thus,

$$M \supseteq B_r(p) \,\dot{\cup}\, B_{\frac{\pi}{\sqrt{K}}-r}(q) \quad \text{(disjoint union)}$$

hence $\operatorname{Vol}(M) \overset{\circledast}{\geq} \operatorname{Vol}(B_r(p)) + \operatorname{Vol}\left(B_{\frac{\pi}{\sqrt{K}}-r}(q)\right)$. From Bishop Vol. Comp.,

$$r \longmapsto \frac{\operatorname{Vol}(B_r(x))}{\operatorname{Vol}(\overline{B_r})} \text{ is non increasing; in particular,}$$

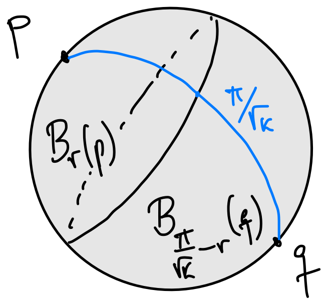
$$\frac{\operatorname{Vol}(B_r(x))}{\operatorname{Vol}(\overline{B_r})} \geq \frac{\operatorname{Vol}\left(B_{\frac{\pi}{\sqrt{K}}}(x)\right)}{\operatorname{Vol}\left(\overline{B_{\frac{\pi}{\sqrt{K}}}}\right)} = \frac{\operatorname{Vol}(M)}{\operatorname{Vol}\left(S^n(1/\sqrt{K})\right)} \quad \text{b/c} \quad \begin{cases} \overline{B_{\frac{\pi}{\sqrt{K}}}} = S^n(1/\sqrt{K}) \\ B_{\frac{\pi}{\sqrt{K}}}(x) = M \end{cases}$$

i.e. $\operatorname{Vol}(B_r(x)) \geq \dfrac{\operatorname{Vol}(M)}{\operatorname{Vol}(S^n(1/\sqrt{K}))}\operatorname{Vol}(\overline{B_r})$. Thus, applying this in $\circledast$:

$$\operatorname{Vol}(M) \geq \frac{\operatorname{Vol}(M)}{\operatorname{Vol}(S^n(1/\sqrt{K}))}\Big(\underbrace{\operatorname{Vol}(\overline{B_r}) + \operatorname{Vol}\left(\overline{B_{\frac{\pi}{\sqrt{K}}-r}}\right)}_{\operatorname{Vol}(S^n(1/\sqrt{K}))}\Big) = \operatorname{Vol}(M), \text{ so all}$$

the inequalities using Bishop Vol. Comp. above are <u>equalities</u>. Thus, from rigidity in the equality case of Bishop Vol. Comp., we have

$$B_r(p) \underset{\text{isom}}{\cong} \overline{B_r} \quad \text{and} \quad B_{\frac{\pi}{\sqrt{\kappa}}-r}(q) \underset{\text{isom}}{\cong} \overline{B_{\frac{\pi}{\sqrt{\kappa}}-r}}, \quad \text{thus} \quad M \underset{\text{isom}}{\cong} S^n(1/\sqrt{\kappa}).$$



$M \underset{\text{isom}}{\cong} S^n(1/\sqrt{\kappa})$

Indeed, there is no room for any $M \setminus \overline{(B_r(p) \cup B_{\frac{\pi}{\sqrt{\kappa}}-r}(q))}$ because that would increase the diameter. $\square$

<u>Open problem:</u> If (M^n, g) has $\mathrm{Ric} \geq (n-1)\kappa > 0$ and $\mathrm{Vol}(M,g) > \frac{1}{2}\mathrm{Vol}(S^n(1/\sqrt{\kappa}))$, then $M \underset{\substack{\text{homeo?}\\ \text{diffeo?}}}{\cong} S^n$.

<u>Exercise:</u> a) Find counter-example with $\mathrm{Vol}(M,g) = \frac{1}{2}\mathrm{Vol}(S^n(1/\sqrt{\kappa}))$.

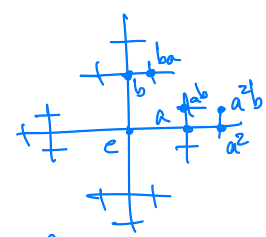
HW5

b) Prove that (M^n, g) as above is simply-connected.

<u>Hint:</u> if M is not simply connected, take its universal covering.

<u>Lecture 23</u> 5/1/2024

A quick taste of Geometric Group Theory:



Cayley graph of $F = \langle a, b \rangle$

- If Γ is finitely generated, fix a finite generating set G, with $e \in G$ and $G^{-1} = G$. Then define <u>growth function</u> for Γ:

$$N_k^G = \#\{g \in \Gamma : g = g_1 \cdots g_k, \text{ with } g_i \in G\}$$

In terms of the <u>Cayley graph</u> with the word metric, this is the cardinality of the closed ball of radius k around $e \in \Gamma$

\# of group elements that can be written as product of k generators in the fixed generating set G.

- If G' is another choice of generating set for Γ as above, then

$$N_k^{G'} \geq N_{Ck}^{G} \quad \text{and} \quad N_k^{G} \geq N_{Dk}^{G'} \quad \text{for some constants } C, D > 0,$$

so can ignore choice of gen. set G for questions below.

- Q: How does N_k grow with k? Polynomially? Exponentially?

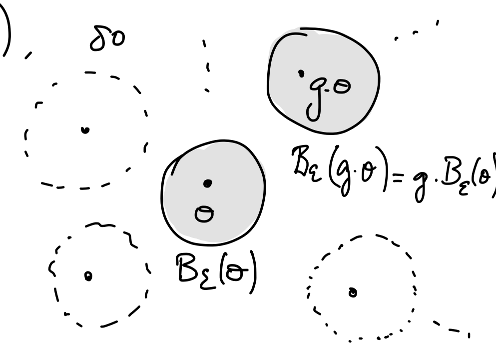
> **Thm** (Milnor '68). If (M^n, g) is complete and has $\mathrm{Ric} \geq 0$, then any finitely generated subgroup $\Gamma < \pi_1 M$ has $N_k \leq C \cdot k^n$.
>
> i.e., "polynomial growth"

Pf: Choose $o \in \widetilde{M}^n$, and let $V(r) = \mathrm{Vol}(B_r(o))$. By Bishop Volume Comp.,

$$V(r) \leq \mathrm{Vol}\left(B_r^{\mathbb{R}^n}(0)\right) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} r^n.$$

Let $G = \{g_1, \dots, g_p\}$ be the fixed generating set for $\Gamma < \pi_1 M$ and $\mu = \max_{1 \leq i \leq p} \mathrm{dist}(o, g_i \cdot o)$.

Then $B_{\mu \cdot k}(o)$ has at least N_k^G distinct points of the form $g \cdot o$, with $g \in \Gamma$. Choose $\varepsilon > 0$ s.t. $g \cdot B_\varepsilon(o) \cap B_\varepsilon(o) = \emptyset$ if $g \neq e$. Then $B_{\mu \cdot k + \varepsilon}(o)$ has at least N_k^G disjoint subsets of the form $g \cdot B_\varepsilon(o)$, so



$$N_k^G \cdot V(\varepsilon) = \mathrm{Vol}\Big(\coprod_{\substack{g = g_1 \cdots g_k \\ g_i \in G}} g \cdot B_\varepsilon(o)\Big) \leq V(\mu k + \varepsilon)$$

Thus

$$N_k^G \leq \frac{V(\mu k + \varepsilon)}{V(\varepsilon)} \overset{\text{Bishop}}{\leq} \frac{\widetilde{C}\,(\mu k + \varepsilon)^n}{V(\varepsilon)} \leq C \cdot k^n. \qquad \square$$

(recall: M compact $\Rightarrow$ $\pi_1 M$ is finitely generated)

> **Thm** (Milnor '68). If (M, g) is a closed Riem. mfld with $\sec < 0$, and $\pi_1 M = \langle G \rangle$, $|G| < \infty$, then $N_k^G \geq a^k$ for some $a > 1$.
>
> i.e., "exponential growth"

Ex: Fundamental group of hyperbolic manifold Σ^n has exponential growth; thus, cannot be π_1 of mfld w/ $\mathrm{Ric} \geq 0$.
So, cannot "improve" the above Thm to $\mathrm{scal} > 0$, as $\Sigma^2 \times S^{n-2}(\varepsilon)$ has $\mathrm{scal} > 0$ for $n \geq 4$ and $\varepsilon > 0$ suff. small, if Σ^2 is a hyperbolic surface.

Conjecture (Milnor, 1968). If (M^n, g) is complete and has $\mathrm{Ric} \geq 0$, then $\pi_1 M$ is finitely generated.

- For $n=3$, it was proven by [Liu, 2013] and indep. [Pan, 2017].

  Inventiones paper, uses minimal surfaces

  Crelle paper, uses Cheeger–Colding theory and $\mathrm{Ric} \geq 0$ limit spaces

- In November 2023, a counter-example (M^7, g) with $\mathrm{Ric} \geq 0$ and $\pi_1 M^7 = \mathbb{Q}/\mathbb{Z}$ was announced by Bruè–Naber–Semola, using a sophisticated gluing method to produce a "smooth fractal structure".

One of the founding achievements of Geometric Group Theory is:

Thm (Gromov '81) A finitely generated group Γ has polynomial growth if and only if Γ is virtually nilpotent ($\exists N \triangleleft \Gamma$ nilpotent with $[\Gamma : N] < \infty$.)

So, if $\Gamma < \pi_1 M$ is fin. gen. and M has $\mathrm{Ric} \geq 0$, then Γ is virtually nilpotent. Conversely, if Γ is fin. gen. and virtually nilpotent, then $\Gamma = \pi_1 M$ for some M with $\mathrm{Ric} \geq 0$ (Wilking '2000)

This paper also shows that if a counter-example M to Milnor's conjecture exists, then it has a covering space $\hat{M} \to M$ with $\pi_1 \hat{M}$ abelian and not fin. gen (e.g., $\pi_1 \hat{M} = \mathbb{Q}/\mathbb{Z}$.)

Stronger results about $\pi_1 M$ can be proven with stronger curvature assumptions:

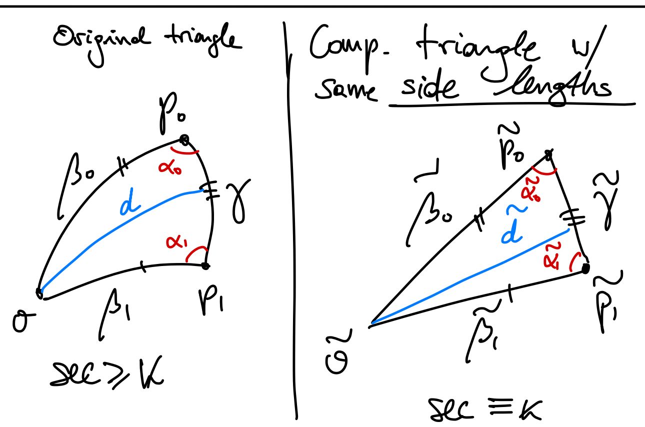
Toponogov Triangle Comparison

See, e.g., Eschenburg's Comparison Geometry Notes for a proof.

