Happy St. Patrick's Day!

Continuous Random Variables

(Say: $\Omega \subseteq \mathbb{R}$)

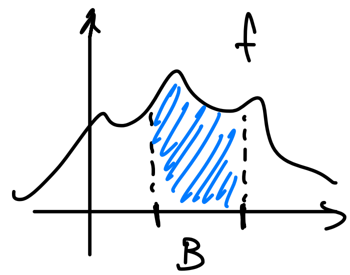
A random variable $X : \Omega \to \mathbb{R}$ is a continuous random variable if there exists a nonnegative function

$$f : \Omega \to [0, +\infty)$$

must satisfy: $\int_{-\infty}^{+\infty} f(x)\,dx = 1$. $\int_\Omega f =$

called probability density function, such that:

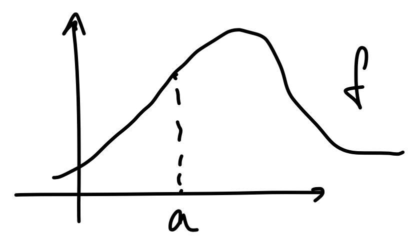
$$P(X \in B) = \int_B f(x)\,dx$$



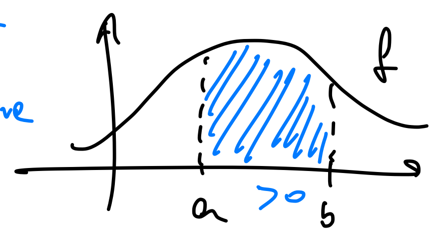
where $B \subset \Omega$ is any (measurable) subset.

Note:

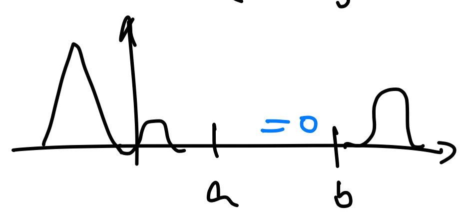
$$P(X = a) = \int_a^a f(x)\,dx = 0$$



$$P(a \le X \le b) = \int_a^b f(x)\,dx$$ ← might be positive



$$-\infty \le a \le b \le +\infty$$



Important: $P(-\infty \le X \le \infty) = 1$

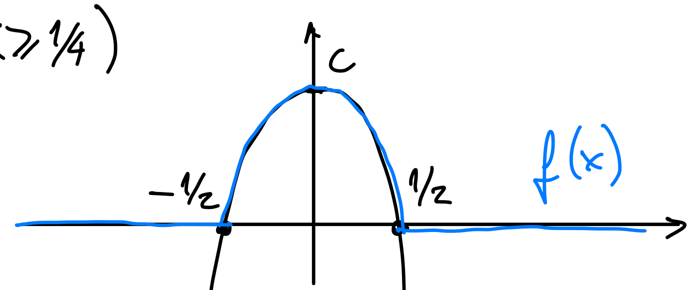
$$\parallel$$

$$\int_{-\infty}^{+\infty} f(x)\,dx$$

Ex: Suppose $X:\mathbb{R}\to[0,+\infty)$ is a random variable with prob. density function $f(x)=\max\{0,\ c(1-4x^2)\}$; where c is some real number.

a) Find the value of $c\in\mathbb{R}$ so that $f(x)$ is a p.d.f.

b) Compute $P(X\geq 0)$ and $P(X\geq 1/4)$



a) Need

$$1=\int_{-\infty}^{+\infty} f(x)\,dx=\underbrace{\int_{-\infty}^{-1/2} f(x)\,dx}_{=0}+\int_{-1/2}^{1/2} f(x)\,dx+\underbrace{\int_{1/2}^{+\infty} f(x)\,dx}_{=0}=\int_{-1/2}^{1/2} f(x)\,dx$$

$$=c\int_{-1/2}^{1/2}(1-4x^2)\,dx=2c\cdot\left(x-4\frac{x^3}{3}\right)\Bigg|_0^{1/2}=2c\left(\frac{1}{2}-\frac{\cancel{4}}{3}\cdot\frac{1}{\cancel{8}\,2}\right)$$

$$=2\left(\frac{1}{2}-\frac{1}{6}\right)c=\frac{2}{3}c \implies \boxed{c=\frac{3}{2}}$$

b)

$$P(\underbrace{X\geq 0}_{\substack{\Updownarrow\\ X\in[0,+\infty)}})=\int_0^{+\infty} f(x)\,dx=\int_0^{1/2}\frac{3}{2}(1-4x^2)\,dx=\cdots=\boxed{\frac{1}{2}}$$

