MAT313

Solutions to Review Problems

1. $$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 2 & 0 \\ 8 & 16 & 4 & 4 & 0 \\ 6 & 20 & 5 & 17 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 4 & 3 & 2 & 0 \\ 0 & -16 & -20 & -12 & 0 \\ 0 & -4 & -13 & 5 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 4 & 3 & 2 & 0 \\ 0 & 4 & 5 & 3 & 0 \\ 0 & 0 & -8 & 8 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 4 & 3 & 2 & 0 \\ 0 & 1 & 5/4 & 3/4 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 = 3x_4 \\ x_2 = -2x_4 \\ x_3 = x_4 \\ x_4 \text{ free} \end{cases}$$

Solutions are: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_4 \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \\ 1 \end{pmatrix}$, i.e. $\operatorname{span}\left\{\begin{pmatrix} 3 \\ -2 \\ 1 \\ 1 \end{pmatrix}\right\}$.

2. $T: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$, $\quad T\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = T\left(\frac{1}{3}\left(\begin{pmatrix} 2 \\ 7 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)\right) = \frac{1}{3}T\begin{pmatrix} 2 \\ 7 \end{pmatrix} - \frac{2}{3}T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -5/3 \\ -7/3 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} - 2\begin{pmatrix} -1 \\ -5/3 \\ -7/3 \end{pmatrix} = \begin{pmatrix} 5 \\ 16/3 \\ 26/3 \end{pmatrix}.$$

$$[T]_{can} = \begin{pmatrix} 5 & -1 \\ 16/3 & -5/3 \\ 26/3 & -7/3 \end{pmatrix}.$$

$\dim \operatorname{Ker} T = 0$

$\dim \operatorname{Im} T = 2$

3. $T: \mathbb{R}^3 \to \mathbb{R}^3 \qquad T(x,y,z) = (z,\ x+y,\ x-y)$

a) $$[T]_{\mathcal{E}_{can}} = \begin{pmatrix} | & | & | \\ Te_1 & Te_2 & Te_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

b) Yes, it is invertible.

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \qquad [T^{-1}]_{\mathcal{E}_{can}} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & +1/2 & -1/2 \\ 1 & 0 & 0 \end{pmatrix}$$

4. $$\begin{cases} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \\ \vdots \\ x_7 + x_8 = 0 \end{cases} \rightsquigarrow \begin{cases} x_1 = -x_8 \\ \cdots \\ x_6 = -x_7 = x_8 \\ x_7 = -x_8 \\ x_8 \text{ free} \end{cases} \rightsquigarrow x = x_8 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

Basis is $\{(-1, 1, -1, 1, -1, 1, -1, 1)\}$.

5. a) Linearly dependent: 3 vectors in 2-dim space

b) Linearly independent;

c) $$(b+c,\ a+c,\ a+b) = (0,0,0) \iff \begin{cases} a+b = 0 \\ a+c = 0 \\ b+c = 0 \end{cases} \iff a=b=c=0$$ lin. indep.

d) lin dep: $2\begin{pmatrix}1&0\\0&1\end{pmatrix}+2\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}2&2\\2&2\end{pmatrix}$.

e) lin dep: $(t-1)-(t)=-\frac{1}{7}(7)$

6. a) $T(x,y)=(x+y,\ 2x+2y)$ $\qquad [T]_{can}=\begin{pmatrix}1&1\\2&2\end{pmatrix}$

$\operatorname{Ker} T=\operatorname{span}\{(1,-1)\}$, $\qquad \operatorname{Im} T=\operatorname{span}\{(1,2)\}$

b) $T(x,y,z,w)=(x,w)$

$\operatorname{Ker} T=\operatorname{span}\{(0,1,0,0),(0,0,1,0)\}$

$\operatorname{Im} T=\operatorname{span}\{(1,0),(0,1)\}$

c) $T(x)=(x,5x)$

$\operatorname{Ker} T=\{0\}$ $\qquad \operatorname{Im} T=\operatorname{span}\{(1,5)\}$.

d) $T(p(t))=\int p(t)\,dt$ $\qquad \int a_0+a_1t+a_2t^2\,dt=a_0t+a_1\frac{t^2}{2}+a_2\frac{t^3}{3}$

$\operatorname{Ker} T=\{0\}$ $\qquad \operatorname{Im} T=\operatorname{span}\{t,t^2,t^3\}$.

e) Same as b): $\operatorname{Ker} T=\operatorname{span}\left\{\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right\}$

$\operatorname{Im} T=\operatorname{span}\{(1,0),(0,1)\}$

7.