Here and throughout: if $K > 0$, then assume all lengths are $< \frac{\pi}{\sqrt{K}}$.

Triangle Version

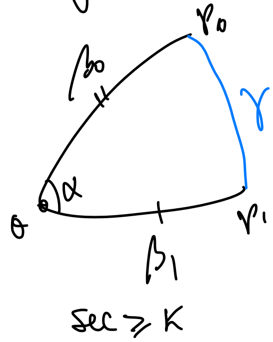
If (M^n, g) has $\sec \geq K$, $o, p_0, p_1 \in M$, $\gamma : [0, L] \to M$ geod from p_0 to p_1, β_i min. geod from o to p_i, then $d = \mathrm{dist}_g(o, \gamma(t)) \geq \tilde{d} = \mathrm{dist}_{\tilde{g}}(\tilde{o}, \tilde{\gamma}(t))$ for all $t \in [0, L]$ and $\alpha_i \geq \tilde{\alpha}_i$.



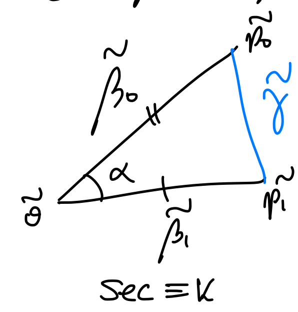
Hinge Version

If (M^n, g) has $\sec \geq K$, $o, p_0, p_1 \in M$, β_i min. geod. from o to p_i, Then $\ell(\gamma) \leq \ell(\tilde{\gamma})$; where $\gamma, \tilde{\gamma}$ are the min. geod. that close the hinge: $\ell(\gamma) = \mathrm{dist}_g(p_0, p_1)$, $\ell(\tilde{\gamma}) = \mathrm{dist}_{\tilde{g}}(\tilde{p}_0, \tilde{p}_1)$.

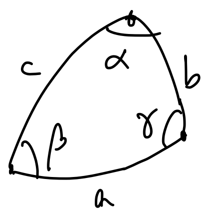
Original triangle



Comp. triangle w/ same hinge: $\ell(\beta_i) = \ell(\tilde{\beta}_i)$, $\alpha = \tilde{\alpha}$



Corollary: A geodesic triangle on a manifold with $\sec \geq 0$ satisfies



(i) $\ell(c)^2 \leq \ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma$ $\quad \ell =$ length

(ii) $\alpha + \beta + \gamma \geq \pi$ $\quad$ If $\sec > 0$, then get strict inequalities.

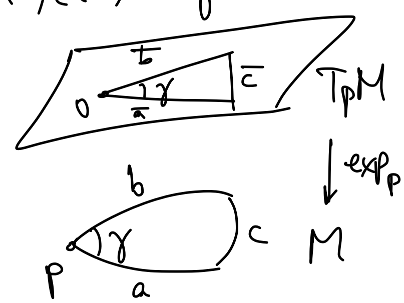
Pf: (i) is immediate:

$$\ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma \overset{\text{Gauss Lemma}}{=} \ell(\bar{a})^2 + \ell(\bar{b})^2 - 2\ell(\bar{a})\ell(\bar{b})\cos\gamma$$

$$\overset{\text{Law of Cosines in } \mathbb{R}^2}{=} \ell(\bar{c})^2$$

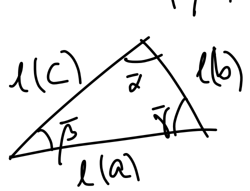
$$\overset{\text{Toponogov (Hinge)}}{\geq} \ell(c)^2.$$

where $a = \exp_p \bar{a}$, $b = \exp_p \bar{b}$, $c = \exp_p \bar{c}$.



(ii) Follows from (i) as in the $\sec \leq 0$ case: build comparison triangle in $\mathbb{R}^2$ with side lengths $\ell(a), \ell(b), \ell(c)$, and angles $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$. Then $\ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma \geq \ell(c)^2$

$$= \ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\bar{\gamma}$$



So $\cos\gamma \leq \cos\bar{\gamma}$ hence $\gamma \geq \bar{\gamma}$. Similarly for α, β and get

$$\alpha + \beta + \gamma \geq \bar{\alpha} + \bar{\beta} + \bar{\gamma} = \pi. \qquad \square$$

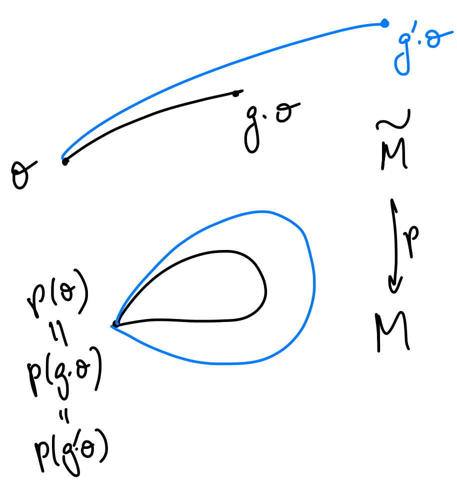
Combining above w/ earlier work on $\sec \leq 0$

As before...

Cor: (M^n, g) has $\sec \geq 0$ (≤ 0) iff $\forall p \in M$, $\exp_p : C_p \subset T_pM \xrightarrow{\cong} M$ is distance non-increasing (non-decreasing).

Thm (Gromov 1978). If (M^n, g) has $\sec \geq 0$, then $\pi_1 M$ can be generated by $\leq \sqrt{2n\pi}\cdot 2^{n-2}$ elements. If (M^n,g) has $\sec \geq -K^2$ and $\operatorname{diam}(M) \leq D$, then $\pi_1 M$ can be generated by $\leq \frac{1}{2}\sqrt{2n\pi}\,\big(2+2\cosh(2KD)\big)^{\frac{n-1}{2}}$. ← Note: If $K \to 0$, then this becomes $\sqrt{2n\pi}\cdot 2^{n-2}$.

Pf: (Case $K=0$). Fix $o \in \widetilde{M}$ and consider the isometric action of $\Gamma = \pi_1 M$, by deck transformations. Define displacement of $g \in \Gamma$: $|g| = \operatorname{dist}(o, g\cdot o)$.



Clearly, a min. geod. from o to $g\cdot o$ in $\widetilde{M}$ projects to geodesic loop based at $p(o) \in M$, which has minimal length in its homotopy class. For any given $R>0$, there are only finitely many $g \in \Gamma$ with $|g| \leq R$, because otherwise an infinite seq. $g_i \in \Gamma$ with $|g_i| \leq R$ would produce an infinite seq. $g_i \cdot o$ of points in $B_R(o)$, which has a limit and contradicts the covering property.

Thus, we can define $g_1 \in \Gamma$ s.t. $|g_1| = \min\limits_{g \in \Gamma} |g|$, and $g_2 \in \Gamma$ with $|g_2| = \min\limits_{g \in \Gamma \setminus \langle g_1 \rangle} |g|$; inductively, define a sequence $g_1, g_2, \ldots \in \Gamma$ of generators ← "short basis" with $|g_1| \leq |g_2| \leq \ldots$ and $|g_{i+1}| = \min\limits_{g \in \Gamma \setminus \langle g_1, \ldots, g_i\rangle} |g|$. (Keep adding elements g_i until a set of generators is achieved!)

Set $\ell_{ij} = \operatorname{dist}(g_i \cdot o, g_j \cdot o)$ for all $i<j$. Then $\ell_{ij} \geq |g_j|$,

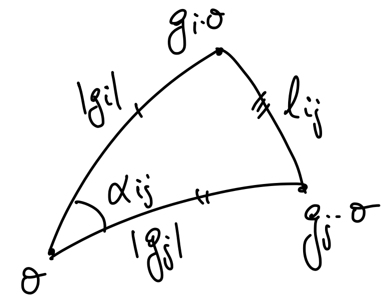
for otherwise $\bar{g} = g_i^{-1} g_j$ would have

$$|\bar{g}| = \ell_{ij} < |g_j| \qquad \text{and} \qquad \langle g_1, \ldots g_i, \ldots, g_j \rangle = \langle g_1, \ldots, g_i, \ldots, \bar{g} \rangle$$

hence contradict the min. choice of g_j above.

Note that all sides of the triangles $o, g_i\cdot o, g_j \cdot o$ are min geodesics.

By Toponogov, applied to the triangle $g_i \cdot o, g_j \cdot o, o$, we have that $\alpha_{ij} \geq \tilde{\alpha}_{ij}$.



sec ≥ 0

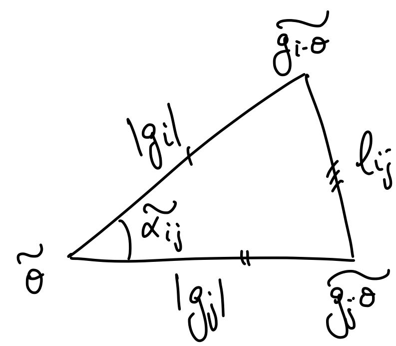
Law of cosines in $\mathbb{R}^2$:

$$l_{ij}^2 = |g_i|^2 + |g_j|^2 - 2|g_i||g_j| \cos \tilde{\alpha}_{ij}$$

$$\Rightarrow \quad \cos(\tilde{\alpha}_{ij}) = \frac{|g_i|^2 + |g_j|^2 - l_{ij}^2}{2|g_i| \cdot |g_j|}$$

$|g_i| \leq |g_j| \leq l_{ij}$ if $i<j$

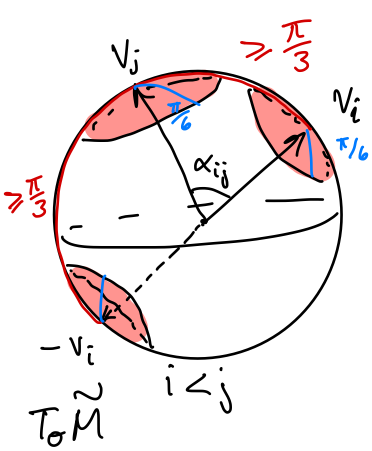
$$\leq \frac{|g_i|^2 + \cancel{(|g_j|^2 - |g_j|^2)}}{2 \cdot |g_i|^2} = \frac{1}{2}$$



sec $\equiv 0$

$$\Rightarrow \quad \alpha_{ij} \geq \tilde{\alpha}_{ij} \geq \frac{\pi}{3}.$$

Let $v_i \in T_o \tilde{M}$ be the unit vector tangent to the min. geod. from o to $g_i \cdot o$. By the above, the distance (on the unit sphere in $T_o \tilde{M}$) between v_i and v_j is $\alpha_{ij} \geq \frac{\pi}{3}$, so the balls of radius $\frac{\pi}{6}$ centered at v_i and v_j must be disjoint. (This already proves there can be only finitely many v_i's, hence finitely many g_i's so $\Gamma = \pi_1 M$ is finitely generated.) Moreover, as $|g_i^{-1}| = |g_i|$, we must also have that distance from $-v_i$ to v_j is $\geq \frac{\pi}{3}$ if $i<j$, therefore the number of v_i's is:



$$\#\{g_i\} = \#\{v_i\} \leq \frac{Vol(\mathbb{RP}^{n-1}(1))}{Vol(B^{S^{n-1}}_{\pi/6}(v))}$$

Volume of the set of $\pm v \in S^{n-1} \subset T_o \tilde{M}$.

