MAT 330/681 Lecture 2 2/3/2021

Quick recap:

- Permutations of n objects: $n!$

- Permutations of n objects (n_1 of them alike, n_2 — " —, ⋮, n_k — " —) $\dfrac{n!}{n_1! \cdots n_k!}$

- Combinations of n objects taken k at a time $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

"choose k objects from n available objects, with the order in which they are chosen being irrelevant"

Ex: From a group of 5 girls and 7 boys, a team of 2 girls and 3 boys must be chosen.

a) How many teams are possible?

$$\underbrace{\binom{5}{2}}_{\overset{\text{teams of girls}}{=\,10}} \underbrace{\binom{7}{3}}_{\overset{\text{teams of boys}}{=\,35}} = \frac{5! = 5.4!}{3!\,2!} \cdot \frac{7!}{\cancel{4!}\,3!} = \frac{5.\,7.\,\cancel{6}.5.\overset{2}{\cancel{4}}.\cancel{3!}}{\cancel{3!}\,\cancel{2!}\,\cancel{3!}} = \boxed{350}$$

b) What if 2 of the boys are fighting and cannot be put in the same team?

"Bad teams"

fighting boy 1 — fighting boy 2 — any of the last 5 remaining boys

$$\underbrace{\binom{2}{2}}_{1}\cdot\underbrace{\binom{5}{1}}_{5} = 5 \text{ bad teams of boys}$$

(out of 35 teams of boys)

Total number of teams available now is

$$\underset{\text{"good" teams of boys}}{30}\cdot\underbrace{\binom{5}{2}}_{10} = \boxed{300}$$

Pascal's Triangle

← Convenient way to arrange binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$ - - -	
$n=0$	$\binom{0}{0}=1$					
$n=1$	$\binom{1}{0}=1$	$\binom{1}{1}=1$				
$n=2$	$\binom{2}{0}=1$	$\binom{2}{1}=2$	$\binom{2}{2}=1$			
$n=3$	$\binom{3}{0}=1$	$\binom{3}{1}=3$	$\binom{3}{2}=3$	$\binom{3}{3}=1$		
$n=4$	$\binom{4}{0}=1$	$\binom{4}{1}=4$	$\binom{4}{2}=6$	$\binom{4}{3}=4$	$\binom{4}{4}=1$	
$n=5$	$\binom{5}{0}=1$	$\binom{5}{1}=5$	$\binom{5}{2}=10$	$\binom{5}{3}=10$	$\binom{5}{4}=5$	$\binom{5}{5}=1$

Pascal's identities:

symmetry "across the middle" of Pascal's triangle

① $$\binom{n}{k} = \binom{n}{n-k}$$

choosing k objects out of n to take.

choosing $n-k$ objects out of n to leave behind.

② $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

sum of two adjacent numbers in Pascal's triangle gives the number "below"

choosing k objects out of n

choices that include object $*$

choices that do not include object $*$.

Binomial Theorem:

For all $n \in \mathbb{N}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Pf: Proof by induction on n:

$n=1$ $$(x+y)^1 = \sum_{k=0}^{1} \binom{1}{k} x^k y^{1-k} = \underbrace{\binom{1}{0} x^0 y^{1-0}}_{k=0} + \underbrace{\binom{1}{1} x^1 y^{1-1}}_{k=1}$$

$$= y + x$$

Step: Suppose the formula holds for $n-1$, and let us

Show that if then must also hold for n.

$$(x+y)^n = (x+y)(x+y)^{n-1}$$

$$\overset{\text{induction hypothesis } (n-1)}{=} (x+y)\left(\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k}\right)$$

$$= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{k+1}y^{n-1-k} + \sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-k}$$

