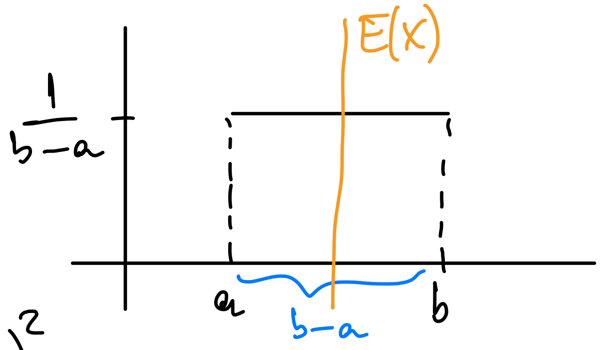
Quick recap: Uniform random variables

$X \sim \text{Uniform}([a,b])$



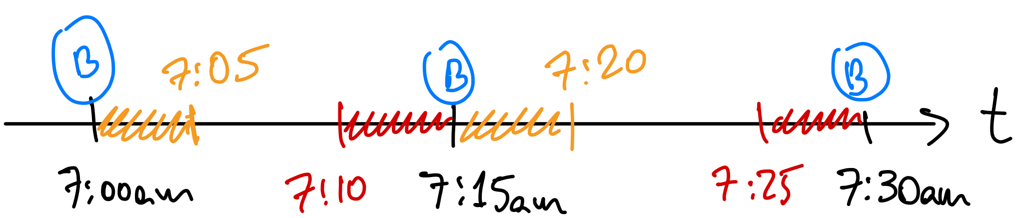
$$E(X) = \frac{a+b}{2}$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{(b-a)^2}{12}$$

Ex: Buses leave every 15 min starting at 7:00 am from the bus stop near your home, and you typically arrive to the bus stop any time between 7:00 am and 7:30 am, uniformly distributed. What is the prob. that

a) you have to wait < 5 min?

b) ——"—— > 10 min?



X = minutes past 7:00 when you arrive to bus stop

$X \sim \text{Uniform}([0,30])$ $\qquad f(x) = \frac{1}{30-0} = \boxed{\frac{1}{30}}$

$$P(10 < X < 15) = \int_{10}^{15} \frac{1}{30}\,dx = \frac{1}{30}\cdot(15-10) = \frac{5}{30} = \frac{1}{6}.$$

$$P(25 < X < 30) = \int_{25}^{30} \frac{1}{30}\,dx = \frac{1}{6}$$

$$\boxed{a)\ \frac{1}{3}}$$

a) $P(0<X<5)+P(15<X<20) = \boxed{\frac{1}{3}}$

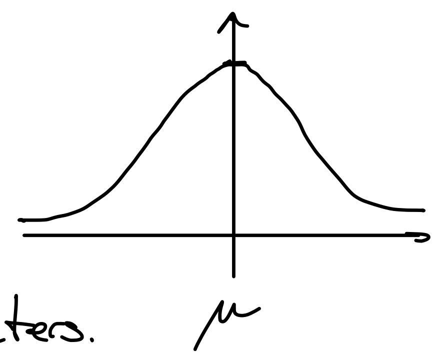
two intervals with the same combined length as the previous two in a).

Normal random Variables

Very important in real life applications

A continuous random variable X is normally distributed if its p.d.f. is of the form

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



where $\sigma > 0$ and $\mu \in \mathbb{R}$ are parameters.

Fact: $E(X)=\mu$, $\quad Var(X)=\sigma^2$.