Volume of each disjoint ball around $\pm v_i \in S^{n-1}$.

Standard computations give:

Volume of spherical ball of radius r is $\geq$ volume of Euclidean ball of radius $\sin r$. $(0 < r < \pi/2)$

- $$\mathrm{Vol}\left(B^{S^{n-1}}_{\pi/6}(v)\right) \geq \mathrm{Vol}\left(B^{\mathbb{R}^{n-1}}_{\underbrace{\sin \pi/6}_{=1/2}}(0)\right) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)2^{n-1}}$$

(Γ = Gamma function)

- $$\mathrm{Vol}\left(\mathbb{R}P^{n-1}(1)\right) = \frac{1}{2}\mathrm{Vol}\left(S^{n-1}(1)\right) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$$

log-concavity of Γ:
$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \leq \sqrt{\frac{n}{2}}$$

So $$\#\{g_i\} = \#\{v_i\} \leq \frac{\pi^{n/2}\,\Gamma\left(\frac{n+1}{2}\right)2^{n-1}}{\Gamma\left(\frac{n}{2}\right)\cdot \pi^{\frac{n-1}{2}}} = \sqrt{\pi}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\,2^{n-1} \leq \sqrt{2n\pi}\cdot 2^{n-2}.$$

For case $\sec \geq -k^2$, see Eschenburg's notes. $\square$

Using Bishop Volume Comparison, Toponogov Triangle Comparison, Critical point theory for distance functions and topological constructions, Gromov proved the following:

Thm (Gromov '1981).
i) If (M^n, g) is a complete mfld with $\sec \geq 0$, then $\sum_{k=0}^{n} b_k(M) \leq C(n)$.
ii) If (M^n, g) is a closed mfld with $\sec \geq -k^2$ and $\mathrm{diam} \leq D$, then $\sum_{k=0}^{n} b_k(M) \leq C(n)^{1+kD}$.

Cannot replace the hypothesis $\sec \geq 0$ to $\mathrm{Ric} > 0$ because:

"Docking station"

Thm (Sha-Yang '90s). $\forall \ell \in \mathbb{N}$, $\#^\ell S^2 \times S^2$ and $\#^k \mathbb{C}P^2 \#^\ell \overline{\mathbb{C}P}^2$ have $\mathrm{Ric} > 0$.

also $\#^\ell S^n \times S^m$ for any $n, m \geq 2$, $\ell \geq 1$.

Thm. (Perelman '97). $\forall \ell \in \mathbb{N}$, $\#^\ell \mathbb{C}P^2$ has a metric with $\mathrm{Ric} > 0$, $\mathrm{diam} = 1$ and $\mathrm{Vol} \geq V > 0$.

Thus, since $b_2(\#^\ell S^2 \times S^2) = 2\ell$ and $b_2(\#^k \mathbb{C}P^2 \#^\ell \overline{\mathbb{C}P}^2) = k + \ell$, only finitely many of these manifolds can have $\sec \geq 0$. Currently,

Note: $\mathrm{scal} > 0$ is preserved by $\#$; indeed by surgeries of codimension ≥ 3.

- only S^4 and $\mathbb{C}P^2$ are known to have $\sec > 0$ and
- only $S^2 \times S^2$ and $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$ are known to have $\sec \geq 0$.

⚠ Related open question: is there a simply-connected closed mfld that admits $\mathrm{scal} > 0$ but does not admit $\mathrm{Ric} > 0$?

Conjecturally, the above is the complete list of simply-connected 4-mflds with $\sec > 0$ and $\sec \geq 0$.

(double disk: )

Note: As $\ell \nearrow +\infty$, Perelman's $\#^\ell \mathbb{C}P^2$ converges to $B^4 \cup B^4$ flat

Lecture 24 5/3/2024

Lie Groups and Lie Algebras

Mostly taken from my book w/ Alexandrino: "Lie groups and geometric aspects of isometric actions" Chapters 1, 2.

Def. A manifold G is a Lie group if it is a group, and the maps

$$G\times G \ni (g,h) \longmapsto gh \in G, \qquad G\ni g \longmapsto g^{-1}\in G$$

are smooth. Equivalently, the map $G\times G\ni (g,h)\longmapsto g\cdot h^{-1}\in G$ is smooth.

Rmk: Equivalently, could define requiring only $(g,h)\mapsto g\cdot h$ is smooth; inversion is then also smooth by Inverse Function Theorem.

Ex: $(\mathbb{R},+)$, $(S^1,\cdot)$

Examples (Classical Lie groups).

$GL(n,F)=\{A\in M_{n\times n}(F) : \det A\neq 0\}$, $\quad F\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$

$SL(n,F)=\{A\in GL(n,F) : \det A = 1\}$

$O(n)=\{A\in GL(n,\mathbb{R}) : A^TA = Id\}$ orthogonal group

$SO(n) = O(n)\cap SL(n,\mathbb{R})$

conjugate transpose (adjoint)

$U(n)=\{A\in GL(n,\mathbb{C}) : A^*A = Id\}$ unitary group

$SU(n) = U(n)\cap SL(n,\mathbb{C})$

$Sp(n)=\{A\in GL(n,\mathbb{H}) : A^*A = Id\}$ symplectic group.

Def. A vector space $\mathfrak{g}$ is a Lie algebra if it is endowed with a bilinear skewsymmetric map $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, s.t. $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$. $\forall X,Y,Z\in\mathfrak{g}$.

(i.e., $[X,Y]=-[Y,X]$)

Ex. $\mathbb{R}^n$ with $[\cdot,\cdot]\equiv 0$.

$\mathfrak{gl}(n,F)=M_{n\times n}(F)$, $\quad [A,B]=AB-BA$

$\mathfrak{sl}(n,F)=\{X\in\mathfrak{gl}(n,F) : \operatorname{tr} X = 0\}$

$\mathfrak{o}(n)=\mathfrak{so}(n)=\{X\in\mathfrak{gl}(n,\mathbb{R}) : X^T+X=0\}$

$\mathfrak{u}(n)=\{X\in\mathfrak{gl}(n,\mathbb{C}) : X^*+X=0\}$

$\mathfrak{su}(n)=\mathfrak{u}(n)\cap\mathfrak{sl}(n,\mathbb{C})$

$\mathfrak{sp}(n)=\{X\in\mathfrak{gl}(n,\mathbb{H}) : X^*+X=0\}$.

Def: Left translation $L_g : G \longrightarrow G$, $x \longmapsto g\cdot x$, Right translation $R_g : G \longrightarrow G$, $x \longmapsto xg$ are both smooth and have smooth inverses $L_{g^{-1}}$ and $R_{g^{-1}}$, hence are diffeomorphisms.

Def. A tensor X over G, e.g., a vector field, k-form, etc, is called <u>left-invariant</u> if $dL_g X = X \circ L_g$, i.e. $X(gh) = d(L_g)_h X(h)$, and similarly <u>right-invariant</u> if $dR_g X = X \circ R_g$. If it is simultaneously left and right invariant, then it is called <u>bi-invariant</u>.

Prop. A left (or right) invariant tensor is smooth.

Pf. (Vector fields) $\mu : G \times G \longrightarrow G$, $\mu(g,h) = g\cdot h$ is smooth, and so is $s : G \longrightarrow TG \times TG$, $s(g) = (0_g, X(e))$, where $g \longmapsto 0_g$ is the zero section of TG. Then $X = d\mu \circ s$ is a composition of smooth maps, hence smooth, and all left-invariant vector fields are of that form.

Def: A group homomorphism $\varphi : G \longrightarrow H$ is a Lie group homomorphism if it is smooth (actually, continuity $\Longrightarrow$ smoothness). A linear transf $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a Lie algebra homomorphism if $[\phi(X), \phi(Y)] = \phi([X,Y])$.

Prop. Given a Lie group G, let $\mathfrak{g} = \{X \in \mathfrak{X}(G) : X \text{ left-invariant}\}$.

(i) $\mathfrak{g}$, endowed with Lie bracket of vector fields, is a Lie algebra.

(ii) Endow T_eG with the Lie bracket $[X,Y] := [\tilde{X}, \tilde{Y}]_e$, where $\tilde{X} = d(L_g)_e X$, $\tilde{Y} = d(L_g)_e Y$. Then $(T_eG, [\cdot,\cdot])$ is a Lie algebra, and the evaluation map $\mathrm{eval}_e : \mathfrak{g} \longrightarrow T_eG$ is a Lie algebra isomorphism.

usually write just dL_g for simplicity

Pf. (i) is straightforward from linearity of $d(L_g)_e$ and noticing X is left-invariant iff it is L_g-related to itself

(ii) eval_e is clearly linear and injective; if $X_e = Y_e$, then $X_g = dL_g X_e = dL_g Y_e = Y_g$, $\forall g$ so $X = Y$. It is also clearly surjective: given $v \in T_eG$, $X = dL_g v \in \mathfrak{g}$ has $X_e = v$. By its definition, it preserves brackets, so it is a Lie algebra isom. $\square$

Def. The Lie algebra of the Lie group G is the above $\mathfrak{g} = T_eG$.

Ex: If (M^n, g) is a complete Riem. mfld, then $G = \mathrm{Isom}(M, g)$ is a Lie group (Myers–Steenrod) and its Lie algebra is the space of Killing vector fields $\mathfrak{g} = \{X \in \mathfrak{X}(M) : \nabla X \text{ skew symmetric}\}$.

Lie group G $\rightsquigarrow$ Lie algebra $\mathfrak{g}$

converse?

Lie's Third Theorem. Let $\mathfrak{g}$ be a (finite-dim.) Lie algebra. There exists a unique connected, simply-connected, Lie group G with Lie algebra isomorphic to $\mathfrak{g}$.

Def. A Lie subgroup $H < G$ is a subgroup which is an immersed submanifold and $H \times H \ni (x, y) \longmapsto xy^{-1} \in H$ is smooth.

Lemma. If $\varphi : G_1 \longrightarrow G_2$ is a Lie gp. homomorphism, given $X \in \mathfrak{g}_1$, there is a unique $Y \in \mathfrak{g}_2$ that is φ-related to X.

Pf. Define $Y = dL_g\, d\varphi_e X_e$. Then:

$$d\varphi_g X_g = d\varphi_g\, dL_g X_e = d(\varphi \circ L_g)_e X_e$$

$$= d(L_{\varphi(g)} \circ \varphi)_e X_e = dL_{\varphi(g)}\, d\varphi_e X_e = Y_{\varphi(g)}$$

Uniqueness follows from left-invariance of Y.

Prop. If $\varphi: G_1 \to G_2$ is a Lie gp homomorphism, then $d\varphi: \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie alg. homomorphism.