(use $i=k+1$) (use $i=k$)

$$= \sum_{i=1}^{n-1}\binom{n-1}{i-1}x^i y^{n-i} + \sum_{i=1}^{n-1}\binom{n-1}{i}x^i y^{n-i}$$

$$+ \underbrace{\binom{n-1}{0}x^0 y^{n-0}}_{\substack{i=0\\(\text{second sum})}} + \underbrace{\binom{n-1}{n-1}x^n y^{n-n}}_{\substack{i=n\\(\text{first sum})}}$$

$$= x^n + y^n + \sum_{i=1}^{n-1}\underbrace{\left[\binom{n-1}{i-1}+\binom{n-1}{i}\right]}_{\binom{n}{i}} x^i y^{n-i}$$

$\leftarrow$ Pascal's identity

$$= \binom{n}{0}x^n + \sum_{i=1}^{n-1}\binom{n}{i}x^i y^{n-i} + \binom{n}{n}y^n$$

$$= \sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i} \qquad \square$$

Multinomial coefficients

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! \, n_2! \cdots n_r!}$$

Permutations of n objects, out of which n_1 are alike, ... , n_r are alike.

here $n = n_1 + n_2 + \cdots + n_r$

Note: $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{k, n-k}$ $\quad r=2$, $n_1 = k$, $n_2 = n-k$

Q: Suppose we have n objects that we must divide into r different categories, with n_1 of them in the first category, ..., n_r in the r^{th} category.

A:

$$\underbrace{\binom{n}{n_1, n_2, \ldots, n_r}}_{\substack{\text{\# of permutations of } n \text{ objects} \\ \text{w/ } n_1 \text{ alike}, \ldots, n_r \text{ alike.}}} = \underbrace{\binom{n}{n_1}}_{\substack{\text{choosing } n_1 \text{ objects that} \\ \text{belong to first category}}} \cdot \underbrace{\binom{n-n_1}{n_2}}_{\substack{\text{choosing } n_2 \text{ objects to} \\ \text{belong to second category}}} \underbrace{\binom{n-n_1-n_2}{n_3}}_{\cdots} \cdots \underbrace{\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}}_{\substack{\text{choosing } n_r \text{ objects to} \\ \text{belong to last category}}}$$

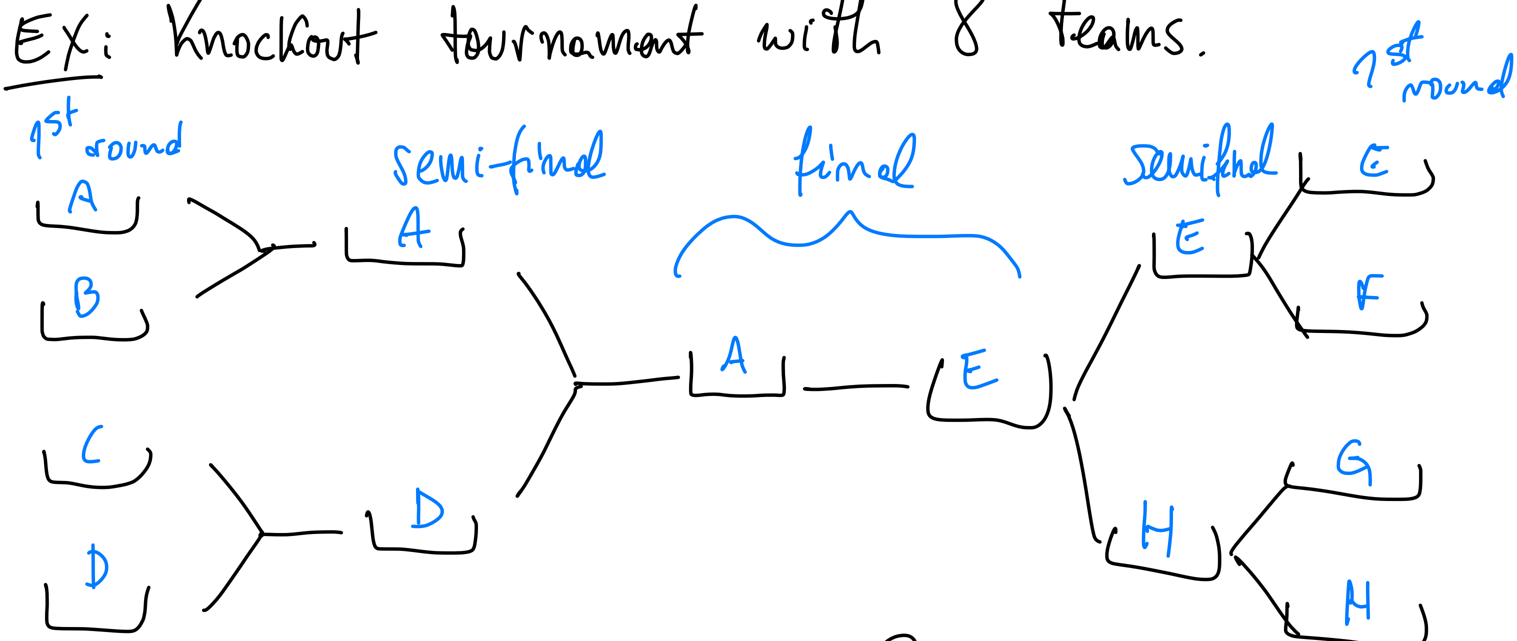
Multinomial theorem:

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1,\ldots,n_r) \\ n_1+\cdots+n_r=n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}.$$

EX: You have a bowl with 10 blueberries, 7 grapes and 4 raspberries. In how many ways can you eat them (one at a time)?

$$\binom{21}{10, 7, 4} = \frac{21!}{10!\,7!\,4!} = 116{,}396{,}280.$$

EX: Knockout tournament with 8 teams.



How many possible outcomes?

$\underline{1^{st} \text{ round}}$: $\binom{8}{2,2,2,2}\dfrac{1}{4!}\cdot 2^4 = \dfrac{8!}{\cancel{2!}\cancel{2!}\cancel{2!}\cancel{2!}}\,\dfrac{\cancel{2^4}}{4!} = \boxed{\dfrac{8!}{4!}}$

choosing 2 teams (a pair) to play each game of the first round

order of the pairs does not matter

1 team wins
1 team loses

outcomes in 1st round

$\underline{2^{nd} \text{ round}}$ (semi-final)

$$\binom{4}{2,2}\frac{1}{2!}\,2^2 = \frac{4!}{2!\cancel{2!}}\,\frac{1}{\cancel{2!}}\,\cancel{2^2} = \boxed{\frac{4!}{2!}}$$

outcomes in 2^{nd} round

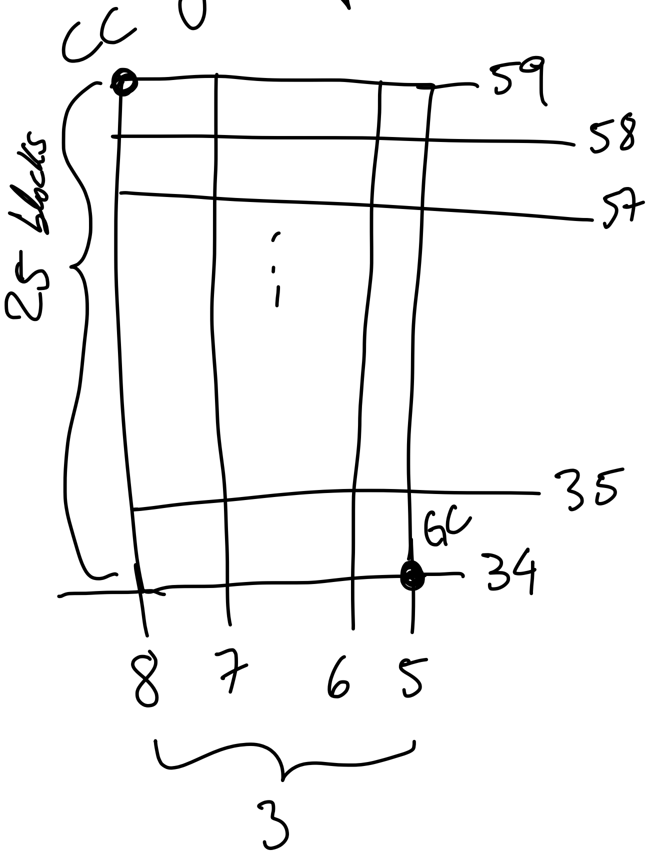
$\underline{3^{rd} \text{ round}}$ (final)

$\binom{2}{2}\cdot 2 = 2$ outcomes.

