MAT 330/681 — Lecture 8 — 3/1/2021

Suppose we have a Bernoulli process, with

(S) Prob. of success $= p$ (F) Prob. of failure $= 1-p$

in each trial.

Q: What is the probability that n successes occur before m failures?

trials.

Points problem: Players play a sequence of matches.

Assume Player A wins any given match with prob. p. S

Player B — " — $1-p$. F

Player A : needs n points to win

Player B : needs m points to win.

Q: what is the prob. that Player A wins the game?

$\boxed{\text{A wins}} \iff \boxed{\text{A wins at least } n \text{ matches in the next } n+m-1 \text{ rounds}}$

($\Leftarrow$) trivial.

($\Rightarrow$) If A wins at most $n-1$ matches in the next $n+m-1$ rounds, then B wins at least m of them. Hence B won and A lost.

Therefore, the prob. that A wins is the same as the prob. that at least n successes occur in the "next" $n+m-1$ trials.

$$\begin{pmatrix}\text{Prob. exactly} \\ k \text{ successes occur} \\ \text{in } n+m-1 \text{ trials}\end{pmatrix} = \binom{n+m-1}{k} p^k (1-p)^{n+m-1-k}$$

$$\begin{pmatrix}\text{Prob. that at} \\ \text{least } n \text{ successes} \\ \text{occur in } n+m-1 \text{ trials}\end{pmatrix} = \sum_{k=n}^{n+m-1} \underbrace{\begin{pmatrix}\text{Prob. exactly} \\ k \text{ successes in} \\ n+m-1 \text{ trials}\end{pmatrix}}_{A \text{ wins}, \; k=n, n+1, \ldots, n+m-1}$$

$$= \boxed{\sum_{k=n}^{n+m-1} \binom{n+m-1}{k} p^k (1-p)^{n+m-1-k}}$$

EX: Suppose you are playing rounds of a game in a casino where the house edge is 2%, that is, your chance of winning any single round is 48%. Before you arrived at the casino, you made a deal with yourself that you will stop playing after either winning or losing 12 rounds. You have already played 15 rounds, of which you won 9 and lost 6. What is the prob. that when you leave, you would have won 12 rounds?

Player A: you $\quad n = 3 \quad p = 0.48$

Player B: Casino $\quad m = 6 \quad 1-p = 0.52$

$$P = \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} p^k (1-p)^{n+m-1-k} = \sum_{k=3}^{8} \binom{8}{k} 0.48^k \, 0.52^{8-k}$$

$$\underline{\underline{= 82.76\%}}$$

Gambler's Ruin Problem

(independent)

Players A and B bet on the outcome of biased coin flips, and the loser pays the winner \$1. The coin outcomes are:

$$P(H) = p \qquad P(H^c) = q = 1-p$$

and A always bets on H, while B always bets on H^c.

Initial wealth: Player A : \$$i$ $\qquad 0 \le i \le N$

Player B : \$$(N-i)$

\$N Total jackpot.

Q: What is the prob. that A wins (B goes bankrupt)?

$$P_i = \begin{pmatrix} \text{Prob. that A wins} \\ \text{if starting with \$} i \end{pmatrix} = P(E)$$

Condition P_i on the first outcome:

$$P_i = P(E) = P(E|H)\underbrace{P(H)}_{p} + P(E|H^c)\underbrace{P(H^c)}_{q}$$

$P(E|H) = P_{i+1}$: If the first outcome is H, then the wealths of players become:

A : \$$(i+1)$ $\qquad$ B: \$$(N-i-1) = \$(N-(i+1))$

$P(E|H^c) = P_{i-1}$: (similarly)

A: \$$(i-1)$ $\qquad$ B: \$$(N-i+1) = \$(N-(i-1))$

Obtain the recursion: $P_i = p \cdot P_{i+1} + q \cdot P_{i-1}$, $i = 1, \dots, N-1$

Boundary conditions: $P_0 = 0$, $P_N = 1$.

$$\underbrace{(p+q)}_{1} \cdot P_i = p\, P_{i+1} + q\, P_{i-1}$$

$$p P_i + q P_i = p\, P_{i+1} + q\, P_{i-1}$$

$$\boxed{P_{i+1} - P_i = \frac{q}{p}\left(P_i - P_{i-1}\right) \qquad i = 1, \dots, N-1}$$

$i = 1$: $P_2 - P_1 = \frac{q}{p}\left(P_1 - \cancel{P_0}^{\,0}\right) = \frac{q}{p} P_1$

$i = 2$: $P_3 - P_2 = \frac{q}{p}\left(P_2 - P_1\right) = \left(\frac{q}{p}\right)^2 P_1$.

