MAT313, Fall 2022

Practice Problems for the Final Exam

1. Find all solutions to the system of linear equations

$$\begin{cases} x_1 + 4x_2 + 3x_3 + 2x_4 = 0 \\ 8x_1 + 16x_2 + 4x_3 + 4x_4 = 0 \\ 6x_1 + 20x_2 + 5x_3 + 17x_4 = 0 \end{cases}$$

2. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map, such that $T(1,2) = (3,2,4)$ and $T(2,7) = (3,-1,1)$.
   (a) Find $T(0,1)$.
   (b) What is the matrix that represents T in the canonical basis?
   (c) What are the dimensions of the subspaces $\ker T$ and $\operatorname{Im} T$?

3. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x,y,z) = (z, x+y, x-y)$.
   (a) Find the matrix that represents T in the canonical basis.
   (b) Is T invertible? If so, find a formula for its inverse.

4. Write a basis for the subspace of $\mathbb{R}^8$ consisting of solutions to the following system:

$$\begin{cases} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \\ x_3 + x_4 = 0 \\ x_4 + x_5 = 0 \\ x_5 + x_6 = 0 \\ x_6 + x_7 = 0 \\ x_7 + x_8 = 0 \end{cases}$$

5. Decide if each of the following sets is linearly dependent or linearly independent:
   (a) $\{(4,6),(1,1),(3,7)\}$ in $\mathbb{R}^2$
   (b) $\{(1,-1),(1,1)\}$ in $\mathbb{R}^2$
   (c) $\{(0,1,1),(1,0,1),(1,1,0)\}$ in $\mathbb{R}^3$
   (d) $\{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}2&2\\2&2\end{smallmatrix}\right)\}$ in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$
   (e) $\{t^2+5t, t-1, t, 7\}$ in $\mathbb{R}[t]_3$

6. Find bases for $\ker T$ and ImT for each of the following linear transformations:

   (a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (x+y, 2x+2y)$
   (b) $T\colon \mathbb{R}^4 \to \mathbb{R}^2$, $T(x,y,z,w) = (x,w)$
   (c) $T\colon \mathbb{R} \to \mathbb{R}^2$, $T(x) = (x, 5x)$
   (d) $T\colon \mathbb{R}[t]_2 \to \mathbb{R}[t]_3$, $T(p(t)) = \int p(t)\,\mathrm{d}t$
   (e) $T\colon \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathbb{R}^2$, $T(A) = (a_{11}, a_{22})$, where $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$

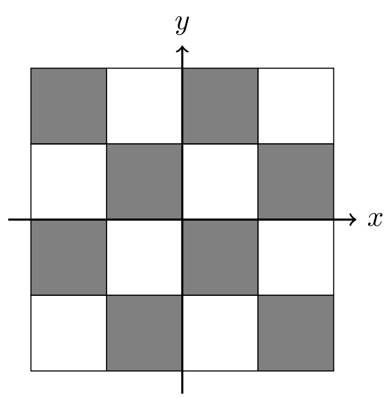
7. Compute the determinant of the matrices $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 5 & 4 & 4 \end{pmatrix}$.

8. Suppose that A is an invertible matrix. Is its transpose A^T also invertible? Justify.

9. Suppose that A is symmetric. Is A invertible? Justify.

10. Suppose that A and B are invertible. Is A^2B^2 invertible? What about $ABAB$?

11. Sketch the image of the checkerboard below under each of linear transformation:



   (a) $T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
   (b) $T = \begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$
   (c) $T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
   (d) $T = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$

12. Find the dimension of the subspace of $\mathbb{R}^5$ given by vectors of the form

$$\big(a-3b+c,\ 2a-6b-2c,\ 3a-9b+c,\ c,\ 2a-6b+6c\big)$$

13. Find the eigenvalues of the following matrices

(a) $\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

14. Find A^{100} where $A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$.

15. Suppose that A is symmetric. Is A^2 symmetric? Justify.

16. Find an orthonormal basis of the subspace $\{(x,y,z) \in \mathbb{R}^3 : x+y+z=0\}$.

17. Find the equation of the 1-dimensional subspace of $\mathbb{R}^2$ orthogonal to the line $y = mx+b$.

18. What are all the possible lengths of the line segments joining vertices of a cube in $\mathbb{R}^3$ whose side lenght is 1? (Bonus: What about for a "hypercube" in $\mathbb{R}^n$?)

19. If A is diagonalizable, prove that $A + c\,\mathrm{Id}$ is also diagonalizable for any $c \in \mathbb{R}$.

20. Find a least-squares solution to $Ax = b$ where $A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$.

21. Find a change of variables $(x_1, x_2) \mapsto (y_1, y_2)$ which eliminates the cross-terms in the quadratic form $Q(x_1,x_2) = 4x_1^2 + 8x_1x_2 + 10x_2^2$.

22. Does the matrix $\begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & 3 \end{pmatrix}$ admit an orthonormal basis of eigenvectors? Justify.

23. Suppose that $p(t)$, $q(t)$, and $r(t)$ are elements of $\mathbb{R}[t]_7$ satisfying the following:

$$\langle p,p\rangle = 4,\ \langle p,q\rangle = 0,\ \langle p,r\rangle = 8,\ \langle q,q\rangle = 1,\ \langle q,r\rangle = 0,\ \langle r,r\rangle = 50.$$

(a) Compute $\langle p, q+r\rangle$.

(b) Compute $\|p+r\|$ and $\|q+r\|$.

(c) Find the orthogonal projection of r onto $\operatorname{span}\{p,q\}$. (Write your answer as a linear combination of p and q).

(d) Find an orthonormal basis of $\operatorname{span}\{p,q,r\}$.

24. Find the singular values of the matrices $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 2 & 2 \end{pmatrix}$.