Let us check that $f(x)$ above is a p.d.f.:

$z = \frac{x-\mu}{\sigma}$ $\quad dz = \frac{dx}{\sigma}$

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{+\infty} e^{-z^2/2} dz}_{I = \sqrt{2\pi}} . = 1$$

Let $I = \int_{-\infty}^{+\infty} e^{-z^2/2} dz$. (>0)

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2/2} dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2/2} dy\right)$$

Fubini

$$\overset{\text{Fubini}}{=} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underbrace{e^{-x^2/2} \cdot e^{-y^2/2}}_{e^{-\frac{x^2+y^2}{2}}} \, dx\,dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{x^2+y^2}{2}} \, dx\,dy$$

Polar

Replace x, y w/ r, θ:

$x^2+y^2=r^2$

$dx\,dy = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^{+\infty} e^{-r^2/2} \, r\,dr\,d\theta = 2\pi \int_0^{+\infty} r e^{-r^2/2} \, dr$$

$u = -r^2/2$

$du = -r\,dr$

$$= 2\pi \left(-e^{-r^2/2} \right) \Big|_0^{+\infty} = 2\pi\big(0-(-1)\big) = 2\pi$$

$\therefore$ $I = \sqrt{2\pi}$ Thus $f(x)$ is a p.d.f.

Affine functions of normal random variables are normal:

$$X \sim \text{Normal}(\mu, \sigma^2) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\boxed{F_X'(x) = f(x)}$$

Let $Y = aX+b$, $a, b \in \mathbb{R}$.

$$F_X(x) = P(X \le x), \quad F_Y(y) = P(Y \le y)$$

Suppose $a > 0$.

$$F_Y(x) = P(Y \le x) = P(aX+b \le x) = P\left(X \le \frac{x-b}{a}\right)$$

$$= F_X\left(\frac{x-b}{a}\right).$$

Taking derivatives on both sides:

$$f_Y(x) = f_X\left(\frac{x-b}{a}\right)\cdot\frac{1}{a} = \frac{1}{a}\,\frac{1}{\sqrt{2\pi}\,\sigma}\,e^{-\left(\frac{x-b}{a}-\mu\right)^2/2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi}\,(a\sigma)}\,e^{-(x-b-a\mu)^2/2(a\sigma)^2}$$

$$= \frac{1}{\sqrt{2\pi}\,(a\sigma)}\,e^{-\left(x-(a\mu+b)\right)^2/2(a\sigma)^2}$$

This is exactly the p.d.f. of a normal random variable:

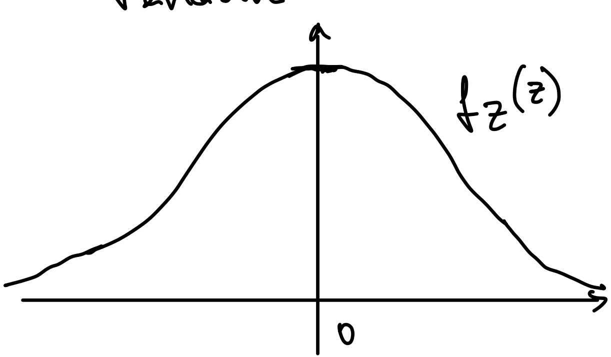
So $Y \sim \text{Normal}\left(a\mu+b,\ a^2\sigma^2\right)$.

<u>Upshot</u>: If $X \sim \text{Normal}(\mu, \sigma^2)$, and $Y = aX+b$, then:

$$Y \sim \text{Normal}\left(a\mu+b,\ a^2\sigma^2\right).$$

Useful application: "standardization".

<u>Def</u>: A <u>standard</u> normal random variable is a normal random variable with $\mu = 0$ and $\sigma = 1$.



$$Z \sim \text{Normal}(0,1).$$

$$\boxed{Z} = aX + b \qquad Z \sim \text{Normal}\big(\underbrace{a\mu + b}_{=0},\ \underbrace{a^2\sigma^2}_{=1}\big).$$

