Jointly distributed Random Variables

Suppose X and Y are discrete random variables

$$p(x,y) = P(X=x, Y=y) \quad \text{joint prob. mass function}$$

$X=x$ and $Y=y$.

$X \backslash Y$	y_1	y_2	---	y_m	marginal prob. on X
x_1	$p(x_1,y_1)$	$p(x_1,y_2)$	---	$p(x_1,y_m)$	$P(X=x_1)$
x_2	$p(x_2,y_1)$	$p(x_2,y_2)$	---	$p(x_2,y_m)$	$P(X=x_2)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_n	$p(x_n,y_1)$	$p(x_n,y_2)$	---	$p(x_n,y_m)$	$P(X=x_n)$
marginal prob. on Y	$P(Y=y_1)$	$P(Y=y_2)$	...	$P(Y=y_m)$	

prob. mass function of X

prob. mass function of Y

Marginal: $$P(X=x_i) = \sum_{j=1}^{m} p(x_i,y_j) = p_X(x_i)$$

$$P(Y=y_j) = \sum_{i=1}^{n} p(x_i,y_j) = p_Y(y_j)$$

Example: Prob. that a pedestrian is hit by a car when crossing a dangerous intersection w/ traffic light

$$H = \begin{cases} 0 & \text{not hit} \\ 1 & \text{hit} \end{cases} \qquad L = \begin{cases} \text{Red} \\ \text{Yellow} \\ \text{Green} \end{cases}$$

H \ L	Red	Yellow	Green	marginal on H
0	0.198	0.09	0.662	0.95
1	0.002	0.01	0.038	0.05
marginal on L	0.2	0.1	0.7	

What is the probability that

a) Pedestrian is not hit and light is yellow?

$$P(H=0, L=\text{yellow}) = 0.09 = 9\%$$

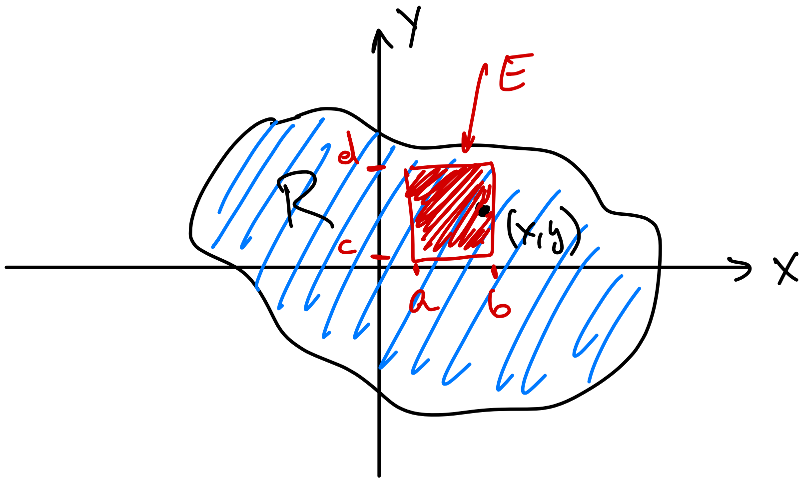
b) Pedestrian is not hit given that light is yellow?

$$P(H=0 \mid L=\text{yellow}) = \frac{P(H=0, L=\text{yellow})}{P(L=\text{yellow})} = \frac{0.09}{0.1} = 0.9 = 90\%$$

c) Pedestrian is hit and light is red or yellow?

$$P(H=1, L=\text{red}) + P(H=1, L=\text{yellow}) = 0.012 = 1.2\%$$

Suppose X and Y are continuous random variables



Joint prob. density function

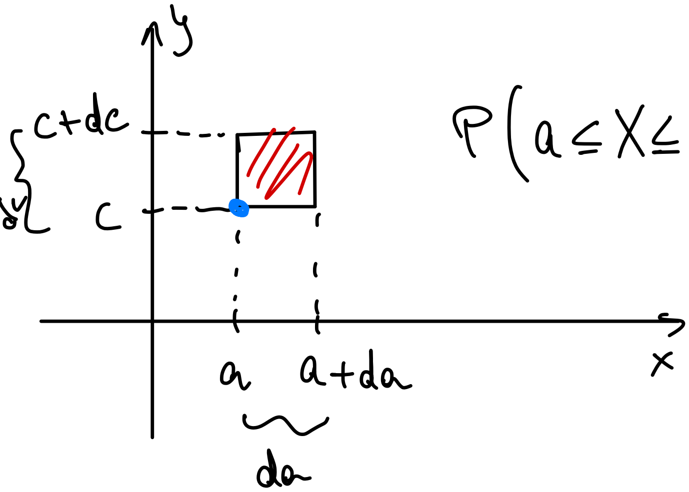
$$f: R \longrightarrow (0, +\infty)$$

$$\iint_R f(x,y)\,dx\,dy = 1.$$

$$P\big((X,Y) \in E\big) = \iint_E f(x,y)\,dx\,dy$$

If $E = [a,b] \times [c,d]$ is a rectangle, then:

$$P\big((X,Y) \in E\big) = \iint_E f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$