$\det A=1\begin{vmatrix}2&3\\4&9\end{vmatrix}-1\begin{vmatrix}1&3\\1&9\end{vmatrix}+1\begin{vmatrix}1&2\\1&4\end{vmatrix}$ $\qquad$ Vandermonde:

$=6-6+2=2$ $\qquad (3-2)(3-1)(2-1)=2$.

$\det B=1\cdot(-3)\cdot 4=-12$

8. Yes. $\det(A^T) = \det(A)$ so $\det A \neq 0 \Leftrightarrow \det A^T \neq 0$

9. No, e.g. $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is symmetric but not invertible.

10. Yes.

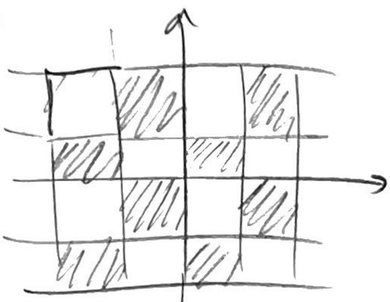
$$(A^2B^2)^{-1} = (B^2)^{-1}(A^2)^{-1} = B^{-2}A^{-2} = (B^{-1})^2(A^{-1})^2$$

$$(ABAB)^{-1} = (AB)^{-1}(AB)^{-1} = \left((AB)^{-1}\right)^2 = (B^{-1}A^{-1})^2$$

$$= B^{-1}A^{-1}B^{-1}A^{-1}.$$

11.

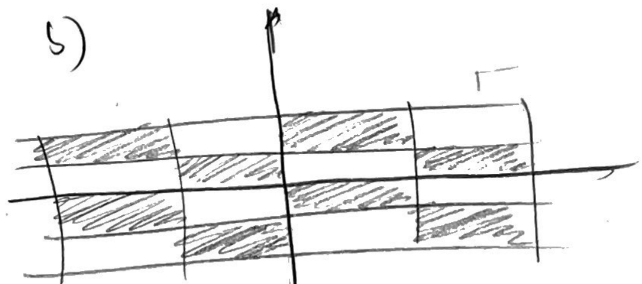
a)



Reflection on y-axis

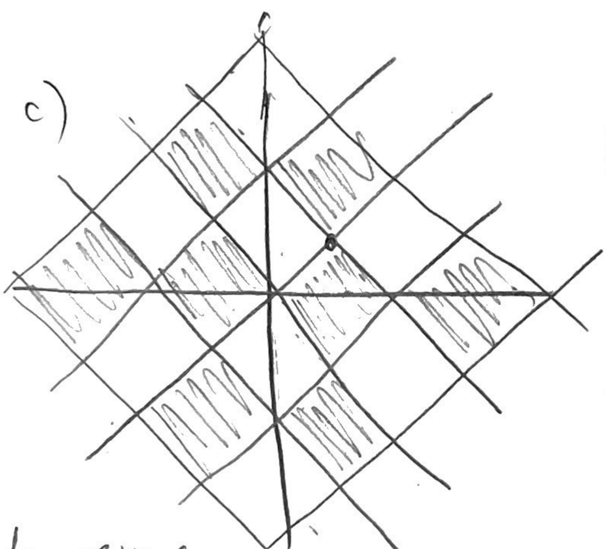
$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ same size

b)



$\begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix}$ stretch by 3x on x-axis, contract by 1/2 on y-axis

c)

