MAT 330/681 Lecture 16 4/5/2021

Prob. Density Function of $g(X)$

X random variable

$f_X(x) = $ p.d.f. of X

$F_X(x) = $ c.d.f. of X

$F_X'(x) = f(x)$

$\xrightarrow{g}$

$g: \mathbb{R} \to \mathbb{R}$

$Y = g(X)$ "new" random var.

$f_Y(y) = $ p.d.f. of Y ?

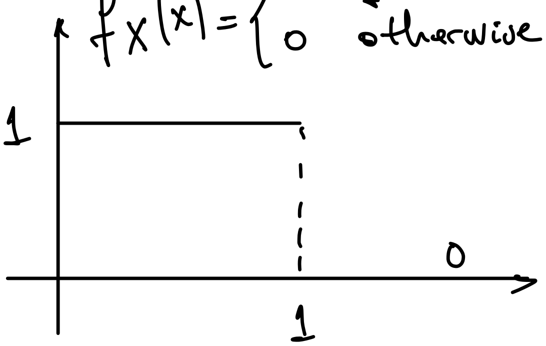
$F_Y(y) = $ c.d.f. of Y ?

$F_Y'(y) = f_Y(y)$

Ex: $g(x) = x^n$ for some $n \in \mathbb{N}$.

$X \sim \text{Uniform}(0,1)$

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$= \int_0^x 1\,dt = x$$

$$F_X(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

$Y = g(X) = X^n$?

$$F_Y(y) = P(Y \leq y) = P(X^n \leq y)$$

$$\overset{(X \geq 0)}{=} P(X \leq y^{1/n})$$

$$= F_X(y^{1/n}) = \begin{cases} y^{1/n} & \text{if } y \leq 1 \\ 1 & \text{if } y \geq 1 \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} y^{1/n} = \frac{1}{n} y^{\frac{1}{n}-1}$$

$$f_Y(y) = \begin{cases} \frac{1}{n} y^{\frac{1}{n}-1} & \text{if } 0 \leq y \leq 1. \\ 0 & \text{otherwise} \end{cases}$$

Thm. If X is a continuous random variable w/ p.d.f. $f_X(x)$ and $g:\mathbb{R}\to\mathbb{R}$ is a monotonic function, differentiable,

increasing or decreasing

then $Y=g(X)$ has p.d.f.

$$f_Y(y)=\begin{cases} f_X\left(g^{-1}(y)\right)\cdot\left|\frac{d}{dy}g^{-1}(y)\right| & \text{if } y=g(x) \text{ for some } x. \\ 0 & \text{otherwise.}\end{cases}$$

Pf. Suppose $g(x)$ is increasing.

$$F_Y(y)=P(Y\le y)=P\left(g(X)\le y\right)\overset{g \text{ increasing}}{=}P\left(X\le g^{-1}(y)\right)=F_X\left(g^{-1}(y)\right).$$

Differentiate in y:

$$f_Y(y)=\frac{d}{dy}F_Y(y)=\frac{d}{dy}F_X\left(g^{-1}(y)\right)\overset{\text{C.R.}}{=}F_X'\left(g^{-1}(y)\right)\cdot\frac{d}{dy}g^{-1}(y)$$

$$=f_X\left(g^{-1}(y)\right)\cdot\frac{d}{dy}g^{-1}(y).$$

If, instead, $g(x)$ is decreasing.

$$F_Y(y)=P(Y\le y)=P\left(g(X)\le y\right)\overset{d}{=}P\left(X\geq g^{-1}(y)\right)=1-P\left(X\le g^{-1}(y)\right)$$

$$=1-F_X\left(g^{-1}(y)\right)$$

Diff. in y:

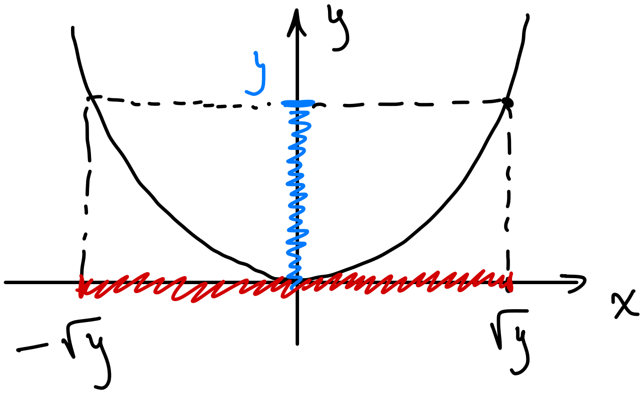
$$f_Y(y)=\frac{d}{dy}F_Y(y)=\frac{d}{dy}\left(1-F_X\left(g^{-1}(y)\right)\right)=-F_X'\left(g^{-1}(y)\right)\frac{d}{dy}g^{-1}(y)$$

$$=-f_X\left(g^{-1}(y)\right)\cdot\frac{d}{dy}g^{-1}(y).$$

Altogether: $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$ if $y \in \operatorname{Im} g$. $\square$

What if $g: \mathbb{R} \to \mathbb{R}$ is not monotonic?

