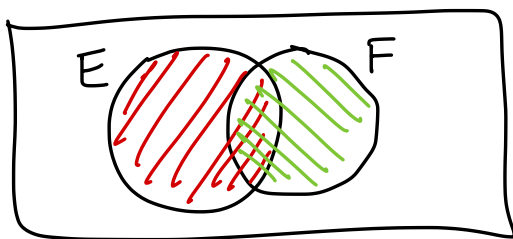


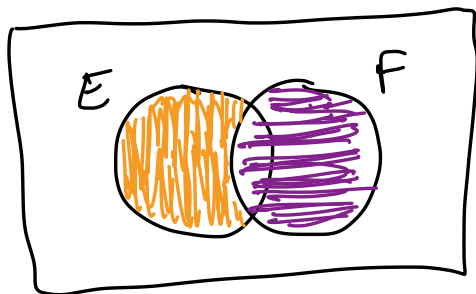
Inclusion-Exclusion principle:



$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

or and

S Pl: $E \cup F = (E \cap F^c) \cup F$



$$P(E \cup F) = P((E \cap F^c) \cup F)$$

← disjoint!

$$\stackrel{(iii)}{=} P(E \cap F^c) + P(F) \quad (1)$$

$$(E \cap F^c) \cup (E \cap F) = E \stackrel{(iii)}{\implies} P((E \cap F^c) \cup (E \cap F)) = P(E)$$

$$P(E \cap F^c) + P(E \cap F)$$

$$\implies P(E \cap F^c) = P(E) - P(E \cap F) \quad (2)$$

By (1) & (2), we conclude $P(E \cup F) = P(E) + P(F) - P(E \cap F)$. □

Amazon's artificial intelligence predicts there is a:

- 50% chance you will like book A
- 40% chance you will like book B
- 30% chance you will like both books A and B.

What is the implied chance you do not like either?

A = event that you like book A

$$P(A) = \frac{1}{2}$$

B = _____ B

$$P(B) = \frac{2}{5}$$

$A \cap B$ = _____ A and B

$$P(A \cap B) = \frac{3}{10}$$

$$P((A \cup B)^c) = P(A^c \cap B^c) = ?$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

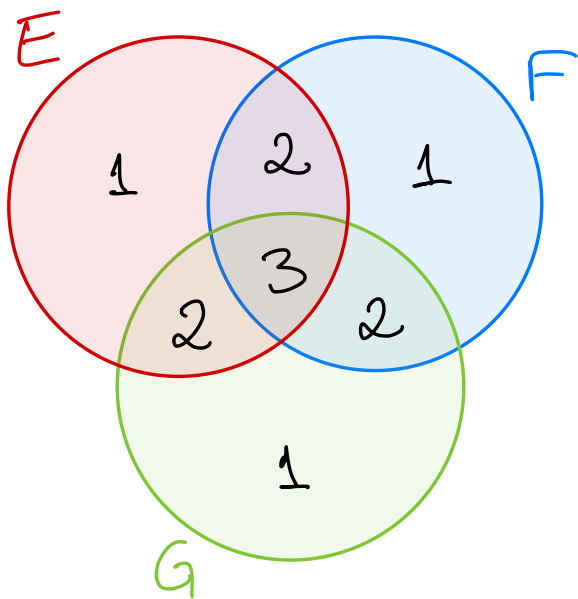
$$= \frac{1}{2} + \frac{2}{5} - \frac{3}{10} = \frac{5+4-3}{10} = \frac{6}{10} = \frac{3}{5}$$

$$P((A \cup B)^c) = 1 - P(A \cup B) = 1 - \frac{3}{5} = \frac{2}{5} \quad (40\%)$$

Note: $P(E^c) = 1 - P(E)$.

Pr: $P(E^c \cup E) = P(S) = 1$
 \parallel
 $P(E^c) + P(E) \Rightarrow P(E^c) = 1 - P(E)$.

Inclusion-Exclusion principle with 3 or more sets:



$$P(E \cup F \cup G) = P(E) + P(F) + P(G) \\ - P(E \cap F) - P(E \cap G) \\ - P(F \cap G) \\ + P(E \cap F \cap G)$$

With more sets: $E_1, E_2, E_3, \dots, E_n$:

$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_{i_1} (-1)^2 P(E_{i_1}) + \sum_{i_1 < i_2} (-1)^3 P(E_{i_1} \cap E_{i_2})$$

$$+ \sum_{i_1 < i_2 < i_3} (-1)^4 P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) + \dots$$

$$\dots + \sum_{i_1 < i_2 < \dots < i_k} (-1)^{k+1} P(E_{i_1} \cap \dots \cap E_{i_k}) \dots + (-1)^{n+1} P(E_1 \cap \dots \cap E_n)$$

$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_{k=1}^n (-1)^{k+1} \sum_{i_1 < \dots < i_k} P(E_{i_1} \cap \dots \cap E_{i_k})$$

Matching problem:

When classes were in person, n students were always on their phones, so the professor collected the phones and placed them in a bucket. Suppose all phones were identical models, and couldn't be distinguished from the outside. At the end of class, when these n students pick a phone at random from the bucket, what is the probability that no one picks their own phone?

• $E_i = i^{\text{th}}$ student picking their own phone.

• $E_1 \cup E_2 \cup \dots \cup E_n \leftarrow$ at least someone picks up their own phone

• $(E_1 \cup E_2 \cup \dots \cup E_n)^c = E_1^c \cap E_2^c \cap \dots \cap E_n^c \leftarrow$ desired probability.