Pf. $X_1, X_2 \in \mathfrak{g}_1 \overset{\text{Lemma}}{\rightsquigarrow} Y_1, Y_2 \in \mathfrak{g}_2$ are the left-inv. extensions of $d\varphi_e X_1$, $d\varphi_e X_2$, so, since Lie brackets of φ-related vector fields are φ-related,

$$Y_i \circ \varphi = d\varphi_e X_i \quad \Rightarrow \quad [d\varphi_e X_1, d\varphi_e X_2]_e = [Y_1, Y_2]_e = d\varphi_e([X_1, X_2]).$$

$\square$

Cor. If $H < G$ is a Lie subgroup, $i: H \hookrightarrow G$ is an injective Lie gp. homomorphism that induces an isomorphism di_e from the Lie algebra $\mathfrak{h}$ of H and the Lie subalgebra $di_e(\mathfrak{h})$ of $\mathfrak{g}$.

Thm. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and $\mathfrak{h} < \mathfrak{g}$ a Lie subalgebra. Then there exists a unique connected Lie subgroup $H < G$ with Lie algebra $\mathfrak{h}$.

Pf. The distribution $D_g = \{X_g = dL_g X \text{ for } X \in \mathfrak{h}\}$ of G is involutive b/c $\mathfrak{h}$ is a Lie algebra. By Frobenius, it is integrable; let H be the leaf through $e \in G$. Since D is left-invariant, L_g maps leaves to leaves, so $L_{h^{-1}}(H) = H$ for all $h \in H$, i.e., H is a subgroup.

$\square$

Uniqueness follows from:

Lemma: Let $G_0 < G$ be the connected component of $e \in G$. Then $G_0 \triangleleft G$ and connected components of G are of the form $g \cdot G_0$ for some $g \in G$. Moreover, $\forall\, U \ni e$ open neighborhood, $G_0 = \bigcup_{n \in \mathbb{N}} \{g_1^{\pm 1} \cdots g_n^{\pm 1} : g_i \in U\}$.

Pf. See my book w/ Alexandrino.

Thm. If G is a connected Lie gp, there is a unique simply connected Lie group $\tilde{G}$ and a Lie gp homomorphism $\pi : \tilde{G} \to G$ which is a covering map.

E.g., $\mathbb{R}^n \to T^n$, $SU(2) \to SO(3)$, $SU(2) \times SU(2) \to SO(4)$

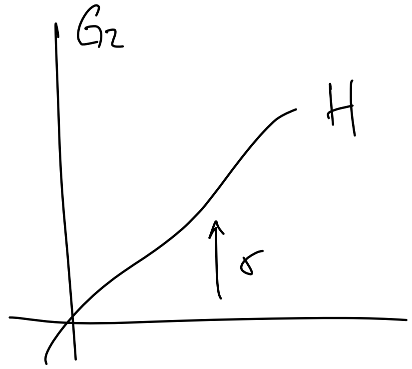
$Sp(2) \to SO(5)$, $SU(4) \to SO(6)$

Prop. A Lie gp homomorphism $\pi : G_1 \to G_2$ between connected groups is a covering map iff $d\pi_e : \mathfrak{g}_1 \to \mathfrak{g}_2$ is an isomorphism.

Pf. A covering map is a local diffeo, so $\Rightarrow$ is clear.
Conversely, if $d\pi_e$ is an isom., by Inv. Funct. Thm., $\exists U \subset G_1$ and $V \subset G_2$ neighborhoods of the identity, s.t. $\pi|_U : U \to V$ is a diffeo. By Lemma, given $h \in G_2$, $h = h_1^{\pm 1} \cdots h_n^{\pm 1}$ w/ $h_i \in V$, and $\exists g_i \in U$ s.t. $\pi(g_i) = h_i$ so $\pi(g_1^{\pm 1} \cdots g_n^{\pm 1}) = h_1^{\pm 1} \cdots h_n^{\pm 1} = h$, so π is a surjective homomorphism. One then easily checks it is a covering map with deck transf. gp. $\operatorname{Ker} \pi$. $\square$

Lemma. If $\varphi, \psi : G_1 \to G_2$ are Lie gp homomorphisms, G_1 connected, and $\theta : \mathfrak{g}_1 \to \mathfrak{g}_2$ a Lie algebra homomorphism, $d\varphi_e = d\psi_e = \theta$, then $\varphi = \psi$.

Pf. Consider the graph of θ, $\mathfrak{h} := \{(X, \theta(X)) : X \in \mathfrak{g}_1\}$, which is a Lie subalgebra of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. By a Theorem above, there exists a unique connected Lie subgroup H of $G_1 \times G_2$ with Lie algebra $\mathfrak{h}$.



Then $\sigma: G_1 \longrightarrow G_1 \times G_2$

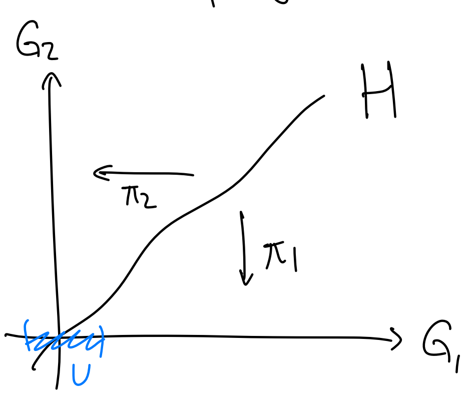
$g \longmapsto (g, \varphi(g))$

is a Lie gp. homom. with $d\sigma(X) = (X, \theta(X))$, a Lie alg. homom., and $\sigma(G_1) \subset G_1 \times G_2$ Lie subgroup with Lie algebra $\mathfrak{h}$. By uniqueness, $\sigma(G_1) = H$. So replacing φ with ψ, if $d\varphi_e = d\psi_e = \theta$, we would obtain the same subgroup of $G_1 \times G_2$, which is the graph of the homomorphism $\varphi = \psi$. $\square$

Thm. If $\theta: \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ is a Lie algebra homomorphism, and G_1 is connected and simply connected, then $\exists \varphi: G_1 \longrightarrow G_2$ unique Lie group homomorphism w/ $d\varphi_e = \theta$.

This uniqueness is crucial in what follows; later references to "uniqueness" point here.

Pf. Let $\mathfrak{h} = \text{graph } \theta$ and $H \subset G_1 \times G_2$ its Lie group, as above.



Then $(\pi_1 \circ i): H \longrightarrow G_1$ is a Lie gp homomorphism, $d(\pi_1 \circ i)_e (X, \theta(X)) = X$, so it is locally invertible near the identity:

$$(\pi_1 \circ i)^{-1}: \underset{G_1}{\overset{\cap}{U}} \longrightarrow \underset{H}{\overset{\cap}{V}} \text{ is a diffeo.}$$

Let $\varphi = \pi_2 \circ (\pi_1 \circ i)^{-1}: U \longrightarrow G_2$. Then φ is a local homomorphism with $d\varphi_e = \theta$, since

$$d\varphi_e X = d\left(\pi_2 \circ (\pi_1 \circ i)^{-1}\right)_e X = d\pi_2 \left(d(\pi_1 \circ i)^{-1} X\right)$$

$$= d\pi_2 (X, \theta(X)) = \theta(X).$$

Since $d(\pi_1 \circ i)$ is an isom., $(\pi_1 \circ i): H \to G_1$ is a covering map. As G_1 is simply-connected, $\pi_1 \circ i$ is a diffeomorphism so can be globally inverted, hence $\varphi: G_1 \to G_2$ can be globally defined as $\varphi = \pi_2 \circ (\pi_1 \circ i)^{-1}$. Uniqueness follows from Lemma. $\square$

<u>Cor.</u> G_1, G_2 connected, simply-connected, $\theta: \mathfrak{g}_1 \to \mathfrak{g}_2$ isomorphism then $\exists!\, \varphi: G_1 \to G_2$ isomorphism s.t. $d\varphi_e = \theta$; i.e., Lie groups G_1 and G_2 as above are <u>isomorphic</u> if and only if their Lie algebras are isomorphic.

<u>Def.</u> A Lie group homomorphism $\lambda: \mathbb{R} \to G$ is called a 1-parameter subgroup of G.

By the above theorems, the Lie algebra homomorphism $\theta: \mathbb{R} \to \mathfrak{g}$, $\theta(t) = tX$ (for some fixed $X \in \mathfrak{g}$) "integrates" to a unique 1-param. subgroup $\lambda_X: \mathbb{R} \to G$ s.t. $\lambda_X'(0) = X$, which is also the integral curve through $e \in G$ of the left-invariant vector field X:

$$\lambda_X'(t) = \frac{d}{ds} \lambda_X(t+s)\Big|_{s=0} = dL_{\lambda_X(t)} \lambda_X'(0) = dL_{\lambda_X(t)} X = X_{\lambda_X(t)}$$

<u>Def.</u> The (Lie) exponential of the Lie group G is

$$\exp: \mathfrak{g} \to G, \quad \exp(X) = \lambda_X(1)$$

<u>Rmk:</u> If $G < GL(n, \mathbb{F})$ is a matrix Lie group, then $\exp: \mathfrak{g} \to G$ coincides with matrix exponentiation $\exp X = e^X = \sum_{k=0}^{+\infty} \frac{X^k}{k!}$.

Lecture 25 5/8/2024

Recap last lecture.

Prop: $\exp : \mathfrak{g} \to G$ satisfies the following properties

(i) $\exp(tX) = \lambda_X(t)$

(ii) $\exp(-tX) = \exp(tX)^{-1}$

(iii) $\exp(t_1X + t_2X) = \exp t_1X \cdot \exp t_2X$

(iv) $\exp : T_eG \to G$ is smooth and $d(\exp)_0 = id$; hence exp is a local diffeo from neighborhood of $0 \in T_eG$ to neighborhood of $e \in G$.

Pf. Let $\lambda(s) = \lambda_X(st)$. Differentiating at $s=0$, we have

$$\lambda'(0) = \frac{d}{ds}\lambda_X(st)\Big|_{s=0} = \lambda_X'(0)\,t = tX.$$

Thus, by uniqueness of the 1-param. subgroup with initial velocity tX, we have $\lambda_X(st) = \lambda_{tX}(s)$. Setting $s=1$ we obtain (i). Since $\lambda_X : \mathbb{R} \to G$ is a Lie group homomorphism, (ii), (iii) follow. See book for pf of (iv), similar to Riem. case.

Prop. The exponential map $\exp : \mathfrak{gl}(n,F) \to GL(n,F)$ coincides with matrix exponentiation $\exp X = e^X = \sum_{k=0}^{+\infty} \frac{X^k}{k!}$.

Pf. The series $e^X = \sum_{k=0}^{+\infty} \frac{X^k}{k!}$ converges uniformly in compacts of $\mathfrak{gl}(n,F)$, and $[X,Y]=0$ if and only if $e^{X+Y} = e^X \cdot e^Y$. Let $\lambda : \mathfrak{gl}(n,F) \to GL(n,F)$, $\lambda(t) = e^{tX}$. Each entry in the matrix e^{tX} is a power series with infinite radius of convergence, so λ is smooth. Differentiating term-by-term, $\lambda'(0) = X$. Since λ is a 1-parameter subgroup of $GL(n,F)$, by uniqueness it follows that $\lambda(t) = \exp tX$. $\square$

Prop. If $\varphi: G_1 \to G_2$ is a Lie gp homomorphism, then

$$\begin{tikzcd} \mathfrak{g}_1 \arrow[r, "d\varphi"] \arrow[d, "\exp^1"'] & \mathfrak{g}_2 \arrow[d, "\exp^2"] \\ G_1 \arrow[r, "\varphi"'] & G_2 \end{tikzcd}$$

commutes.

