Lecture 19

1. Semidefinite programming

A *semidefinite program* (SDP) is an optimization problem of the form

$$\begin{aligned} \min \quad \langle C, X\rangle \quad \text{s.t.} \quad & \langle A_i, X\rangle = b_i, \qquad i = 1, \dots, m, \\ & X \succeq 0 \end{aligned} \tag{1}$$

where $A_i, C \in \mathrm{Sym}^2(\mathbb{R}^n)$ are given $n \times n$ symmetric matrices, and $b_i \in \mathbb{R}$. Recall that the inner product in the vector space $\mathrm{Sym}^2(\mathbb{R}^n)$ of symmetric $n \times n$ matrices is given by $\langle X, Y\rangle = \operatorname{tr} XY$.

In other words, an SDP is an optimization problem whose target function is *linear* and whose feasible set is a *spectrahedron.*

Exercise 1. [1] Recognize that the following problem is an SDP on the matrix $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix}$.

$$\begin{aligned} \min \quad 2x_{11} + 2x_{12} \quad \text{s.t.} \quad & x_{11} + x_{22} = 1, \\ & X \succeq 0. \end{aligned}$$

Find the optimal solution by recognizing (geometrically) the feasible set as a subset of $\mathbb{R}^2$.

Solution to Exercise 1. The above is an SDP of the form (1) with

$$C = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ b_1 = 1.$$

The feasible set is the spectrahedron determined by the polynomial inequality $x_{11}(1 - x_{11}) \geq x_{12}^2$, which can be easily seen (complete the square!)[2] to be a closed disk in the $(x_{11}, x_{12}, 1 - x_{11})$-plane with center $(1/2, 0, 1/2)$ and radius $1/2$. Thus, the optimal solution is

$$X = \begin{pmatrix} \frac{2-\sqrt{2}}{4} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{2+\sqrt{2}}{4} \end{pmatrix},$$

where the target function achieves its minimum value $1 - \sqrt{2}$.

[1] This exercise is taken from “Semidefinite Optimization and Convex Algebraic Geometry”, MOS-SIAM Series on Optimization, edited by G. Blekherman, P. Parrilo, and R. Thomas.

[2] To find that this is equivalent to $\left(x_{11} - \frac{1}{2}\right)^2 + x_{12}^2 \leq \frac{1}{4}$.