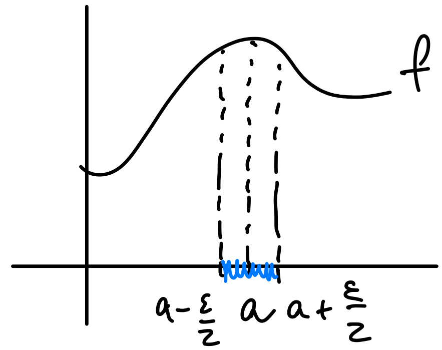
$$P(\underbrace{X\geq 1/4}_{\substack{\Updownarrow\\ X\in[1/4,+\infty)}})=\int_{1/4}^{+\infty} f(x)\,dx=\int_{1/4}^{1/2}\frac{3}{2}(1-4x^2)\,dx=\frac{3}{2}\left(x-\frac{4x^3}{3}\right)\Bigg|_{1/4}^{1/2}=\cdots$$

$$\cdots=\boxed{\frac{5}{32}}$$

Cumulative Distribution Function (C.D.F.)

X cont. random variable, $f : \Omega \to [0, +\infty)$ prob. density function (p.d.f.)

$$F(x) := P(\underbrace{X \le x}_{\Updownarrow \; X \in (-\infty, x]}) = \int_{-\infty}^{x} f(t)\,dt$$



Fund. Thm. Calculus: $F'(x) = f(x)$.

cf.

$$\underbrace{P\left(a - \frac{\varepsilon}{2} \le X \le a + \frac{\varepsilon}{2}\right)}_{\text{prob. of "finding" } X \text{ in a small region of size } \varepsilon > 0 \text{ near } a} = \int_{a-\varepsilon/2}^{a+\varepsilon/2} f(x)\,dx \overset{\text{Taylor}}{=} \boxed{f(a)} \cdot \varepsilon + O(\varepsilon^2)$$

size of region

"density"

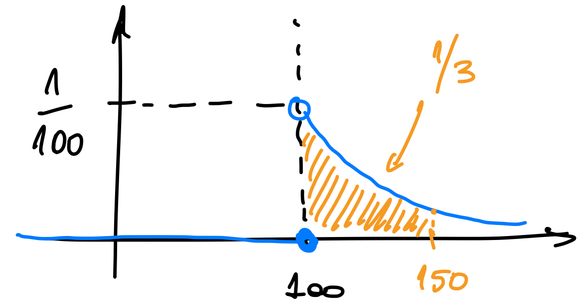
$$\frac{d}{d\varepsilon} \int_a^{a+\varepsilon/2} f(x)\,dx = \frac{1}{2} f\left(a + \frac{\varepsilon}{2}\right).$$

$$\lim_{\varepsilon \to 0} \frac{1}{2}\left[f(a+\varepsilon/2) + f(a-\varepsilon/2)\right] = f(a)$$

$$\frac{d}{d\varepsilon} \int_{a-\varepsilon/2}^{a} f(x)\,dx = -\frac{d}{d\varepsilon} \int_a^{a-\varepsilon/2} f(x)\,dx = +\frac{1}{2} f\left(a - \frac{\varepsilon}{2}\right)$$

Ex: Suppose the lifetime (in hours) of a certain circuit in an electronic device is a random variable w/ p.d.f.

$$f(x) = \begin{cases} 0 & \text{if } x \le 100 \\ \dfrac{100}{x^2} & \text{if } x > 100 \end{cases}$$



Suppose the device has 5 circuits. What is the prob. that exactly 2 of these circuits will need to be replaced within the first 150 hours of operation? (Assume the circuits fail independently)

X = lifetime of a circuit. (cont. random variable)

$$P(\underbrace{X \le 150}_{\Updownarrow \; X \in (-\infty, 150]}) = \int_{-\infty}^{150} f(x)\,dx = \int_{100}^{150} \frac{100}{x^2}\,dx = 100\left(-\frac{1}{x}\right)\Big|_{100}^{150}$$

$$= 100\left(-\frac{1}{150} + \frac{1}{100}\right) = 100\left(\frac{-2+3}{300}\right) = \boxed{\frac{1}{3}}$$

prob. that 1 circuit fails in the first 150 hours.