Pf. $\alpha(t) = \varphi(\exp^1(tX))$ and $\beta(t) = \exp^2(d\varphi_e\, tX)$ are 1-param. subgps of G_2 and $\alpha'(0) = \beta'(0) = d\varphi_e X$, so $\alpha = \beta$ by uniqueness.

Cor. If $H < G$, then $i \circ \exp^H = \exp^G \circ\, di_e$ where $i: H \hookrightarrow G$ is inclusion.

Campbell - Baker - Hausdorff formulas:

$$\exp(tX)\exp(tY) = \exp\left(t(X+Y) + \frac{t^2}{2}[X,Y] + O(t^3)\right)$$

$$\exp(tX)\exp(tY)\exp(-tX) = \exp\left(tY + t^2[X,Y] + O(t^3)\right)$$

Pf. See Spivak vol 1.

Def. A representation of a group G on a vector space V is a group homomorphism $\varphi: G \to \mathrm{Aut}(V)$.

Def. The adjoint representation Ad of a Lie group G on its Lie algebra $\mathfrak{g}$ is $Ad: G \to \mathrm{Aut}(\mathfrak{g})$, $Ad(g)X = d(a_g)_e X$, where $a_g: G \to G$ is conjugation by g, i.e., $a_g(x) = gxg^{-1}$.

By the Chain rule, we have $Ad(g) = d(L_g)_{g^{-1}} \circ d(R_{g^{-1}})_e$.

Note that

$$Ad(g)X = \frac{d}{dt} a_g(\exp tX)\Big|_{t=0} = \frac{d}{dt}\, g \exp(tX)\, g^{-1}\Big|_{t=0}$$

so, since exp and Lie gp/alg. homomorphisms commute,

$$\exp(t\,Ad(g)X) = a_g(\exp tX) = g \cdot \exp tX\, g^{-1}$$

so, setting $t=1$, $\quad \exp(\mathrm{Ad}(g)X) = g\cdot\exp X\cdot g^{-1}$

Differentiating again, we have: $\mathrm{ad}(X):\mathfrak{g}\to\mathfrak{g}$, $\mathrm{ad}(X)\,Y = d\,\mathrm{Ad}_e(X)\,Y$ which is a Lie algebra representation $\mathrm{ad}:\mathfrak{g}\to\mathrm{End}(\mathfrak{g})$.

By the Chain Rule,

$$\mathrm{ad}(X)Y = \frac{d}{dt}\mathrm{Ad}(\exp tX)\,Y\Big|_{t=0}$$

Since exp and Lie gp/alg. homomorphisms commute,

$$\mathrm{Ad}(\exp(tX)) = \exp(t\,\mathrm{ad}(X))$$

so, setting $t=1$, we see that the following diagram commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"'] & \mathrm{End}(\mathfrak{g}) \arrow[d, "\exp"] \\ G \arrow[r, "\mathrm{Ad}"] & \mathrm{Aut}(\mathfrak{g}) \end{tikzcd}$$

note $\mathrm{Aut}(\mathfrak{g})\subset GL(\mathfrak{g})$ is the closed subgroup given by isomorphisms which are Lie algebra isomorphisms (i.e., preserve the bracket), thus it is a Lie subgroup of $GL(\mathfrak{g})$, with Lie algebra $\mathrm{End}(\mathfrak{g})$.

Prop. $\mathrm{ad}(X)Y = [X,Y]$.

Pf. By CBH formulas, $\exp(tX)\exp(tY)\exp(-tX) = \exp\left(tY + t^2[X,Y] + O(t^3)\right)$, so with $g=\exp(tX)$,

$$\exp\left(\mathrm{Ad}(g)\,tY\right) = g\cdot\exp(tY)\cdot g^{-1} = \exp\left(tY + t^2[X,Y]+O(t^3)\right)$$

Since exp is locally injective near $0\in\mathfrak{g}$,

$$\mathrm{Ad}(g)\,tY = tY + t^2[X,Y] + O(t^3)$$

so dividing by t and differentiating at $t=0$, we have

$$\mathrm{ad}(X)Y = \frac{d}{dt}\mathrm{Ad}(\exp tX)\,Y\Big|_{t=0} = \frac{d}{dt}\,Y + t[X,Y] + O(t^3)\Big|_{t=0} = [X,Y]. \qquad \square$$

Def. The center of a Lie gp G is $Z(G) = \{g \in G : ghg^{-1} = h, \forall h \in G\}$ and the center of a Lie alg. $\mathfrak{g}$ is $Z(\mathfrak{g}) = \{X \in \mathfrak{g} : [X,Y] = 0 \ \forall Y \in \mathfrak{g}\}$.

Prop: If G is connected, then $Z(G) = \operatorname{Ker} \operatorname{Ad}$ is a normal Lie subgroup of G, with Lie algebra $Z(\mathfrak{g}) = \operatorname{Ker} \operatorname{ad}$.

Pf. If $g \in Z(G)$, then $a_g = id$ so $\operatorname{Ad}(g) = id$. Conversely, if $g \in \operatorname{Ker} \operatorname{Ad}$, then $g(\exp tX)g^{-1} = \exp(tX)$ for all $X \in \mathfrak{g}$, so g commutes with all elements in a neighborhood of $e \in G$, hence with all elements in $G_0 = G$. Since $Z(G) \trianglelefteq G$ is closed, it is an embedded Lie subgroup. Since $d(\operatorname{Ad})_e = \operatorname{ad}$, it follows that its Lie algebra is $\operatorname{Ker} \operatorname{ad}$. $\square$

Rmk: If $\pi : \hat{G} \to G$ is a covering of connected Lie gps, then $\operatorname{Ker} \pi$ is a discrete subgroup of $Z(\hat{G})$.

Lecture 26 5/10/2024

Def. A Riem. metric $\langle \cdot, \cdot \rangle$ on a Lie group G is *left-invariant* if $L_g : G \to G$ is an isometry $\forall g \in G$, i.e. $\forall g, h \in G$, $\forall X, Y \in T_h G$,

$$\langle d(L_g)_h X, d(L_g)_h Y \rangle_{gh} = \langle X, Y \rangle_h$$

Similarly, it is right-invariant if $R_g : G \to G$ is an isometry $\forall g \in G$. Note that an inner product $\langle \cdot, \cdot \rangle_e$ on $T_e G$ defines a unique left-invariant metric on G: $\langle X, Y \rangle_g = \langle d(L_{g^{-1}})_g X, d(L_{g^{-1}})_g Y \rangle_e$

A metric on G is bi-invariant if it is left- and right-invariant.

Prop. Compact Lie groups admit bi-invariant metrics.

Pf. Let $\omega \in \Omega^n(G)$ be a right-invariant volume form, e.g., obtained by setting $\omega_g := R_{g^{-1}}^* \omega_e$, for a given volume form $\omega_e \in \Lambda^n T_eG$; i.e., $\omega_g(X_1, \dots, X_n) = \omega_e(d(R_{g^{-1}})_g X_1, \dots, d(R_{g^{-1}})_g X_n)$. Let $\langle \cdot, \cdot \rangle$ be a right-invariant metric, e.g., $\langle X, Y \rangle_g = \langle d(R_{g^{-1}})_g X, d(R_{g^{-1}})_g Y \rangle_e$ for an arbitrary inner product $\langle \cdot, \cdot \rangle$ on T_eG. Define $\forall X, Y \in T_xG$

$$Q : T_xG \times T_xG \to \mathbb{R}, \qquad Q(X,Y)_x = \int_G \langle dL_g X, dL_g Y \rangle_{gx} \, \omega.$$

Then Q is left-invariant because, setting $f(g) := \langle dL_g X, dL_g Y \rangle_{gx}$,

$$Q(dL_h X, dL_h Y)_{hx} = \int_G \langle dL_g dL_h X, dL_g dL_h Y \rangle_{ghx} \, \omega$$

$$= \int_G f(gh) \, \omega = \int_G R_h^* (f\omega) = \int_{R_h(G)} f\omega = Q(X,Y)_x$$

and Q is right-invariant because:

$$Q(dR_h X, dR_h Y)_{xh} = \int_G \langle dL_g dR_h X, dL_g dR_h Y \rangle_{g(xh)} \, \omega$$

$$= \int_G \langle dR_h dL_g X, dR_h dL_g Y \rangle_{(gx)h} \, \omega = \int_G \langle dL_g X, dL_g Y \rangle_{gx} \, \omega = Q(X,Y)_x$$

$R_h \circ L_g = L_g \circ R_h$ so also $dR_h dL_g = dL_g dR_h$

$\square$

Rmk: Some noncompact Lie groups do not admit bi-invariant metrics, but some do (e.g. $\mathbb{R}^n \times G$)

compact.

<u>Prop</u>. The left-invariant metric $\langle \cdot,\cdot \rangle$ on G obtained from the inner product $\langle \cdot,\cdot \rangle_e$ on T_eG is bi-invariant iff $\langle Ad(g)X, Ad(g)Y \rangle_e = \langle X, Y \rangle_e$

i.e., $Ad(g): \mathfrak{g} \to \mathfrak{g}$ is an isometry.

<u>Pf</u>. Exercise

<u>Cor</u>. $\left\{ \begin{matrix} \text{Bi-invariant} \\ \text{metrics on } G \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Ad-invariant inner} \\ \text{products on } \mathfrak{g} \end{matrix} \right\}$

<u>Prop</u>. Let Q be a bi-invariant metric on G, $X, Y, Z \in \mathfrak{g}$. Then

(i) $Q([X,Y],Z) = -Q(Y,[X,Z])$, i.e., $ad(X): \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric

(ii) $\nabla_X Y = \frac{1}{2}[X,Y]$

(iii) $R(X,Y)Z = \frac{1}{4}[Z,[X,Y]]$.

(iv) $Q(R(X \wedge Y), Z \wedge W) = \frac{1}{4} Q([X,Y],[Z,W])$

In particular, (G,Q) has $R \geq 0$ and hence $\sec \geq 0$.