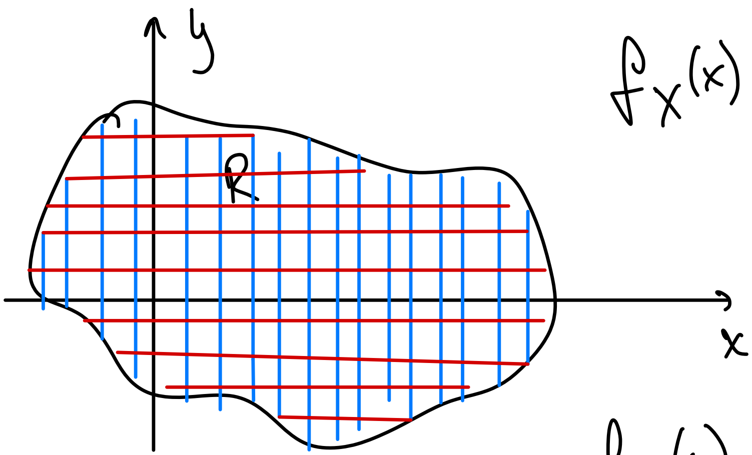
$$P\big(a \le X \le a+da,\ c \le Y \le c+dc\big) = \int_a^{a+da} \int_c^{c+dc} f(x,y)\,dy\,dx$$

$$\simeq \underbrace{f(a,c)}_{\text{density}}\,da\,dc$$

Cumulative Distr. Function: $F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(t,s)\,ds\,dt$

Fund. Thm. of Calculus: $f(a,c) = \dfrac{\partial^2 F}{\partial y\,\partial x}(a,c).$

Marginal distributions:



$$f_X(x) = \int_{-\infty}^{+\infty} \underline{f(x,y)\,dy} \qquad \text{p.d.f. of } X$$

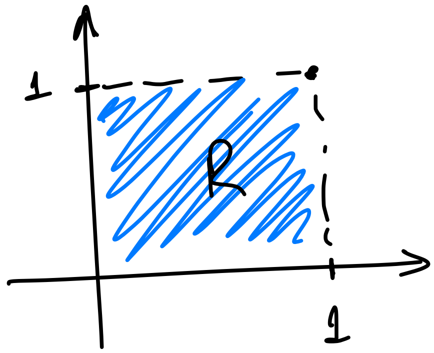
$$f_Y(y) = \int_{-\infty}^{+\infty} \underline{f(x,y)\,dx} \qquad \text{p.d.f. of } Y$$

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx = \int_a^b \underbrace{\left(\int_{-\infty}^{+\infty} f(x,y)\,dy \right)}_{f_X(x)} dx$$

$$P(c \leq Y \leq d) = \int_c^d f_Y(y)\,dy = \int_c^d \underbrace{\left(\int_{-\infty}^{+\infty} f(x,y)\,dx \right)}_{f_Y(y)} dy$$

Example: Suppose X and Y are jointly distributed with p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{7}(x+y)^2 & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$



a) Check that $f_{X,Y}(x,y)$ is a p.d.f.

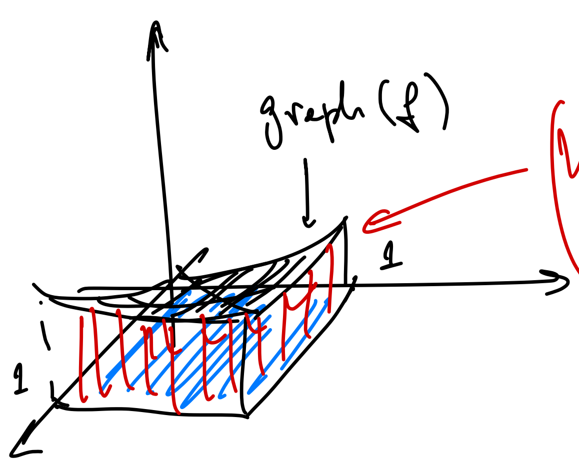
b) Compute $P\left(X<\frac{1}{2}, Y<\frac{3}{4}\right)$.

c) Compute $E(X)$.

a) $$\int_0^1\int_0^1 \frac{6}{7}(x+y)^2\,dx\,dy = \frac{6}{7}\int_0^1\int_0^1 x^2+2xy+y^2\,dx\,dy$$