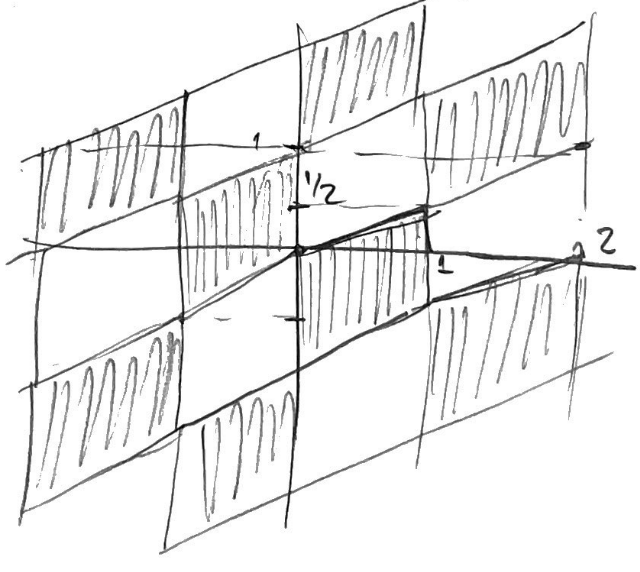


$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

each square has side $\sqrt{2}$ now..

d) $\begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}$

Vertical sheer



12.
$$\begin{pmatrix} a-3b+c \\ 2a-6b-2c \\ 3a-9b+c \\ c \\ 2a-6b+6c \end{pmatrix} = a\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 2 \end{pmatrix} + b\begin{pmatrix} -3 \\ -6 \\ -9 \\ 0 \\ -6 \end{pmatrix} + c\begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \\ 6 \end{pmatrix}$$

lin. dep.

$\dim V = 2.$

13.

a) $\det\begin{pmatrix} 4-\lambda & 3 \\ 3 & 4-\lambda \end{pmatrix} = (4-\lambda)^2 - 9 = 16 - 8\lambda + \lambda^2 - 9 = \lambda^2 - 8\lambda + 7$
$= (\lambda-1)(\lambda-7)$

Eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 7$.

b) $\det\begin{pmatrix} -\lambda & 1 \\ 1 & 4-\lambda \end{pmatrix} = -\lambda(4-\lambda) - 1 = +\lambda^2 - 4\lambda - 1$

Eigenvalues are $\quad \lambda = \dfrac{4 \pm \sqrt{16+4}}{2} = 2 \pm \dfrac{\sqrt{20}}{2} = 2 \pm \sqrt{5}$

$\lambda_1 = 2 + \sqrt{5}, \quad \lambda_2 = 2 - \sqrt{5}$

c) $\det\begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix} = (2-\lambda)\left((1-\lambda)^2 - 1\right)$

$= (2-\lambda)(1 - 2\lambda + \lambda^2 - 1) = (2-\lambda)(\lambda)(\lambda-2)$

Eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 0$
(mult. 2)

14. $A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$ From previous ex. $\lambda_1 = 1,\ \lambda_2 = 7$

$$E_1 = \operatorname{Ker}\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

$$E_2 = \operatorname{Ker}\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

$$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$P^{-1} = P^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$A = PDP^{-1} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}$$

$$A^{100} = PD^{100}P^{-1} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 7^{100} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & \frac{7^{100}}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{7^{100}}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1+7^{100}}{2} & \frac{7^{100}-1}{2} \\ \frac{7^{100}-1}{2} & \frac{1+7^{100}}{2} \end{pmatrix}}_{\text{note: also symmetric!}}$$

15. Yes; $A^T = A$

$$\Downarrow$$

$$(A^2)^T = (AA)^T = A^T A^T = AA = A^2.$$

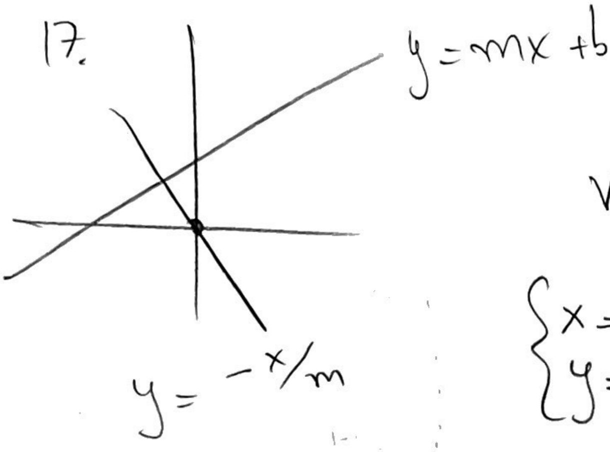
16. $\{(x,y,z) \in \mathbb{R}^3 : x+y+z = 0\} = \operatorname{span}\{(1,0,-1),(0,1,-1)\}$

