Lecture 12

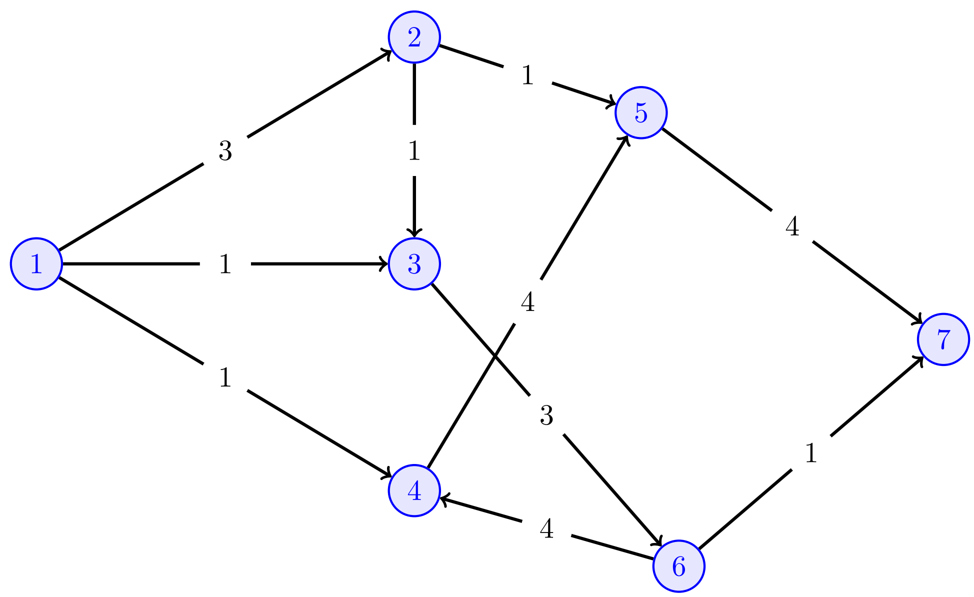
1. Applications of Linear Programming, Part I

1.1. **Network max flow.** A network consists of nodes (vertices) and links (edges), and may represent lots of interesting real-world scenarios where one may want to maximize flow (or, as we will see later, to find the most efficient way of cutting edges to interrupt the flow), such as:

(1) a railway system through which supplies and passengers move, like the NYC subway system, or the freight or passenger train systems in a given country;[1]
(2) an electrical wiring diagram for a certain circuit, through which electrical current flows;
(3) a wired communication network between computers, through which bits of data flow;
(4) a wireless communication network between Wi-Fi devices, or between satellites, through which data flows;
(5) a system of financial transactions, through which money flows between different investments;
(6) a contagion diagram for an infectious disease, e.g. COVID-19, spreading through individuals;
(7) a trafficking scheme for moving illegal narcotics between various locations.

We assume that each link has a bounded capacity, that is, only a limited amount of material (passengers, bits of data, US dollars,...) can be transported along each edge, and only in a given direction indicated by the arrow.

For example, consider the following network, where the numbers along links indicate their capacity; e.g., the link from node 2 to node 3 supports up to 1 unit of material any given time.[2]