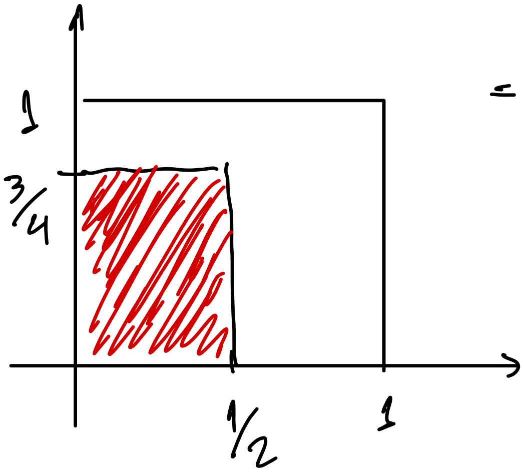
$$= \frac{6}{7}\int_0^1 \left(\frac{x^3}{3}+x^2y+xy^2\right)\Big|_0^1 dy = \frac{6}{7}\int_0^1 \frac{1}{3}+y+y^2\,dy$$

$$= \frac{6}{7}\left(\frac{y}{3}+\frac{y^2}{2}+\frac{y^3}{3}\right)\Big|_0^1 = \frac{6}{7}\Big(\underbrace{\frac{1}{3}+\frac{1}{2}+\frac{1}{3}}_{\frac{2}{3}+\frac{1}{2}=\frac{4+3}{6}}\Big) = \boxed{1}$$



(Volume under the graph of $f(x,y)$ over region R) $= 1$

$$P\left(X<\tfrac{1}{2}, Y<\tfrac{3}{4}\right) = \int_0^{3/4}\int_0^{1/2} \frac{6}{7}(x+y)^2\,dx\,dy = \frac{6}{7}\int_0^{3/4}\left(\frac{x^3}{3}+x^2y+xy^2\right)\Bigg|_0^{1/2} dy$$



$$= \frac{6}{7}\int_0^{3/4} \frac{1}{24}+\frac{y}{4}+\frac{y^2}{2}\,dy = \frac{6}{7}\left(\frac{y}{24}+\frac{y^2}{8}+\frac{y^3}{6}\right)\Bigg|_0^{3/4}$$

$$= \frac{\cancel{6}}{7}\left(\frac{3}{4\cdot\underset{4}{\cancel{24}}}+\frac{9\cdot 3}{16\cdot\underset{4}{\cancel{8}}}+\frac{27}{64\cdot\cancel{6}}\right) = \frac{1}{7}\left(\frac{\overset{12}{\cancel{3}}}{\underset{64}{\cancel{16}}}+\frac{27}{64}+\frac{27}{64}\right)$$

$$= \frac{33}{224} = \underline{0.1473}$$

c)

$$E(X) = \int_0^1 x\,\underbrace{f_X(x)}_{=\int_0^1 f(x,y)\,dy}\,dx = \int_0^1\int_0^1 x\,f(x,y)\,dx\,dy$$

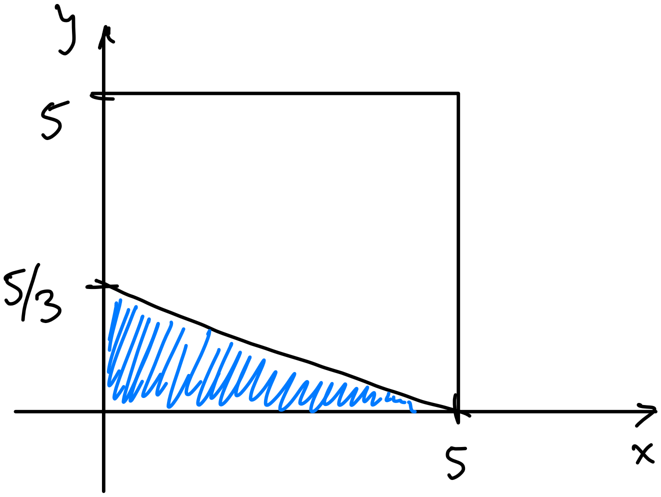
$$= \frac{6}{7}\int_0^1\int_0^1 x(x+y)^2\,dx\,dy = \frac{6}{7}\int_0^1\int_0^1 x^3+2x^2y+xy^2\,dx\,dy$$

$$= \frac{6}{7}\int_0^1\left(\frac{x^4}{4}+\frac{2x^3}{3}y+\frac{x^2}{2}y^2\right)\Bigg|_0^1 dy = \frac{6}{7}\int_0^1 \frac{1}{4}+\frac{2}{3}y+\frac{1}{2}y^2\,dy$$

$$= \frac{6}{7}\left(\frac{y}{4}+\frac{y^2}{3}+\frac{y^3}{6}\right)\Bigg|_0^1 = \frac{6}{7}\left(\frac{1}{4}+\frac{1}{3}+\frac{1}{6}\right) = \boxed{\frac{9}{14}}$$

