Lecture 17

1. Review of Linear Algebra

Recall from last lecture that $\lambda \in \mathbb{R}$ is an *eigenvalue* for an $n \times n$ matrix A if the linear equation $Ax = \lambda x$ has a nontrivial solution $x \neq 0$. Equivalently, eigenvalues are the roots of the characteristic polynomial $p(\lambda) = \det(\lambda \operatorname{Id} - A)$. The matrix A is diagonalizable if its eigenvectors form a basis of $\mathbb{R}^n$; equivalently, if there exists an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. We will make frequent use of the following important algebraic results.

Theorem 1. *If A is symmetric, that is, $A^T = A$, then A is orthogonally diagonalizable, that is, there exists a matrix P such that $P^T P = \operatorname{Id}$ and $A = PDP^{-1} = PDP^T$, where D is diagonal.*

Theorem 2. *The characteristic polynomial $p(\lambda)$ of an $n \times n$ matrix A satisfies*

$$p(\lambda) = \det(\lambda \operatorname{Id} - A) = \sum_{k=0}^{n} (-1)^k \operatorname{tr}(\wedge^k A)\, \lambda^{n-k},$$

where $\operatorname{tr}(\wedge^k A)$ is the sum of all principal minors[1] of A of size k.

2. Positive-semidefinite matrices

A symmetric $n \times n$ matrix A is *positive-semidefinite* if for all $x \in \mathbb{R}^n$ we have $x^T A x \geq 0$, we write $A \succeq 0$ for short. Note that $q(x) = x^T A x$ is the quadratic form associated to A, so $A \succeq 0$ means that $q(x) \geq 0$ for all $x \in \mathbb{R}^n$. More generally, $A \succeq 0$ has several equivalent characterizations:

Theorem 3. *If A is an $n \times n$ symmetric matrix of rank r, then the following are equivalent:*
(1) *A is positive-semidefinite, i.e., $A \succeq 0$;*
(2) *All eigenvalues of A are nonnegative;*
(3) *There exists a $r \times n$ matrix Q such that[2] $A = Q^T Q$;*
(4) *There exists a symmetric and positive-definite $n \times n$ matrix[3] R such that $A = R^2$;*
(5) *All principal minors of A are nonnegative;*
(6) *All principal minors of A of size at most r are nonnegative.*

Exercise 1. Use Theorem 1 to prove the equivalence (1) $\iff$ (2).

The equivalence (1) $\iff$ (5) is very useful for computations, and is known as *Sylvester's criterion.*

Similarly, A is called *positive-definite*, written $A \succ 0$, if for all $x \in \mathbb{R}^n \setminus \{0\}$, we have $x^T A x > 0$. There is an analogous version of Theorem 3 giving equivalent characterizations of $A \succ 0$, a notable simplification is that in (5) and (6) it suffices to have that *leading* principal minors are positive, i.e., those consisting of rows and columns $\{1, \dots, k\}$ for all $1 \leq k \leq n$.

Exercise 2. Use Sylvester's criterion to determine if the following matrices are positive-semidefinite. If so, check if they are positive-definite.

$$A = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 2 \\ 1 & -1 & 3 & 1 \\ 0 & 2 & 1 & 4 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Exercise 3. Describe geometrically the set $S = \left\{ (x, y, z) \in \mathbb{R}^3 : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \succeq 0 \right\}$.

---

[1] Recall that a principal minor of A of size k is the determinant of a submatrix of A obtained by selecting rows $\{i_1, \dots, i_k\}$ and columns $\{i_1, \dots, i_k\}$.

[2] This is often called the Cholesky factorization.

[3] We often write $R = \sqrt{A}$.

Generalizing the example above, note that the set $\mathcal{C}_{PSD} := \{X \in \mathrm{Sym}^2(\mathbb{R}^n) : X \succeq 0\}$ of positive-semidefinite $n \times n$ matrices form a convex cone in the vector space $\mathrm{Sym}^2(\mathbb{R}^n)$ of symmetric $n \times n$ matrices. The natural inner product in $\mathrm{Sym}^2(\mathbb{R}^n)$ is given by $\langle X, Y\rangle = \operatorname{tr} XY$; so, given $A \in \mathrm{Sym}^2(\mathbb{R}^n)$ and $b \in \mathbb{R}$, the affine equation $\langle A, X\rangle = b$ determines a hyperplane in $\mathrm{Sym}^2(\mathbb{R}^n)$.

A subset $S \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a *spectrahedron* if it is of the form

$$S = \big\{X \in \mathrm{Sym}^2(\mathbb{R}^n) : \langle A_i, X\rangle = b_i, 1 \leq i \leq m, \text{ and } X \succeq 0\big\},$$

for some $A_i \in \mathrm{Sym}^2(\mathbb{R}^n)$, $1 \leq i \leq m$. Equivalently, S is a spectrahedron if it can be described by a *linear matrix inequality*, that is,

$$S = \big\{x \in \mathbb{R}^d : F_0 + x_1 F_1 + \cdots + x_d F_d \succeq 0\big\},$$

where $F_j \in \mathrm{Sym}^2(\mathbb{R}^n)$, $0 \leq j \leq d$.

Exercise 4. Prove that spectrahedra are convex.

Exercise 5. Prove that polyhedra are spectrahedra.

Exercise 6. Describe geometrically the following spectrahedra in $\mathbb{R}^2$:

a) $S = \left\{(x,y) \in \mathbb{R}^2 : \begin{pmatrix} 1+x & y \\ y & 1-x \end{pmatrix} \succeq 0\right\}$,

b) $S = \left\{(x,y) \in \mathbb{R}^2 : \begin{pmatrix} 1+x & & & \\ & 1-x & & \\ & & 1+y & \\ & & & 1-y \end{pmatrix} \succeq 0\right\}$,

c) $S = \left\{(x,y) \in \mathbb{R}^2 : \begin{pmatrix} 1 & x & y \\ x & 1 & x \\ y & x & 1 \end{pmatrix} \succeq 0\right\}$.