• Inclusion-exclusion principle:

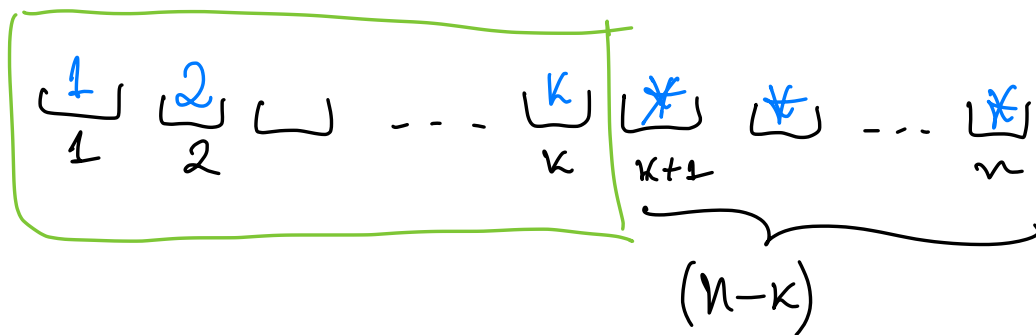
$$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{k=1}^n (-1)^{k+1} \sum_{i_1 < \dots < i_k} P(E_{i_1} \cap \dots \cap E_{i_k})$$

$$P(E_{i_1} \cap \dots \cap E_{i_k}) = \frac{(n-k)!}{n!}$$

← permutations where $i_1 \dots i_k$ select their own phone.

← total # of permutations

(WLOG: $i_1=1, i_2=2, \dots, i_k=k$)



How many permutations of $1\ 2\ 3\ \dots\ n$ fix the first k slots?

$(n-k)!$

$$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{k=1}^n (-1)^{k+1} \sum_{i_1 < \dots < i_k} \underbrace{P(E_{i_1} \cap \dots \cap E_{i_k})}_{\frac{(n-k)!}{k!}}$$

There are $\binom{n}{k}$ collections of indices $i_1 < \dots < i_k$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \frac{(n-k)!}{n!} = \sum_{k=1}^n \frac{(-1)^{k+1}}{k!}$$

$$\begin{aligned}
 P\left(\left(\bigcup_{i=1}^n E_i\right)^c\right) &= 1 - P\left(\bigcup_{i=1}^n E_i\right) = 1 - \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \dots + \frac{(-1)^{n+1}}{n!}\right) \\
 &= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + \frac{(-1)^n}{n!} \\
 &= \sum_{j=2}^n \frac{(-1)^j}{j!} = \sum_{j=0}^n \frac{(-1)^j}{j!}
 \end{aligned}$$

$$\begin{aligned}
 j=0: & \frac{(-1)^0}{0!} = 1 \\
 j=1: & \frac{(-1)^1}{1!} = -1
 \end{aligned}$$

Upshot:

$$\text{As } p(n) = \sum_{j=0}^n \frac{(-1)^j}{j!}$$

$n=5$

$$p(5) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} = \frac{11}{30} \approx 36.6\%$$

$n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} p(n) = \sum_{j=0}^{+\infty} \frac{(-1)^j}{j!} = e^{-1} = \frac{1}{e} \approx 0.3679\dots$$

$$e^x = \sum_{j=0}^{+\infty} \frac{x^j}{j!}$$

Ex: If you pick a number at random between 1 and 1000, what is the probability that it is divisible by 7 or by 11? How about divisible by 7, or 11, or 13?

A = your number is divisible by 7

B = _____ 11

C = _____ 13

$$P(A \cup B) = ?$$

$$P(A \cup B \cup C) = ?$$

$$P(A) = \frac{142}{1000}, \quad P(B) = \frac{90}{1000}, \quad P(C) = \frac{76}{1000}$$

How many numbers between 1 and 1000 are multiples of 7?

$\lfloor x \rfloor =$ largest integer $\leq x$.

$$\left\lfloor \frac{1000}{7} \right\rfloor = \lfloor 142.857 \rfloor = 142$$

Similarly, $\left\lfloor \frac{1000}{11} \right\rfloor = \lfloor 90.9091 \rfloor = 90$.

$$\left\lfloor \frac{1000}{13} \right\rfloor = \lfloor 76.92 \rfloor = 76$$

$$P(A \cap B) = \frac{\left\lfloor \frac{1000}{77} \right\rfloor}{1000} = \frac{12}{1000}$$

being divisible by 7 and by 11 \iff being divisible by 77.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{142}{1000} + \frac{90}{1000} - \frac{12}{1000}$$

$$= \frac{220}{1000} = \boxed{22\%}$$

$$P(A \cup B \cup C) = \underbrace{P(A) + P(B) + P(C)}_{k=1} - \underbrace{(P(AB) + P(AC) + P(BC))}_{k=2}$$

$$+ \underbrace{P(ABC)}_{k=3}$$

$$= \frac{142}{1000} + \frac{90}{1000} + \frac{76}{1000} - \left(\frac{12}{1000} + \frac{6}{1000} + \frac{10}{1000} \right)$$

$$P(BC) = \frac{\left\lfloor \frac{1000}{11 \cdot 13} \right\rfloor}{1000} = \frac{6}{1000}$$

$$= \frac{308}{1000} - \left(\frac{28}{1000} \right)$$

$$= \frac{280}{1000} = \frac{28}{100} = \boxed{28\%}$$

$$P(AC) = \frac{\left\lfloor \frac{1000}{7 \cdot 13} \right\rfloor}{1000} = \frac{10}{1000}$$

$$P(ABC) = 0.$$

$$7 \cdot 11 \cdot 13 = \underline{\underline{1001}} > 1000.$$