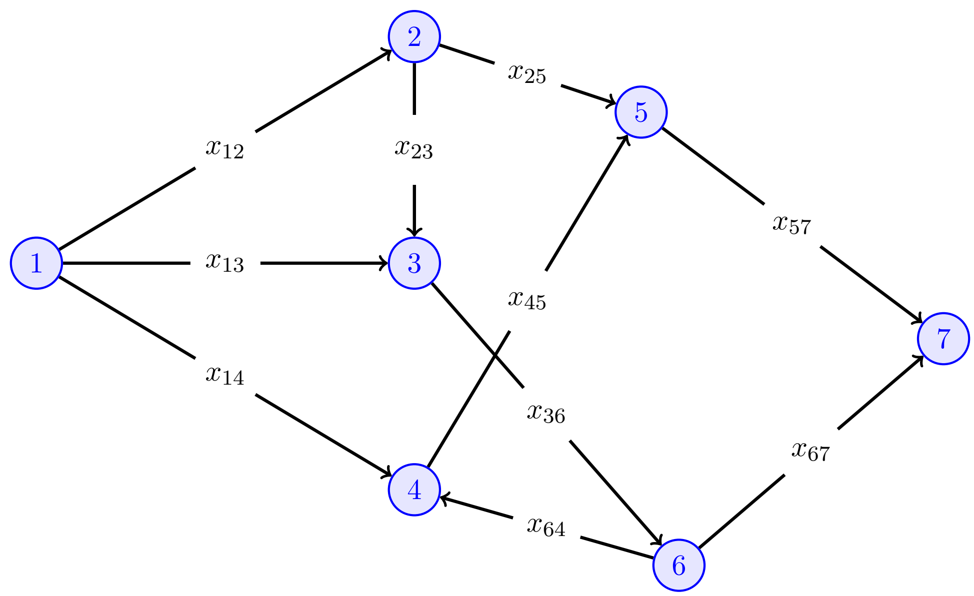


Exercise 1. What is the maximum amount of material that can be transported from node 1 to node 7 along the above network, respecting the capacity limits of all links, and assuming that no material can be stored in any intermediate node?

Solution to Exercise 1. Let x_{ij}, $i \neq j$, denote the amount of material to be transported between nodes i and j in the network, for which a link is available, as shown in the diagram below:

[1] The origins of the max flow problem we are discussing today go back to an effort to analyze capabilities of the Soviet rail network, see Alexander Schrijver "On the history of the transportation and maximum flow problems", Mathematical Programming vol 91, p. 437–445 (2002), `https://link.springer.com/article/10.1007/s101070100259`

[2] This example is modified from Sec. 2.2 in "Understanding and Using Linear Programming", by Jiri Matousek and Bernd Gärtner (Springer).



The directionality and capacity bounds of the links give the following constraints:

$$
\begin{aligned}
&0 \le x_{12} \le 3 && 0 \le x_{36} \le 3 \\
&0 \le x_{13} \le 1 && 0 \le x_{45} \le 4 \\
&0 \le x_{14} \le 1 && 0 \le x_{64} \le 4 \\
&0 \le x_{23} \le 1 && 0 \le x_{57} \le 4 \\
&0 \le x_{25} \le 1 && 0 \le x_{67} \le 1
\end{aligned}
$$

Since no material can be stored in any node, the total amount that *flows in* must *flow out*, which gives the following balancing constraints at interior nodes:

$$
\begin{aligned}
j = 2: &\quad x_{12} = x_{23} + x_{25} \\
j = 3: &\quad x_{13} + x_{23} = x_{36} \\
j = 4: &\quad x_{14} + x_{64} = x_{45} \\
j = 5: &\quad x_{25} + x_{45} = x_{57} \\
j = 6: &\quad x_{36} = x_{64} + x_{67}
\end{aligned}
$$

or, more succinctly, for each interior node, i.e., for each $2 \le j \le 6$,

$$\sum_i x_{ij} = \sum_k x_{jk}$$

We want to maximize the amount of material flowing into the network from the source node 1, which, given the above constraints, is of course equal to the amount of material flowing out of the network at the sink node 7. That quantity is

$$x_{12} + x_{13} + x_{14} = x_{57} + x_{67}$$

We thus arrive at the following LP:

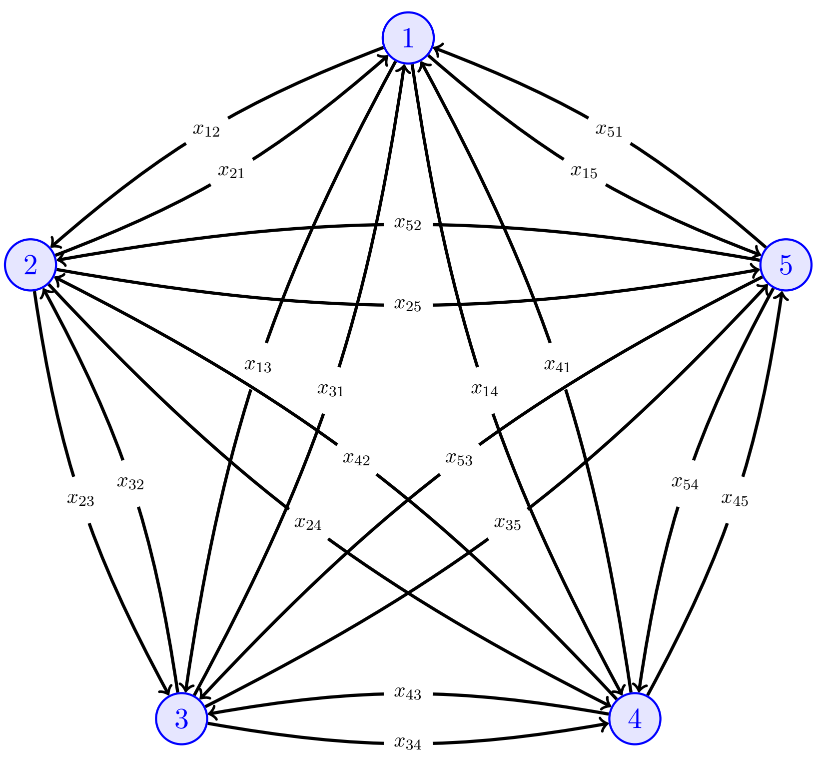
$$\max \quad x_{12}+x_{13}+x_{14} \quad \text{s.t.} \quad \begin{aligned} &x_{12} = x_{23}+x_{25},\\ &x_{13}+x_{23} = x_{36},\\ &x_{14}+x_{64} = x_{45},\\ &x_{25}+x_{45} = x_{57},\\ &x_{36} = x_{64}+x_{67},\\ &0 \le x_{12} \le 3, \quad 0 \le x_{36} \le 3,\\ &0 \le x_{13} \le 1, \quad 0 \le x_{45} \le 4,\\ &0 \le x_{14} \le 1, \quad 0 \le x_{64} \le 4,\\ &0 \le x_{23} \le 1, \quad 0 \le x_{57} \le 4,\\ &0 \le x_{25} \le 1, \quad 0 \le x_{67} \le 1.\end{aligned}$$

The above problem is bounded and feasible. The maximum value is 4 and it is attained at

$$(x_{12}, x_{13}, x_{14}, x_{23}, x_{25}, x_{36}, x_{45}, x_{64}, x_{57}, x_{67}) = (2, 1, 1, 1, 1, 2, 3, 2, 4, 0),$$

see the Mathematica file `lecture12.nb` for details. To read more about the Max Flow problem, see the Wikipedia page `https://en.wikipedia.org/wiki/Maximum_flow_problem`.

1.2. **Currency arbitrage.** Consider now a network given by a complete graph with 5 vertices, where every pair of vertices is joined by two directed edges (one in each direction), see figure below.



Differently from the previous problem, we now consider the situation in which there is *gain* or *loss* when material is transported along any given edge. This situation models trading 5 currencies in an "ideal" foreign exchange market, for example, suppose we trade the following currencies:

1: USD (US dollars)
2: EUR (Euros)
3: GBP (Pounds)
4: JPY (Yen)
5: BRL (Reais)

The spot exchange rates below were collected from Google Finance on October 17, 2023:

	1	2	3	4	5
1	–	1.05678	1.21749	0.00667535	0.198248
2	0.94627	–	1.15197	0.006316	0.187628
3	0.82145	0.868092	–	0.00548347	0.162816
4	149.819	158.313	182.402	–	29.7058
5	5.04528	5.3297	6.14255	0.0336761	–

For example, according to those rates, USD 1 buys EUR 0.94627. In other words, moving along the edge from 1 to 2 multiplies the amount of material being transported by 0.94627. Similarly, moving along the edge from 2 to 1 multiplies the amount of material being transported by 1.05678.

Exercise 2. Set up a linear program in the variables x_{ij}, $i \neq j$, to find an arbitrage opportunity in which USD 100.00 are traded between the above 5 currencies at the above spot rates (i.e., transported through this network), leaving nonnegative balances in all currencies, and resulting in a profit of USD 5.00. Can we arrange for the same investment to generate a profit of USD 5,000,000.00? Can we arrange for an investment of USD 0.00 to generate a profit of USD 5,000,000.00? Explain.