$\underline{\text{Total}}$: $\underbrace{\dfrac{8!}{4!}}_{\substack{1^{st}\\ \text{round}}}\ \underbrace{\dfrac{4!}{2!}}_{\substack{2^{nd}\\ \text{round}}}\ \underbrace{2}_{\substack{3^{rd}\\ \text{round}}} = \boxed{8!} = 40,320.$

$\underline{\text{Note}}$: If $n = 2^m$ teams are playing, after m rounds there are $n!$ total possible outcomes.

Ex: The CUNY Graduate Center is on 34^{th} St. and 5^{th} Ave, and Columbus circle is on 59^{th} St. and 8^{th} Ave. If you only walk on streets and avenues (and not Broadway!), in how many ways can you get from CUNY GC to Columbus circle?



- How many blocks do we need to walk north?

$$59 - 34 = \underline{\underline{25}} \qquad N$$

- How many blocks do we need to walk west?

$$8 - 5 = \underline{\underline{3}} \qquad W$$

NNWNN . . .

25 N
3 W
$\underline{\underline{28}}$

Answer: $\binom{28}{3} = \binom{28}{25} = \dfrac{28!}{25!\,3!} = \boxed{3{,}276}$

Q: How many positive integer solutions are there to the equation $x_1 + x_2 = 5$?

$$x_1 + x_2 + x_3 + x_4 = 2021$$

$1 + 4 = 5$
$2 + 3 = 5$
$3 + 2 = 5$
$4 + 1 = 5$

4 total.

$$1 + 1 + 1 + 1 + 1 = 5$$

5−1=4 slots for the separating bar

Upshot: To separate n units into r parts (that contain at least some unit) we need to place $r-1$ separating bars in $n-1$ of the available slots.

$$1 + 1 + 1 + 1 + 1 + \cdots = 2021$$

2021−1=2020 slots for separating bars

$$1 + 1 + 1 + \cdots + 1 = 2021$$

$$\binom{2020}{3} = \frac{2020!}{3!\,2017!} = 1,371,695,140$$

Prop: The equation $x_1 + x_2 + \cdots + x_r = n$ has $\binom{n-1}{r-1}$ different positive integer solutions.

Cor: The equation $x_1 + x_2 + \cdots + x_r = n$ has $\binom{n+r-1}{r-1}$ different nonnegative integer solutions.

Pf:

$y_1 = x_1 + 1, \quad y_2 = x_2 + 1, \quad \ldots, \quad y_j = x_j + 1, \ldots, \quad y_r = x_r + 1$

$$y_1 + y_2 + \cdots + y_r = \underbrace{x_1 + x_2 + \cdots + x_r}_{n} + r = n + r$$

$$\#\ \text{solutions of } y_1 + \cdots + y_r = n + r \ (y_j > 0) \overset{\text{Prop}}{=} \binom{n+r-1}{r-1} = \#\ \text{solutions of } x_1 + x_2 + \cdots + x_r = n \ (x_j \geq 0)$$

EX: An investor has \$20,000 to invest in 4 possible investments. Each investment must be in units of \$1,000. In how many ways can the investments be allocated?

Sol:

$$x_1 + x_2 + x_3 + x_4 = 20, \qquad x_i \geq 0$$

$$n = 20, \quad r = 4$$

$$\binom{n+r-1}{r-1} = \binom{23}{3} = 1{,}771.$$