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \rightsquigarrow u_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix} \rightsquigarrow u_2 = \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} \\ -\frac{1}{2}\sqrt{\frac{2}{3}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \sqrt{2/3} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\|v\|_2^2 = \frac{1}{4} + 1 + \frac{1}{4} = \frac{3}{2}$$

17.



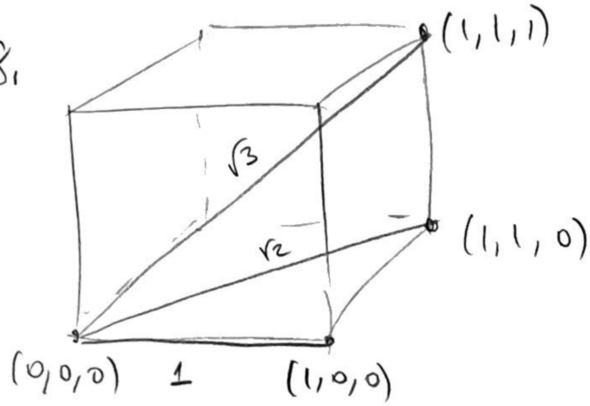
Tangent vector: $\vec{v} = (1, m)$

$$v^{\perp} = \text{span}\{(-m, 1)\}$$

$$\begin{cases} x = -m \cdot t \\ y = 1 \cdot t \end{cases} \overset{m \neq 0}{\Longrightarrow} \quad y = t = \frac{x}{-m} \Rightarrow \boxed{y = -\frac{x}{m}}$$

If $m = 0$, then y-axis: $\boxed{x = 0}$

18.



Possible lengths:

$$\|(1,1,1)\| = \sqrt{3}$$
$$\|(1,1,0)\| = \sqrt{2}$$
$$\|(1,0,0)\| = 1$$

In $\mathbb{R}^n$: $\|(1, \dots, 1)\| = \sqrt{n}$, $\|(1, \dots, 1, 0, \dots, 0)\| = \sqrt{j}$, $\underline{1 \leq j \leq n}$

i.e.: $\sqrt{n}, \sqrt{n-1}, \sqrt{n-2}, \dots, \sqrt{2}, 1$.

19. If A is diagonalizable, $\exists P$ s.t.

$A = PDP^{-1}$ where D is diagonal

Then $A + cI = PDP^{-1} + c\,PP^{-1}$

$$= P(D + c\,Id)P^{-1}$$

So $A + cI$ is also diagonalizable b/c $D + c\,Id$ is diagonal.

20. $A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$ $b = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$ $A^T A = \begin{pmatrix} 17 & 1 \\ 1 & 5 \end{pmatrix}$

$$\hat{x} = (A^T A)^{-1} A^T b = \frac{1}{84} \begin{pmatrix} 5 & -1 \\ -1 & 17 \end{pmatrix} \begin{pmatrix} 14 \\ 16 \end{pmatrix} = \begin{pmatrix} 9/14 \\ 43/14 \end{pmatrix}$$

21. $Q(x_1, x_2) = (x_1 \ \ x_2) \begin{pmatrix} 4 & 4 \\ 4 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$A = \begin{pmatrix} 4 & 4 \\ 4 & 10 \end{pmatrix}$ $\det(A - \lambda Id) = (4-\lambda)(10-\lambda) - 16 = 40 - 14\lambda + \lambda^2 - 16$

$= \lambda^2 - 14\lambda + 24 = (\lambda - 12)(\lambda - 2)$

$E_{12} = \ker \begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \right\}$