<u>Pf</u>. Differentiate $Q(Ad(\exp tX)Y), Ad(\exp tX)Z) = Q(Y,Z)$ at $t=0$,

$$Q(ad(X)Y, Z) + Q(Y, ad(X)Z) = 0$$

$$Q([X,Y],Z) + Q(Y,[X,Z]) = 0.$$

To compute the Levi-Civita connection, use Koszul formula:

X, Y, Z are left-invariant.

$$Q(\nabla_Y X, Z) = \frac{1}{2}\Big(X(\overset{0}{\cancel{Q(Y,Z)}}) + Y(\overset{0}{\cancel{Q(X,Z)}}) - Z(\overset{0}{\cancel{Q(X,Y)}})$$

$$- Q([X,Z],Y) - Q([Y,Z],X) - Q([X,Y],Z)\Big)$$

$$= \frac{1}{2} Q([Y,X],Z) + \frac{1}{2}\Big(\underbrace{Q(ad(Z)X,Y) + Q(X, ad(Z)Y)}_{=0}\Big)$$

i.e. $\nabla_Y X = \frac{1}{2}[Y,X]$; so $\nabla_X Y = \frac{1}{2}[X,Y]$. Then

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z]$$

$$= -\frac{1}{4}[[X,Y],Z] - \frac{1}{4}\Big(\underbrace{[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X]}_{=0 \quad \text{(Jacobi)}}\Big)$$

$$= \frac{1}{4}[Z,[X,Y]].$$

$$Q(R(X\wedge Y), Z\wedge W) = Q(R(X,Y)W, Z)$$

$$= \frac{1}{4} Q([W,[X,Y]], Z)$$

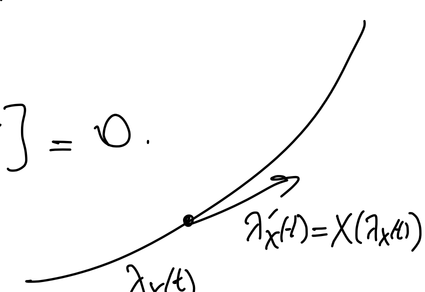
$$= \frac{1}{4} Q([X,Y],[Z,W]).$$

Rmk. If G is connected and $\langle \cdot,\cdot \rangle$ is a left-invariant metric on G s.t. $\langle \mathrm{ad}(X)Y, Z\rangle = -\langle Y, \mathrm{ad}(X)Z\rangle$, then $\langle \cdot,\cdot\rangle$ is bi-invariant.

Prop: $\exp: \mathfrak{g} \to G$ and $\exp_e: T_eG \to G$ agree on Lie groups with bi-invariant metric. In particular, $\exp$ is surjective on compact connected Lie groups.

Pf. The 1-parameter subgroups $\lambda_X : \mathbb{R} \to G$ are geodesics since $\lambda_X(t) = \exp(tX)$ are integral curves of X and hence

$$\frac{D}{dt}\lambda_X'(t) = \frac{D}{dt} X(\lambda_X(t)) = \nabla_{\lambda_X'(t)} X = \nabla_X X = \frac{1}{2}[X,X] = 0.$$



Conversely, the geodesic $\gamma: (-\varepsilon, \varepsilon) \to G$ with $\gamma(0) = e$, $\gamma'(0) = X$ is $\gamma(t) = \exp(tX)$, so can be extended to $\gamma: \mathbb{R} \to G$. Thus, $\exp$ and $\exp_e$ coincide.

If G is a compact Lie gp, then it has a bi-invariant metric, and $\exp_e = \exp$, so $\exp_e : T_eG \to G$ is globally defined b/c $\exp: \mathfrak{g} \to G$ is, hence G is complete by Hopf-Rinow. Thus, $\exp_e : T_eG \to G$ is surjective, so $\exp: \mathfrak{g} \to G$ is surjective. $\square$

<u>Rmk.</u> $\exp: \mathfrak{sl}(2,\mathbb{R}) \to SL(2,\mathbb{R})$ is not surjective, so $SL(2,\mathbb{R})$ does not admit a bi-invariant metric.

<u>Def</u> The Killing form of $\mathfrak{g}$ is $B: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by

$$B(X,Y) = \operatorname{tr}\left(\operatorname{ad}(X) \circ \operatorname{ad}(Y)\right).$$

symmetric b/c $\operatorname{tr} AB = \operatorname{tr} BA$.

The Lie group is called semisimple if B is nondegenerate.

<u>Prop.</u> B is Ad-invariant

<u>Pf</u> If $\varphi: \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism, then

$$\operatorname{ad}(\varphi(X))\,\varphi(Y) = [\varphi(X), \varphi(Y)] = \varphi([X,Y]) = (\varphi \circ \operatorname{ad}(X))(Y)$$

so $\operatorname{ad}(\varphi(X)) = \varphi \circ \operatorname{ad}(X) \circ \varphi^{-1}$. Thus,

$$\begin{aligned} B(\varphi(X), \varphi(Y)) &= \operatorname{tr}\left(\operatorname{ad}(\varphi(X)) \circ \operatorname{ad}(\varphi(Y))\right) \\ &= \operatorname{tr}\left(\varphi \operatorname{ad}X\, \varphi^{-1} \varphi \operatorname{ad}Y\, \varphi^{-1}\right) \\ &= \operatorname{tr}\left(\operatorname{ad}X \operatorname{ad}Y\right) = B(X,Y). \end{aligned}$$

Apply the above to $\varphi = \operatorname{Ad}(g)$. $\square$

E.g., on $SU(n)$,

$$B(X,Y) = -2n\, \underbrace{\operatorname{Re}\operatorname{tr}(XY^*)}_{Q(X,Y) \text{ bi-inv. metric on } SU(n)}$$

on $O(n)$,

$$B(X,Y) = -(n-2)\, \underbrace{\operatorname{tr} XY^T}_{Q(X,Y) \text{ bi-inv metric on } SO(n)}$$

Cor. Let G be a semisimple Lie gp. with negative-definite Killing form. Then $-B$ is a bi-invariant metric.

Rmk. The Ricci tensor of a Lie group G with a bi-invariant metric is $Ric(X,Y) = tr\, R(\cdot, X)Y = tr\, \frac{1}{4}[Y,[\cdot,X]] = -\frac{1}{4}B(X,Y)$.

$\boxed{\text{Thm. A semisimple connected Lie gp. is compact iff } B < 0.}$

Pf. If $B<0$, then $-B>0$ is a bi-invariant metric on G. As $\exp$ and $\exp_e$ agree, by Hopf-Rinow, $(G,-B)$ is a complete Riem. mfld with $Ric(X,Y) = -\frac{1}{4}B(X,Y)$, i.e., $Ric \geq \frac{1}{4}(-B) > 0$. Thus, by Myers theorem, G is compact (and $\pi_1 G$ is finite).

Conversely, if G is compact, then it admits a bi-invariant metric Q. Thus, if $\{e_1, \dots, e_n\}$ is an o.n.b. of $\mathfrak{g}$,

$$B(X,X) = \sum_{i=1}^{n} Q(ad(X)\, ad(X)\, e_i, e_i)$$

$$= -\sum_{i=1}^{n} Q(ad(X)e_i, ad(X)e_i) \leq 0$$

If $\|ad(X)e_i\|^2 = 0$ for some $X \in \mathfrak{g}$, $X \neq 0$, then $B(Y,X) = 0$ for all $Y \in \mathfrak{g}$, so $\operatorname{Ker} B \neq \{0\}$ contradicting the assumption that G is semisimple. Thus, $B<0$. $\square$

Cor. If G is a semisimple connected compact Lie group, then $(G,-B)$ is an Einstein manifold with Einstein constant $1/4$.

Rmk. $\mathfrak{g}$ is semisimple iff $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, where $\mathfrak{g}_i \lhd \mathfrak{g}$ are simple Lie algebras, i.e., non commutative simple ideals of $\mathfrak{g}$.

Thm. If $\mathfrak{g}$ has a bi-invariant metric Q, then $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is the orthogonal direct sum of simple ideals (some may be Abelian). The connected simply-connected Lie gp $\tilde{G}$ with Lie algebra $\mathfrak{g}$ is the product of normal Lie subgroups $\tilde{G} = G_1 \times \cdots \times G_k$, s.t. $G_i = \mathbb{R}$ if $\mathfrak{g}_i$ is Abelian, and G_i is compact if $\mathfrak{g}_i$ is not Abelian.

Pf. See book.

Cor. If $\mathfrak{g}$ has a bi-invariant metric, then $\mathfrak{g} \cong Z(\mathfrak{g}) \oplus \overbrace{[\mathfrak{g},\mathfrak{g}]}^{\text{"semisimple part"}}$.

Cor (Weyl). If G is a compact Lie gp. with finite center, then $\pi_1 G$ is finite and hence every Lie gp. with Lie algebra $\mathfrak{g}$ is compact.

Pf.

$$\left.\begin{array}{l} G \text{ compact} \Rightarrow \mathfrak{g} \text{ has bi-inv. metric.} \\ |Z(G)| < \infty \Rightarrow Z(\mathfrak{g}) = \{0\} \end{array}\right\} \Rightarrow \mathfrak{g} \text{ is semisimple.}$$

$$\Downarrow$$

$$-B > 0$$

$(G, -B)$ is Einstein w/ $Ric \geq \frac{1}{4}$, so $|\pi_1 G| < +\infty$ by Myers.

Thus, $\tilde{G}$ is compact, and any Lie gp. with Lie algebra $\mathfrak{g}$ is a quotient of $\tilde{G}$, hence also compact. $\square$

By the above, the classification of <u>compact</u> Lie groups reduces to the classification of <u>simple</u> Lie groups. Killing + Élie Cartan.

Lecture 27 5/15/2024

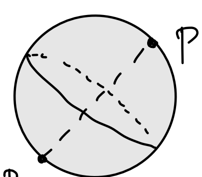
From last time: if G is a compact Lie gp, it admits a bi-inv. metric Q and (G, Q) has $R \geq 0$; in particular $\sec \geq 0$.

Homogeneous Spaces

Def: (M^n, g) is a homogeneous space if it has a transitive action by isometries: $\exists\, G < \mathrm{Isom}(M^n, g)$ s.t. $G(p) = M$.

If $H = G_p = \{ g \in G : g \cdot p = p \}$, then $M = G(p) = G/H$.

Ex: $S^n = \dfrac{O(n+1)}{O(n)} = \dfrac{SO(n+1)}{SO(n)}$, $\mathbb{R}P^n = \dfrac{SO(n+1)}{S(O(n)O(1))}$ $= \left\{ A = \begin{pmatrix} A_1 & \\ & \pm 1 \end{pmatrix} ; \begin{matrix} A_1 \in O(n) \\ \det A = 1 \end{matrix} \right\}$ $\cong SO(n) \rtimes \mathbb{Z}_2$.



$O(n+1) \curvearrowright S^n \subset \mathbb{R}^{n+1}$

isotropy at $p \in S^n = O(n)$

$\mathbb{C}P^n = \dfrac{U(n+1)}{U(n)U(1)} = \dfrac{SU(n+1)}{S(U(n)U(1))}$

$U(n+1) \curvearrowright S^{2n+1} \subset \mathbb{C}^{n+1}$

(commutes w/ $U(1)$ action) $\downarrow$

$U(n+1) \curvearrowright \mathbb{C}P^n$

isotropy at $[e] \in \mathbb{C}P^n$: $U(n) \curvearrowright e^\perp \cong \mathbb{C}^n$
$U(1) \curvearrowright e \cong \mathbb{C}$

$\mathbb{H}P^n = \dfrac{Sp(n+1)}{Sp(n)}$

$Sp(n+1) \curvearrowright S^{4n+3} \subset \mathbb{H}^{n+1}$

commutes w/ $Sp(1)$ $\downarrow$

$Sp(n+1) \curvearrowright \mathbb{H}P^n$

isotropy at $[e] \in \mathbb{H}P^n$: $Sp(n) \curvearrowright e^\perp \cong \mathbb{H}^n$
$Sp(1) \curvearrowright e \cong \mathbb{C}$

$\mathbb{C}aP^2 = \dfrac{F_4}{Spin(9)}$ (The above comprise the compact rank one symmetric spaces "CROSS" for short).