Ex: Suppose $g(x) = x^2$, $\quad Y = g(X) = X^2$

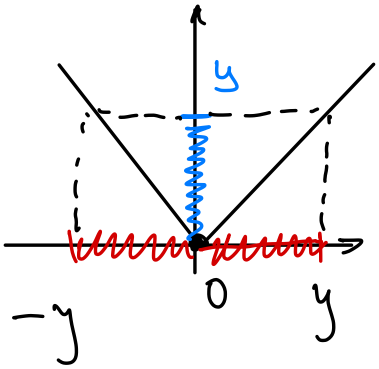


$$F_Y(y) = P(Y \le y) = P(X^2 \le y)$$
$$= P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= P(X \le \sqrt{y}) - P(X \le -\sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

Diff. in y:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(F_X(\sqrt{y}) - F_X(-\sqrt{y})\right)$$
$$= F_X'(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - F_X'(-\sqrt{y}) \cdot \left(-\frac{1}{2\sqrt{y}}\right)$$
$$= \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$
$$= \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right)$$

What if $g:\mathbb{R}\to\mathbb{R}$ is not differentiable?



$g(x)=|x|$

$$Y=g(X)=|X|$$

$$F_Y(y)=P(Y\le y)=P(|X|\le y)=P(-y\le X\le y)$$
$$=F_X(y)-F_X(-y)$$

Diff. in y:

$$f_Y(y)=\frac{d}{dy}F_Y(y)=\frac{d}{dy}\big(F_X(y)-F_X(-y)\big)=f_X(y)+f_X(-y)$$

Lognormal random variables

Motivation: Suppose the price of a stock on day n is S_n.

$$\$:\quad S_1\longrightarrow S_2\longrightarrow S_3\longrightarrow\ ---\ \longrightarrow S_{n-1}\longrightarrow S_n\longrightarrow ---$$

Days Day 1 Day 2 Day 3

rate of return $\sim \text{Normal}(\mu,\sigma^2)$

$$Y=\frac{S_n}{S_{n-1}}=e^X$$

ratio between prices on consecutive days.

use exp. to account for compounding effect in pricing.

Def: If $X\sim \text{Normal}(\mu,\sigma^2)$, then $Y=e^X$ is a lognormal random variable, w/ parameters μ,σ^2.

What is the pdf of Y?

$$Y = g(X), \quad g(x) = e^x \rightsquigarrow g^{-1}(y) = \log y \quad \text{(increasing)}$$

$$\frac{d}{dy} g^{-1}(y) = \frac{1}{y}$$

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(g^{-1}(y)-\mu)^2}{2\sigma^2}} \cdot \frac{1}{y}$$

$$\boxed{= \frac{1}{\sqrt{2\pi}\,\sigma y} e^{-\frac{(\log y - \mu)^2}{2\sigma^2}}}$$

Using the above relation $Y = e^X$, we can compute:

$$E(Y) = e^{\mu + \frac{\sigma^2}{2}}$$

mean (expected value)

$$P(Y \leq e^{\mu}) = \frac{1}{2} = P(Y \geq e^{\mu})$$

median is <u>smaller</u> than mean for lognormal rand. var.

(cf. when $\mu = 0$

Pareto random variables

$\alpha > 0$

Def: If $X \sim \text{Exponential}(\lambda)$, then $Y = \alpha e^{X}$ is a Pareto random variable w/ parameters α and λ.

Recall: $P(X \geq x) = e^{-\lambda x}$

$$P(Y>y) = P(\alpha e^{X} > y) = P\left(e^{X} > \frac{y}{\alpha}\right) = P\left(X > \log \frac{y}{\alpha}\right)$$

$$= e^{-\lambda \log \frac{y}{\alpha}} = e^{\log\left(\frac{y}{\alpha}\right)^{-\lambda}} = \left(\frac{y}{\alpha}\right)^{-\lambda} = \left(\frac{\alpha}{y}\right)^{\lambda}$$

$$F_Y(y) = P(Y \leq y) = 1 - P(Y>y) = 1 - \left(\frac{\alpha}{y}\right)^{\lambda} = 1 - \alpha^{\lambda} y^{-\lambda}$$

Diff. in y:

$$f_Y(y) = \begin{cases} \lambda \alpha^{\lambda} y^{-(\lambda+1)} & \text{if } y \geq \alpha. \\ 0 & \text{if } y < \alpha. \end{cases}$$

Expected Value:

$$E(Y) = \int_{\alpha}^{+\infty} \lambda \alpha^{\lambda} y^{-\lambda}\, dy = \lambda \alpha^{\lambda} \cdot \frac{\alpha^{1-\lambda}}{\lambda - 1} = \frac{\lambda \alpha}{\lambda - 1}.$$

$$\text{Var}(Y) = E(Y^2) - E(Y)^2 = \frac{\lambda \alpha^2}{(\lambda-2)(\lambda-1)^2}.$$

"Pareto's Law" / "80-20 rule":

- 80% of the outcomes are caused by 20% of the actions
- 80% of the profits go to only 20% of the people (*)

Mathematically, this is only true if

$$\alpha = \log_4 5 \cong 1.161.$$

sharpe ratio

However, the following is a mathematically rigorous version of this principle: (*)

Profits are distributed according to a Pareto distribution ($\alpha \geq 1$)	$\Longleftrightarrow$	$\exists\, p \in (0, 1/2)$ s.t. $100p$ % of people receive $100(1-p)$ % of all profits

"80-20 rule"

$$p = 0.2 \Longleftrightarrow \alpha = \log_4 5$$

b/c

$$1 - \frac{1}{\alpha} = \frac{\log(1-p)}{\log p}$$