Solution to Exercise 2. To generate a profit of USD 5.00 with an initial investment of USD 100.00, we attempt to maximize the amount of USD being bought, that is, $\max \sum_i r_{i1}x_{i1}$, where r_{i1} is the exchange rate for buying USD with currency i. With the above spot rates, this is

$$\max \quad 1.05678x_{21} + 1.21749x_{31} + 0.00667535x_{41} + 0.198248x_{51}. \tag{1}$$

The requirement that nonnegative balances are left in every currency corresponds to the constraint that the amount of currency flowing out must be always smaller than the amout of currency flowing in. For currency j, this is

$$\sum_i x_{ji} \leq \sum_i r_{ij}x_{ij}$$

or, with the above rates r_{ij}, which are the entries in the *transpose* of the table above,

$$x_{21} + x_{23} + x_{24} + x_{25} \leq 0.94627x_{12} + 1.15197x_{32} + 0.006316x_{42} + 0.187628x_{52} \tag{2}$$

$$x_{31} + x_{32} + x_{34} + x_{35} \leq 0.82145x_{13} + 0.868092x_{23} + 0.00548347x_{43} + 0.162816x_{53} \tag{3}$$

$$x_{41} + x_{42} + x_{43} + x_{45} \leq 149.819x_{14} + 158.313x_{24} + 182.402x_{34} + 29.7058x_{54} \tag{4}$$

$$x_{51} + x_{52} + x_{53} + x_{54} \leq 5.04528x_{15} + 5.3297x_{25} + 6.14255x_{35} + 0.0336761x_{45} \tag{5}$$

Moreover, we have the constraint that USD 100.00 are invested, and the desired profit is USD 5.00, so we also include

$$\begin{aligned} &x_{12} + x_{13} + x_{14} + x_{15} = 100 \\ &1.05678x_{21} + 1.21749x_{31} + 0.00667535x_{41} + 0.198248x_{51} - (x_{12} + x_{13} + x_{14} + x_{15}) \leq 5, \end{aligned} \tag{6}$$

which, of course, can be combined into a single inequality. Finally, we have the constraint

$$x_{ij} \geq 0. \tag{7}$$

The LP consisting of (1) – (7) is bounded and feasible, with optimal value 105, achieved by setting

$$x_{14} = 100, \quad x_{21} = 95.5733, \quad x_{35} = 1{,}455.15, \quad x_{43} = 265{,}370.00, \quad x_{52} = 509.377, \quad x_{54} = 8{,}428.94,$$

and the remaining $x_{ij} = 0$. See `lecture12.nb` for details.

Replacing the right-hand side of the constraints in (6) with 100 and 5×10^6, or 0 and 5×10^6, respectively, we obtain other bounded and feasible LPs, so these arbitrage opportunities also exist (the optimal solution x_{ij} can be found in `lecture12.nb`, see figure below for the scheme that generates USD 5,000,000.00 from a USD 0.00 investment). If we do not impose a constraint of the form (6), where profit is bounded from above (or, alternatively, daily trading limits $x_{ij} \leq c_{ij}$), then the resulting LP is feasible but *unbounded*, that is, the arbitrage found would yield *infinite profit*.

An explanation for such unrealistic arbitrage opportunities is that if there exists a cycle of trades $i \mapsto j \mapsto k \mapsto i$ that multiplies the amount of currency i by a factor > 1, then, in principle, it could be used indefinitely to generate arbitrarily large profits. This is the case with the above exchange rates, e.g., trading $5 \mapsto 4 \mapsto 3 \mapsto 5$.

Some underlying assumptions in this model render it completely implausible as a trading strategy:

- We assumed the bid-ask spread is equal to zero on all currencies (perfect liquidity);
- We assumed unbounded and immediate availability of funds in all traded currencies (very large amounts are needed to produce meaningful profits, even if positions offset each other);
- We assumed simultaneous instant transactions without fees are possible (real buy/sell orders generally do not execute simultaneously, transaction fees usually apply).

For a more thorough discussion, see:

- Smith, Rachel. "A Discussion of Linear Programming and its Application to Currency Arbitrage Detection" Undergraduate thesis at University of Redlands: `https://inspire.redlands.edu/work/ns/46f3971d-8d2a-4537-afbb-c91bf859c18e`
- Triangular Arbitrage (Wiki). `https://en.wikipedia.org/wiki/Triangular_arbitrage`