$Y \sim \text{Binomial}\left(5, \frac{1}{3}\right)$

$$P(Y=2) = \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = \cdots = \boxed{\frac{80}{243}}$$

Recall: if X is discrete rand. var.

$$E(X) = \sum_x x\,p(x), \quad E(g(X)) = \sum_x g(x)p(x), \quad Var(X) = E(X^2) - E(X)^2$$

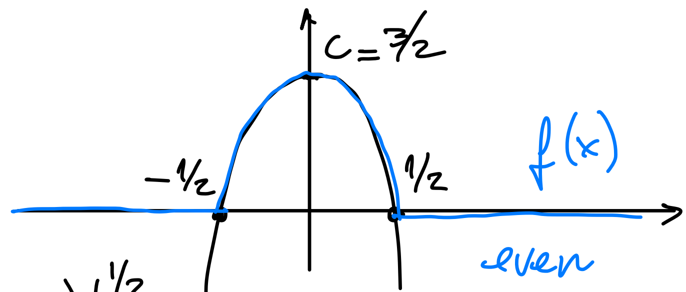
For a cont. random variable X with p.d.f $f(x)$:

$$E(X)=\int_{-\infty}^{+\infty} x\cdot f(x)\,dx, \qquad E(g(X))=\int_{-\infty}^{+\infty} g(x)f(x)\,dx$$

(as before: $Var(X)=E(X^2)-E(X)^2$.)

Revisit the 1st example: $f(x)=\max\left\{0,\ \frac{3}{2}(1-4x^2)\right\}$

$$E(X)=\int_{-1/2}^{1/2} x\,\underbrace{\frac{3}{2}(1-4x^2)}_{f(x)}\,dx =$$



$$=\frac{3}{2}\int_{-1/2}^{1/2} x-4x^3\,dx=\frac{3}{2}\left(\frac{x^2}{2}-x^4\right)\Big|_{-1/2}^{1/2}=0.$$

$$E(X^2)=\int_{-1/2}^{1/2} x^2\,\underbrace{\frac{3}{2}(1-4x^2)}_{f(x)}\,dx=3\int_0^{1/2} x^2-4x^4\,dx=\cancel{3}\left(\frac{x^3}{\cancel{3}}-\frac{3\cdot 4x^5}{5}\right)\Big|_0^{1/2}$$

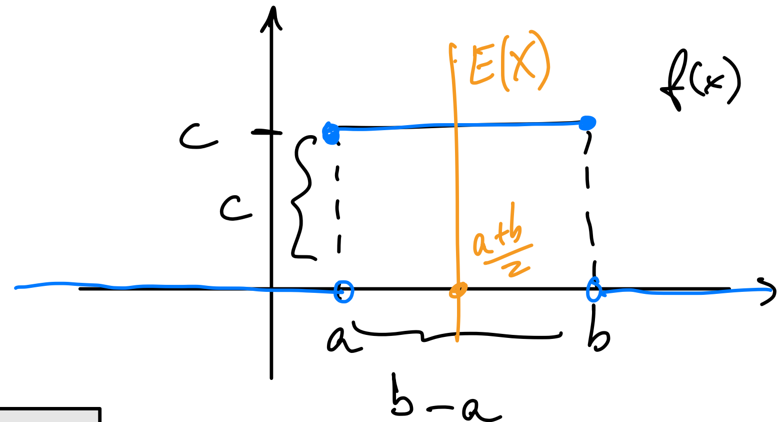
$$=\frac{1}{8}-\frac{\cancel{12}^{\,3}}{5}\,\frac{1}{\cancel{32}_{\,8}}=\frac{1}{8}\left(1-\frac{3}{5}\right)=\frac{1}{8}\cdot\frac{2}{5}=\boxed{\frac{1}{20}}$$ ← second moment

$$Var(X)=E(X^2)-\underbrace{E(X)^2}_{=0}=\boxed{\frac{1}{20}}$$

$$\sigma_X=\frac{1}{\sqrt{20}}=\frac{1}{2\sqrt{5}}.$$

Uniform distribution

A continuous random variable X is <u>uniformly distributed</u> if its p.d.f. only assumes 2 values: 0 and c.



$$1 = \int_{-\infty}^{+\infty} f(x)\,dx = (b-a)\cdot c$$

$$\Rightarrow \quad c = \frac{1}{b-a}$$

$$\boxed{f(x) = \begin{cases} 0 & \text{if } x \notin [a,b] \\ \frac{1}{b-a} & \text{if } x \in [a,b]. \end{cases}}$$

"all points are equally likely"

$$E(X) = \int_a^b x \underbrace{\frac{1}{b-a}}_{f(x)} dx = \frac{1}{b-a}\cdot \frac{x^2}{2}\Big|_a^b = \frac{1}{b-a}\cdot\frac{b^2-a^2}{2} = \frac{a+b}{2}.$$

average of a & b.

$$Var(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \int_a^b x^2 \underbrace{\frac{1}{b-a}}_{f(x)} dx = \frac{1}{b-a}\ \frac{x^3}{3}\Big|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

$$Var(X) = \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(a^2+ab+b^2).4}{3.4} - \frac{(a^2+2ab+b^2).3}{4.3}$$

$$= \frac{1}{12}\left(a^2+b^2-2ab\right) = \boxed{\frac{(a-b)^2}{12}} = \frac{(b-a)^2}{12}$$

b − a is the length of the interval!