Prop. If G is a cpct Lie gp, and $H < G$ have Lie algebras $\mathfrak{h} < \mathfrak{g}$, let $\mathfrak{m} = \mathfrak{h}^{\perp_Q}$, so $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then $T_{eH}\, G/H \cong \mathfrak{m}$ and the isotropy representation $H \curvearrowright T_{eH}\, G/H$ is precisely $\mathrm{Ad}: H \to \mathrm{Aut}(\mathfrak{m})$.
$v \mapsto dh_{(eH)} v$

so $\mathfrak{m}$ is $\mathrm{Ad}(H)$-inv.

Cor. In the above situation: $\left\{\begin{matrix} G\text{-inv. metrics} \\ \text{on } G/H \end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix} Ad(H)\text{-inv. inner} \\ \text{products on } \mathfrak{m} \end{matrix}\right\}$

Def. The chosen bi-inv. metric Q on G is $Ad(H)$-inv, hence induces a G-inv. metric on G/H, called <u>normal homogeneous</u>, and $G \to G/H$ is a Riem. submersion w/ totally geodesic fibers.

Prop. If $\pi: (M,g) \to (N,\check{g})$ is a Riem. submersion, then

$$\sec_{\check{g}}(X\wedge Y) = \sec_g(\overline{X}\wedge\overline{Y}) + \frac{3}{4}\|[\overline{X},\overline{Y}]^v\|^2.$$

$\overline{X}$ is the hor. lift of X, so $d\pi(\overline{X}) = X$, and $(\cdot)^v$ is the vertical component

In particular, if $\sec_g \geq 0$, then $\sec_{\check{g}} \geq 0$.

<u>Rmk:</u> $\nabla_{\overline{X}}\overline{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\overline{X},\overline{Y}]^v$.

$\boxed{\text{Cor. Every compact homogeneous space admits metrics w/ } \sec \geq 0.}$

<u>Fact.</u> The moduli space of G-inv. metrics w/ $\sec \geq 0$ on a cpct homog. space is path-connected. ← Essentially, it is star-shaped around a chosen normal hom. metric.

<u>Ex:</u> $$\begin{array}{c} S^{2n+1} \subset \mathbb{C}^{n+1} \\ \downarrow \\ \mathbb{C}P^n \end{array}$$

Let $V_z = (iz_1, \ldots, iz_{n+1})$ be the vertical direction.

$$\left\langle \tfrac{1}{2}[\overline{X},\overline{Y}], V\right\rangle = \left\langle \nabla^{S^{2n+1}}_{\overline{X}}\overline{Y}, V\right\rangle = \left\langle \nabla^{\mathbb{C}^{n+1}}_{\overline{X}}\overline{Y}, V\right\rangle =$$

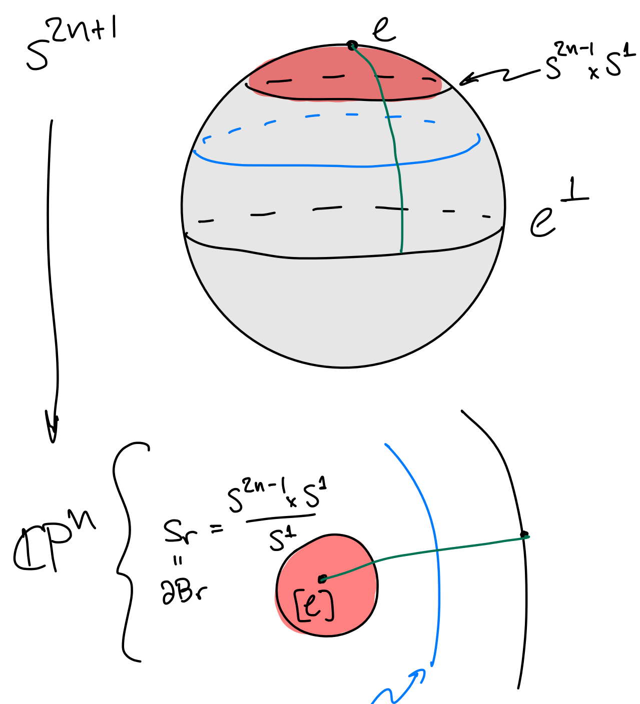
$$= \overline{X}\underbrace{\langle \overline{Y}, V\rangle}_{=0} - \left\langle \overline{Y}, \nabla^{\mathbb{C}^{n+1}}_{\overline{X}} V\right\rangle = \left\langle \overline{Y}, \nabla^{\mathbb{C}^{n+1}}_{i\overline{X}} iV\right\rangle$$

$$= \mathrm{II}^{S^{2n+1}}(\overline{Y}, i\overline{X}) = \langle \overline{Y}, i\overline{X}\rangle.$$

So $\frac{1}{2}[\overline{X},\overline{Y}]^v = \langle \overline{Y}, i\overline{X}\rangle V$ and $\frac{1}{4}\|[\overline{X},\overline{Y}]^v\|^2 = \langle \overline{Y}, i\overline{X}\rangle^2$, so

$\sec_{\mathbb{C}P^n}(X\wedge Y) = 1 + 3\langle \overline{Y}, i\overline{X}\rangle^2$ for all $\{X,Y\}$ orthonormal.

In particular, $1 \leq \sec_{\mathbb{C}P^n} \leq 4$. One easily computes $\mathrm{Ric}_{\mathbb{C}P^n} = (2n+2)g_{\mathbb{C}P^n}$.



The sphere of radius $r = \arctan\sqrt{2n-1}$ is minimal!

When $r \nearrow \pi/2$, the dist. sphere collapses to $\mathbb{CP}^{n-1} = \mathrm{Cut}(p)$; and $S^1 \to S^{2n-1} \to \mathbb{CP}^{n-1}$ is precisely the Hopf bundle.

Recall $S^{2n+1} = [0,\pi/2] \times \underbrace{S^{2n-1}\times S^1}$ ← can think of S^1 acting fiberwise...

$$g = dr^2 + \sin^2 r\, g_{S^{2n-1}} + \cos^2 r\, g_{S^1}.$$

The projection $\pi : S^{2n+1} \to \mathbb{CP}^n$ is a Riem. submersion hence a submetry:

$$\pi(B_r(p)) = B_r(\pi(p))$$

and the boundary of these geod. balls are distance spheres. For $0<r<\pi/2$,

$$g_{\mathbb{CP}^n} = dr^2 + \underbrace{\sin^2 r \left(g|_{\mathcal{H}} + \cos^2 t\, g|_{\mathcal{V}} \right)}_{g_r}$$

where $g_{S^{2n-1}} = g|_{\mathcal{H}} + g|_{\mathcal{V}}$ is the metric on S^{2n-1}, and g_r is the submersion metric from the S^1 action on $S^{2n-1}\times S^1$. ("Cheeger deformation")

Note: These dist. spheres on $\mathbb{CP}^n$ are orbits of an action by $SU(n)$ on $e^\perp \subset \mathbb{C}^{n+1}$.

Similar picture for $\mathbb{HP}^n$, but slightly different for $\mathbb{C}a\mathbb{P}^2$, since $\nexists\, S^N \to \mathbb{C}a\mathbb{P}^2$.

Besides CROSSes, few other examples of closed mflds w/ $\sec>0$ are known; actually only in dimensions 6, 7, 12, 13, and 24. All of them have large symmetry group... why?

purposefully vague. Choose your preferred notion.

Grove Symmetry Program: Classify closed mflds w/ $\sec>0$ and large isometry gps.

While attempting to classify, one may also narrow the places to search for new examples; this is how the last example was discovered $T_1S^4 \# \Sigma^7$.

It has an isometric action w/ 1-dim. orbit space...

"Model" result:

Thm (Hsiang–Kleiner, Grove–Wilking). If (M^4, g) is closed simply-connected with $\sec_g > 0$ and $S^1 \curvearrowright (M^4, g)$ isometric action, then M^4 is (equivariantly) diffeomorphic to S^4 or $\mathbb{CP}^2$.

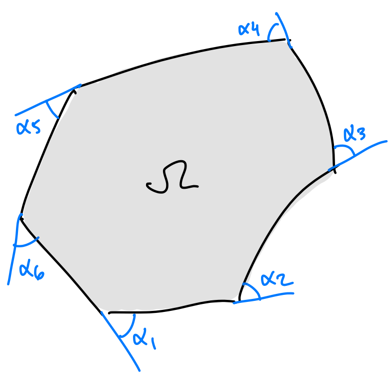
Qual prep

About 49% smooth manifolds, differential forms, etc (1st semester)

49% Riem. geometry (2nd semester)

2% Left overs: Gauss–Bonnet, Lie groups

Recall Gauss-Bonnet: (M^2, g) compact oriented manifold, $\Omega \subset M^2$ compact domain w/ piecewise smooth boundary $\partial\Omega$, then



$$\int_\Omega \sec\, dA + \underbrace{\int_{\partial\Omega} k_g\, ds}_{= \sum_{i=1}^{k} \int_{\gamma_i} k_g\, ds + \sum_{i=1}^{k} \alpha_i} = 2\pi \chi(\Omega).$$

if $\partial\Omega = \gamma_1 \cup \dots \cup \gamma_k$, where α_i is change in angle of tangent vector when moving from γ_i to γ_{i+1} (mod k).

If γ has $\|\dot\gamma\| = 1$, then

$k_g = \|\nabla_{\dot\gamma}\dot\gamma\|$ so $k_g = 0 \iff \gamma$ is a geodesic.

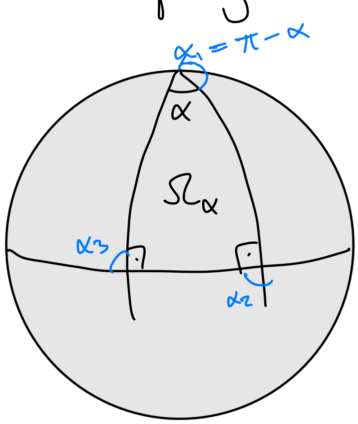
$dA = \mathrm{vol}_g$ is volume form of (M^2, g)

ds = arclength along $\partial\Omega$.

Important cases: $\partial\Omega = \emptyset$ $\quad \int_M \sec\, dA = 2\pi\chi(M)$

$\partial\Omega$ piecewise geodesic $\quad \int_\Omega \sec\, dA + \sum \alpha_i = 2\pi\chi(\Omega)$

Ex: Spring 2021 #10



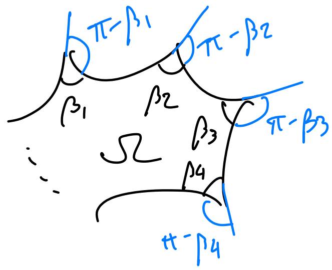
$\Omega_\alpha \subset S^2(r)$ is a triangle on the round sphere with $\sec \equiv \frac{1}{r^2}$ and interior angles $\alpha, \frac{\pi}{2}, \frac{\pi}{2}$.