$$\frac{3+4+2}{12}$$

Example: $X, Y \sim \text{Uniform}(0,5)$ $\qquad f_X(x) = f_Y(y) = \frac{1}{5}.$



$$f_{X,Y}(x,y) = c = \frac{1}{25}$$

$$\int_0^5\int_0^5 f_{X,Y}(x,y)\,dx\,dy = 1.$$

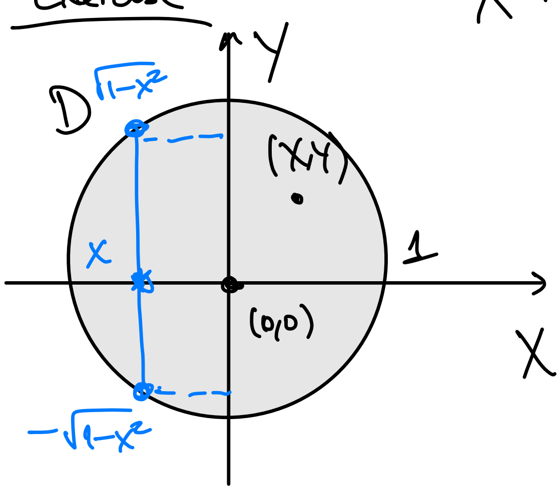
$$25 \cdot c = 1 \Rightarrow c = \frac{1}{25}$$

$$P(\underbrace{X+3Y<5}_{R}) = \iint_R f_{X,Y}(x,y)\,dx\,dy = \frac{1}{25}\,\text{Area}(R) = \frac{1}{\cancel{25}}\left(\cancel{\frac{5}{3}}\cdot\cancel{5}\cdot\frac{1}{2}\right)$$

$$= \frac{1}{6}.$$

cf. $\displaystyle\int_0^5\int_0^{\frac{5}{3}-\frac{x}{3}} \frac{1}{25}\,dy\,dx = \cdots = \frac{1}{6}.$

Exercise: $X^2+Y^2 \le 1$



(X,Y) is distributed uniformly on D

a) Find $f_X(x)$ and $f_Y(y)$.

b) Find the probability that $\rho = \text{dist}\big((X,Y),(0,0)\big) \le a$

c) $E(\rho) = ?$

$$f(x,y) \equiv \frac{1}{\text{Area}(D)} = \frac{1}{\pi}$$

a) $$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\, dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy = \frac{1}{\pi}\left(\sqrt{1-x^2} + \sqrt{1-x^2}\right)$$

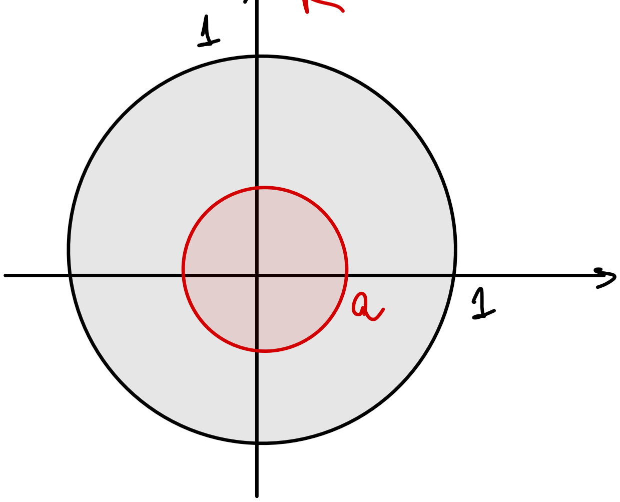
$$= \frac{2}{\pi}\sqrt{1-x^2}$$

$$x^2 + y^2 = 1 \rightsquigarrow y^2 = 1 - x^2$$

$$y = \pm\sqrt{1-x^2}$$

$$\boxed{f_X(x) = \frac{2}{\pi}\sqrt{1-x^2}} \qquad \boxed{f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}}$$

b) $$P(\underbrace{\rho \le a}_{R}) \overset{\text{Unif. d}}{=} \frac{\text{Area}(R)}{\text{Area}(D)} = \frac{\pi a^2}{\pi} = a^2 \longleftarrow$$ Cumulative distr. function of ρ.



$$F_\rho(a) = a^2 \Longrightarrow f_\rho(a) = \frac{d}{da} F_\rho(a) = 2a$$

c) $$E(\rho) = \int_0^1 a\, f_\rho(a)\, da$$

$$= \int_0^1 a \cdot 2a\, da = \int_0^1 2a^2\, da$$

$$= 2 \cdot \left.\frac{a^3}{3}\right|_0^1 = \boxed{\frac{2}{3}}$$