$i = 3$: $P_4 - P_3 = \frac{q}{p}\left(P_3 - P_2\right) = \left(\frac{q}{p}\right)^3 P_1$

$\vdots$

i : $P_i - P_{i-1} = \left(\frac{q}{p}\right)^{i-1} \cdot P_1$

$i = N$: $P_N - P_{N-1} = \frac{q}{p}\left(P_{N-1} - P_{N-2}\right) = \dots = \left(\frac{q}{p}\right)^{N-1} \cdot P_1$.

Adding the first $(i-1)$ equations:

$$P_i - P_1 = P_1\left(\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right)^3 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right)$$

$$P_i = P_1\underbrace{\left(\underbrace{\left(\frac{q}{p}\right)^0}_{1} + \left(\frac{q}{p}\right)^1 + \left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right)^3 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right)}_{\text{partial sum of a geometric series of ratio } = \frac{q}{p}.}$$

$$P_i = \begin{cases} P_1 \cdot i, & \text{if } \frac{q}{p} = 1 \quad \text{fair coin } \left(p = q = \frac{1}{2}\right) \\ P_1 \cdot \dfrac{1-(q/p)^i}{1-(q/p)}, & \text{if } \frac{q}{p} \neq 1 \quad \text{biased coin } (p \neq q) \end{cases}$$

Using that $P_N = 1$:

$$1 = P_N = \begin{cases} P_1 \cdot N & \text{if } \frac{q}{p} = 1 \\ P_1 \cdot \dfrac{1-(q/p)^N}{1-(q/p)} & \text{if } \frac{q}{p} \neq 1. \end{cases}$$

Solving for P_1:

$$P_1 = \begin{cases} 1/N, & \text{if } \frac{q}{p} = 1 \\ \dfrac{1-q/p}{1-(q/p)^N}, & \text{if } \frac{q}{p} \neq 1. \end{cases}$$

Putting everything together:

$$P_i = \begin{cases} \dfrac{i}{N}, & \text{if } p = \frac{1}{2} \\[2ex] \dfrac{1-(q/p)^i}{1-(q/p)^N}, & \text{if } p \neq \frac{1}{2} \end{cases}$$

Similarly the prob. that Player B wins: (same initial wealth assumptions)

$$Q_i = \begin{cases} \dfrac{N-i}{N}, & \text{if } q = \frac{1}{2} \\[2ex] \dfrac{1-(p/q)^{N-i}}{1-(p/q)^N}, & \text{if } q \neq \frac{1}{2}. \end{cases}$$

Note that $P_i + Q_i = 1$ always.

This means that: the probability that either A or B goes bankrupt is $=1$, so the game ends in finite time with prob. 1.

(This does not mean that game cannot go on forever!)

this happens w/ prob. 0.

Ex: Suppose each round costs \$1.00 at a slot machine in which you win \$2.00 with probability $p = 40\%$, and lose your \$1.00 with probability $q = 60\%$. You have \$10.00 to play and decide you will keep playing until you either lose all your money or double it. What is the prob. you lose all your money?

$p = 0.4 \qquad q = 0.6. \qquad \frac{p}{q} = \frac{0.4}{0.6} = \frac{2}{3}$

Initial Wealth:

Player A: you $\qquad \$i = \$10.00$

Player B: slot machine $\qquad \$(N-i) = \$10.00$

$(\$N = \$20.00)$

$$\left(\begin{array}{l}\text{Prob. that you}\\ \text{lose all your money}\end{array}\right) = Q_{10} = \frac{1-(2/3)^{10}}{1-(2/3)^{20}} = \frac{59{,}049}{60{,}073}$$

$$= \boxed{98.3\%}$$