"Verify Gauss–Bonnet" ① $\partial\Omega_\alpha$ is piecewise geodesic

② $\displaystyle\int_{\Omega_\alpha} \sec\, dA = \frac{\text{Area}(\Omega_\alpha)}{r^2} = \frac{\alpha}{2\pi}\frac{\text{Area}(S^2(r))}{2r^2} = \frac{\alpha \cdot 4\pi r^2}{2\pi \cdot 2r^2} = \alpha.$

③ $\sum \alpha_i = (\pi - \alpha) + \frac{\pi}{2} + \frac{\pi}{2} = 2\pi - \alpha$ ④ $\chi = 1$.

Spring 2019 #3



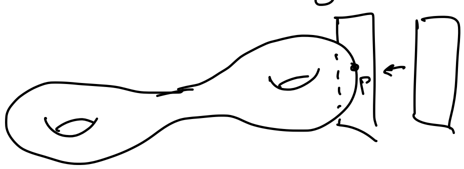
Prove that area of hyperbolic polygon w/ n geodesic sides and interior angles β_i, $1 \le i \le n$, is:

$$\text{Area}(\Omega) = (n-2)\pi - \sum_{i=1}^{n} \beta_i$$

$$\underbrace{\int_\Omega \sec\, dA}_{=-\text{Area}(\Omega)} + \sum_{i=1}^{n} \underset{\pi-\beta_i}{\underset{\|}{\alpha_i}} = 2\pi\chi(\Omega) = 2\pi \implies \text{Area}(\Omega) = n\pi - \sum \beta_i - 2\pi.$$

Fall 2022 #10.

Prove that $\Sigma_g^2 \subset \mathbb{R}^3$ w/ genus $g \ge 2$ has an open set with $\sec < 0$.



① $\Sigma_g^2 \subset \mathbb{R}^3$ is closed embedded so $\exists p \in \Sigma_g^2$ at maximal distance from $0 \in \mathbb{R}^3$, and thus $\sec_p > 0$. (bring parallel planes from infinity)

By continuity, $\sec_p > 0$ on $B_\varepsilon(p)$.

② $\int_{\Sigma_g^2} \sec = 2\pi\chi(\Sigma_g^2) = 2\pi(2-2g) < 0$ so $\exists U \subset \Sigma_g^2$ w/ $\text{Area}(U) > 0$ s.t. $\int_U \sec < 0$.

<u>Rmk:</u> If (M^2, g) is complete, noncompact, and $\int_M \sec < +\infty$, then $\int_M \sec \le 2\pi\chi(M)$.

(e.g., on $M = \mathbb{R}^2$ flat, $0 < 2\pi$.)

Some computational questions:

Fall 2022 #1. Compute sec of $\left(\mathbb{R}^2, e^{2f}(dx^2 + dy^2)\right)$ where $f: \mathbb{R}^2 \to \mathbb{R}$ is smooth.

$X = e^{-f}\frac{\partial}{\partial x}$, $Y = e^{-f}\frac{\partial}{\partial y}$ is global o.n. frame.

$$[X,Y]\phi = XY\phi - YX\phi = e^{-f}\frac{\partial}{\partial x}\left(e^{-f}\frac{\partial\phi}{\partial y}\right) - e^{-f}\frac{\partial}{\partial y}\left(e^{-f}\frac{\partial\phi}{\partial x}\right)$$

$$= e^{-f}\left(e^{-f}\left(-\frac{\partial f}{\partial x}\right)\frac{\partial\phi}{\partial y}\right) + e^{-2f}\frac{\partial^2\phi}{\partial x\partial y} - e^{-f}\left(e^{-f}\left(-\frac{\partial f}{\partial y}\right)\frac{\partial\phi}{\partial x}\right) - e^{-2f}\frac{\partial^2\phi}{\partial y\partial x}$$

$$= e^{-2f}\left(\frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right)\phi = e^{-f}\left(\frac{\partial f}{\partial y}X - \frac{\partial f}{\partial x}Y\right)\phi$$

so $[X,Y] = e^{-f}\left(\frac{\partial f}{\partial y}X - \frac{\partial f}{\partial x}Y\right)$.

Need
$$\sec(X\wedge Y) = \frac{\langle R(X,Y)Y,X\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2} = \frac{\langle \nabla_X\nabla_Y Y - \nabla_Y\nabla_X Y - \nabla_{[X,Y]}Y, X\rangle}{e^{-2f}\cdot e^{-2f}}$$

By Koszul:
$$\langle \nabla_Y X, Z\rangle = \frac{1}{2}\Big(\underbrace{X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))}$$
these vanish on an orthonormal frame!
$$- g([X,Z],Y) - g([Y,Z],X) - g([X,Y],Z)\Big).$$

Before computing a lot...

$g(X,X) = 1$ so $0 = X\, g(X,X) = 2\,g(\nabla_X X, X)$

$0 = Y\, g(X,X) = 2\,g(\nabla_Y X, X)$

so only need $g(\nabla_X X, Y) = \dots$ and $g(\nabla_Y X, Y) = \dots$ to determine $\nabla_X X$ and $\nabla_Y X$. similarly for $\nabla_X Y$ and $\nabla_Y Y$.

$g(Y,Y) = 1$ so $0 = X\, g(Y,Y) = 2\,g(\nabla_X Y, Y)$

$0 = Y\, g(Y,Y) = 2\,g(\nabla_Y Y, Y)$

$g(X,Y) \equiv 0$ so $0 = X\, g(X,Y) = g(\nabla_X X, Y) + g(X, \nabla_X Y)$

$0 = Y\, g(X,Y) = g(\nabla_Y X, Y) + g(X, \nabla_Y Y)$.

don't need to apply Koszul again

$$\nabla_Y Y = \langle \nabla_Y Y, X\rangle X = \frac{1}{2}\Big(-g([Y,X],Y) - g([Y,X],Y)\Big)X$$
$$= g\left(e^{-f}\left(\frac{\partial f}{\partial y}X - \frac{\partial f}{\partial x}Y\right), Y\right)X = -e^{-f}\frac{\partial f}{\partial x}X.$$

$$\nabla_X X = \langle \nabla_X X, Y\rangle Y = \frac{1}{2}\Big(-g([X,Y],X) - g([X,Y],X)\Big) Y$$

$$= -g\left(e^{-f}\left(\frac{\partial f}{\partial y}X - \frac{\partial f}{\partial x}Y\right), X\right) Y = -e^{-f}\frac{\partial f}{\partial y}\,Y$$

$$\nabla_X Y = \langle \nabla_X Y, X\rangle X = \frac{1}{2}\Big(-g([Y,X],X) - g([Y,X],X)\Big) X$$

$$= g\left(e^{-f}\left(\frac{\partial f}{\partial y}X - \frac{\partial f}{\partial x}Y\right), X\right) X = e^{-f}\frac{\partial f}{\partial y}\,X.$$

$$\nabla_Y X = \nabla_X Y + [Y,X] = e^{-f}\frac{\partial f}{\partial y}X - e^{-f}\left(\frac{\partial f}{\partial y}X - \frac{\partial f}{\partial x}Y\right) = e^{-f}\frac{\partial f}{\partial x}\,Y.$$

$$\sec(X\wedge Y) = \frac{\langle R(X,Y)Y, X\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2} = \frac{\langle \nabla_X\nabla_Y Y - \nabla_Y\nabla_X Y - \nabla_{[X,Y]}Y, X\rangle}{e^{-2f}\cdot e^{-2f}}$$

$$= e^{4f}\cdot\left\langle \nabla_X\left(-e^{-f}\frac{\partial f}{\partial x}X\right) - \nabla_Y\left(e^{-f}\frac{\partial f}{\partial y}X\right) - \nabla_{e^{-f}\frac{\partial f}{\partial y}X - e^{-f}\frac{\partial f}{\partial x}Y}Y, X\right\rangle$$

$$= e^{4f}\Bigg(\Big\langle -X\left(e^{-f}\frac{\partial f}{\partial x}\right)X - e^{-f}\frac{\partial f}{\partial x}\,\nabla_X X - Y\left(e^{-f}\frac{\partial f}{\partial y}\right)X - e^{-f}\frac{\partial f}{\partial y}\nabla_Y X$$

orthogonal to X.

$$- e^{-f}\frac{\partial f}{\partial y}\nabla_X Y + e^{-f}\frac{\partial f}{\partial x}\nabla_Y Y,\ X\Big\rangle\Bigg)$$

$$= e^{4f}\left(-e^{-f}\frac{\partial}{\partial x}\left(e^{-f}\frac{\partial f}{\partial x}\right) - e^{-f}\frac{\partial}{\partial y}\left(e^{-f}\frac{\partial f}{\partial y}\right) - \left(e^{-f}\frac{\partial f}{\partial y}\right)^2 - \left(e^{-f}\frac{\partial f}{\partial x}\right)^2\right)$$

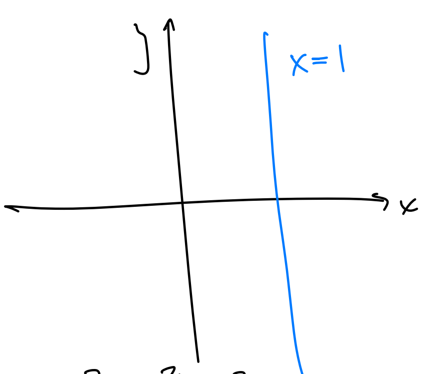
$$= e^{3f}\left(\underline{e^{-f}\left(\frac{\partial f}{\partial x}\right)^2} - e^{-f}\frac{\partial^2 f}{\partial x^2} + \underline{e^{-f}\left(\frac{\partial f}{\partial y}\right)^2} - e^{-f}\frac{\partial^2 f}{\partial y^2}\right) - e^{2f}\left(\underline{\left(\frac{\partial f}{\partial y}\right)^2} + \underline{\left(\frac{\partial f}{\partial x}\right)^2}\right)$$

$$= e^{2f}\left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right) = -e^{2f}\Delta f.$$

The above is "faster" with a moving frames formalism, but also doable with usual Riem. geom. techniques.

Fall 2022 #7 Compute sec of $(\mathbb{R}^2, dx^2 + e^{2y}dy^2)$; show $x=$const are geodesics

Recall $g = dr^2 + f(r)^2 d\theta^2$ has $\sec = -\frac{f''}{f}$. In the above, we can use an arclength parameter $ds^2 = e^{2y}dy^2$, so $\frac{ds}{dy} = e^y$ and $s(y) = e^y$. Thus, $(\mathbb{R}^2, dx^2 + e^{2y}dy^2)$ is isometric to the flat upper half-plane $(\mathbb{R}\times(0,+\infty), dx^2+ds^2)$; in particular, $\sec \equiv 0$. The curves $x=$const. are geodesics b/c they map via the isometry $(x,y)\longmapsto(x,e^y)$ to straight lines in the flat upper half plane, which are geodesics.



$dx^2 + e^{2y}dy^2$

$(x,y)\mapsto(x,e^y)$

$(x,\ln s) \leftarrow\!\!\shortmid (x,s)$

$dx^2 + ds^2, \quad s>0$

Note: neither metric is complete!