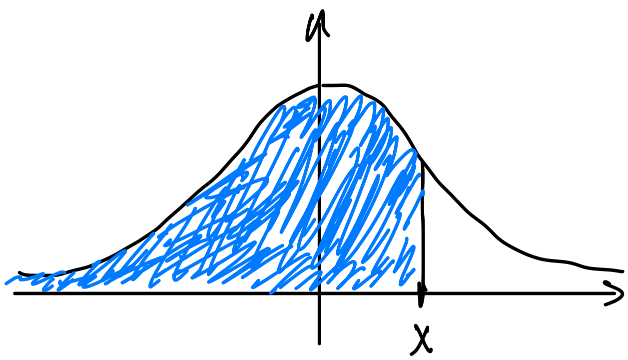
$$\left\{\begin{array}{l} a\mu + b = 0 \Rightarrow b = -a\mu = -\mu/\sigma \\ a\sigma = 1 \Rightarrow a = 1/\sigma \end{array}\right.$$

$$= \frac{1}{\sigma}X - \frac{\mu}{\sigma}$$

$$= \boxed{\frac{X-\mu}{\sigma}}$$

Using "standardization" to compute probabilities

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\, dz$$



$$P(Z \le z) = \Phi(z)$$

$(x > 0)$

Table 5.1 Area $\Phi(x)$ Under the Standard Normal Curve to the Left of X.

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Ex: The height of maple trees at age 10 is estimated to be normally distributed with $\mu = 200$ cm and $\sigma^2 = 64$ cm. What is the probability that a 10-yr old maple tree has height

a) smaller than 204 cm?

b) smaller than 180 cm?

c) greater than 210 cm?

H = height of a 10-yr. old maple tree $\sim \text{Normal}(\underbrace{200}_{\mu}, \underbrace{64}_{\sigma^2})$

$\mu = 200, \quad \sigma = 8.$

a)
$$P(H < 204) = P\left(Z < \frac{204-200}{8}\right) = P(Z < 0.5) = \Phi(0.5)$$

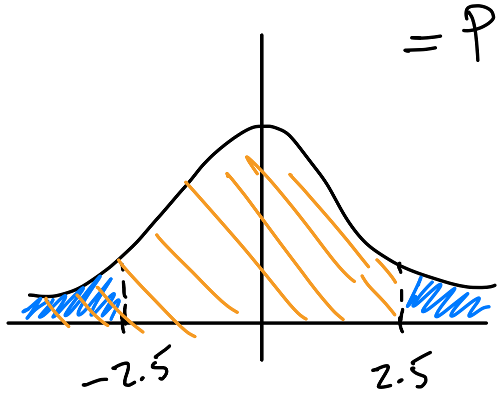
$$\overset{\text{Table}}{=} 0.6915$$

$$= \underline{\underline{69.15\%}}$$

$$Z = \frac{H-200}{8} \sim \text{Normal}(0,1)$$

b)
$$P(H < 180) = P\left(Z < \frac{180-200}{8}\right) = P\left(Z < -\frac{5}{2}\right)$$

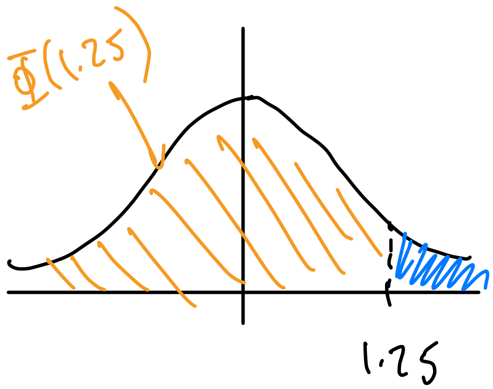
$$= P(Z < -2.5) = 1 - \underline{P(Z < 2.5)}$$



$$\overset{\text{Table}}{=} 1 - 0.9938$$

$$= 0.0062 = \underline{\underline{0.62\%}}$$

c) $P(H > 210) = P\left(Z > \frac{210 - 200}{8}\right) = P(Z > 1.25)$



$$= 1 - \Phi(1.25) \underset{\text{Table}}{=} 1 - 0.8944 = 0.1056$$

$$= \underline{\underline{10.56\%}}$$