$E_2 = \ker \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \right\}$

$P = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$ $P^{-1} = P^T = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$

$y = P^T x \Rightarrow \boxed{\begin{cases} y_1 = \dfrac{x_1}{\sqrt{5}} + \dfrac{2x_2}{\sqrt{5}} \\ y_2 = \dfrac{-2x_1}{\sqrt{5}} + \dfrac{x_2}{\sqrt{5}} \end{cases}}$ $\boxed{Q(x_1, x_2) = 12y_1^2 + 2y_2^2}$

$$12\left(\frac{x_1^2 + 4x_1x_2 + 4x_2^2}{5}\right) + 2\left(\frac{4x_1^2 - 4x_1x_2 + x_2^2}{5}\right) = 4x_1^2 + 8x_1x_2 + 10x_2^2$$

22. No, because it is not symmetric.
A matrix is orthogonally diagonalizable if and only if it is symmetric.

23. a) $\langle p, q+r\rangle = \langle p, q\rangle + \langle p, r\rangle = 0 + 8 = 8$

b) $\|q+r\|^2 = \|q\|^2 + 2\langle q, r\rangle + \|r\|^2$

$= 1 + 50 = 51 \Rightarrow \|q+r\| = \sqrt{51}$

$\|p+r\|^2 = \|p\|^2 + 2\langle p, r\rangle + \|r\|^2$

$= 4 + 16 + 50 = 70 \Rightarrow \|p+r\| = \sqrt{70}$

Since $\{p, q\}$ are orthogonal:

c) $\dfrac{\langle r, p\rangle}{\langle p, p\rangle} p + \dfrac{\langle r, q\rangle}{\langle q, q\rangle} q = \dfrac{8}{4} p + 0 = 2p.$

d) $\dfrac{p}{\|p\|} = \dfrac{p}{2}$

$q - \dfrac{\langle q, p\rangle}{\langle p, p\rangle} p = q - 0 \rightsquigarrow \dfrac{q}{\|q\|} = q.$

$r - \dfrac{\langle q, r\rangle}{\langle q, q\rangle} q - \dfrac{\langle r, p\rangle}{\langle p, p\rangle} p = r - 0 - \dfrac{8}{4} p = r - 2p.$

$\|r-2p\|^2 = \langle r, r\rangle - 4\langle r, p\rangle + 4\langle p, p\rangle$
$= 50 - 32 + 16 = 34$

$\left\{ \dfrac{p}{2}, q, \dfrac{r-2p}{\sqrt{34}} \right\}.$

24. $A = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1/2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5/4 & 1 \\ 1 & 5 \end{pmatrix}$$

$$\det(A^TA - \lambda \mathrm{Id}) = (5/4 - \lambda)(5-\lambda) - 1 = \frac{25}{4} - \left(5 + \frac{5}{4}\right)\lambda + \lambda^2 - 1$$

$$= \frac{21}{4} - \frac{25}{4}\lambda + \lambda^2.$$

$$\lambda = \frac{\frac{25}{4} \pm \sqrt{\frac{625}{16} - 21}}{2} = \frac{25}{8} \pm \frac{1}{2}\sqrt{\frac{625-336}{16}} = \frac{25 \pm 17}{8}$$

$$\lambda_1 = \frac{42}{8} = \frac{21}{4}, \quad \lambda_2 = \frac{8}{8} = 1.$$

$\boxed{\sigma_1 = \frac{\sqrt{21}}{2}, \quad \sigma_2 = 1}$ are the singular values.

$2A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 2 & 2 \end{pmatrix}$ has singular values $\underline{\sqrt{21} \text{ and } 2.}$

$$(2A^T 2A) = 4 A^T A \qquad \det(4A^TA - \lambda \mathrm{Id}) = 0 \iff \det\left(A^TA - \frac{\lambda}{4}\mathrm{Id}\right) = 0$$

$$\iff \frac{\lambda}{4} = \frac{21}{4} \text{ or } 1$$

$$\iff \lambda = 21 \text{ or } \lambda = 4$$

$$\iff \sigma = \sqrt{21} \text{ or } \sigma = 2.$$