Lecture 21

1. Semialgebraic sets

A *basic closed semialgebraic set* is a set $S \subset \mathbb{R}^n$ given by simultaneous polynomial inequalities,

$$S = \{x \in \mathbb{R}^n : p_1(x) \geq 0, \dots, p_r(x) \geq 0\},$$

where $p_i(x) \in \mathbb{R}[x]$, $1 \leq i \leq r$, are polynomials. A general *semialgebraic set* is a finite Boolean combination of basic closed semialgebraic sets.

Exercise 1. Show (write a proof) that a finite *intersection* of basic closed semialgebraic sets is basic closed semialgebraic, but show (by finding a counter-example) that the same is not true for a finite *union*.

Solution to Exercise 1. By induction, it suffices to show that the intersection of a pair of sets

$$S_1 = \{x \in \mathbb{R}^n : p_1(x) \geq 0, \dots, p_r(x) \geq 0\}$$
$$S_2 = \{x \in \mathbb{R}^n : q_1(x) \geq 0, \dots, q_m(x) \geq 0\}$$

is basic closed semialgebraic. This is clear since

$$S_1 \cap S_2 = \{x \in \mathbb{R}^n : p_1(x) \geq 0, \dots, p_r(x) \geq 0, q_1(x) \geq 0, \dots, q_m(x) \geq 0\}.$$

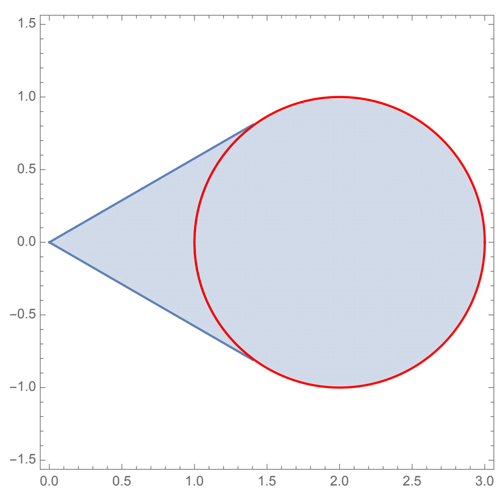
The union of two basic closed semialgebraic sets need not be basic closed semialgebraic. A simple example is $\{(x,y) \in \mathbb{R}^2 : x \geq 0\} \cup \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$. (Why?) Another similar counter-example is the convex set S in Exercise 1 of the Lecture 20 (see figure below) given by the union of

- the disk S_1 of radius 1 centered at $(2,0)$,
- the triangle S_2 with vertices $(0,0)$ and $\big(3/2, \pm\sqrt{3}/2\big)$.

Clearly, S_1 and S_2 are basic closed semialgebraic, since

- $S_1 = \big\{(x,y) \in \mathbb{R}^2 : 1 - (x-2)^2 - y^2 \geq 0\big\}$,
- $S_2 = \big\{(x,y) \in \mathbb{R}^2 : x \geq 0,\ \frac{1}{\sqrt{3}}x + y \geq 0,\ \frac{1}{\sqrt{3}}x - y \geq 0\big\}$.

Since part of the boundary of $S = S_1 \cup S_2$ is the circle of radius 1 centered at $(2,0)$, if S is *basic* closed semialgebraic, i.e., $S = \{(x,y) \in \mathbb{R}^2 : p_1(x,y) \geq 0, \dots, p_r(x,y) \geq 0\}$, then there exists j such that $p_j(x,y) = h(x,y)(1-(x-2)^2-y^2)^k$ for some odd $k \geq 1$ and $h(x,y)$ not divisible by $(1-(x-2)^2-y^2)$. On the other hand, unless $h(x,y)$ is divisible by $(1-(x-2)^2-y^2)$, the points in the interior of S that also lie on the circle of radius 1 centered at $(2,0)$ cannot be interior points of S. (Why?) This contradiction implies that S cannot be basic closed semialgebraic.[1]



[1] For a general statement obstructing the algebraic boundary (i.e., the Zariski-closure of the topological boundary) of basic semialgebraic sets from containing interior points, see "Algebraic Boundaries of Convex Semi-Algebraic Sets" by R. Sinn, Lemma 2.2.4 (page 19). `https://d-nb.info/1052418252/34`

An important theoretic result about semialgebraic sets is the following:

Theorem 1 (Tarski–Seidenberg). *If $S \subset \mathbb{R}^n \oplus \mathbb{R}^m$ is a semialgebraic set and $\pi\colon \mathbb{R}^n \oplus \mathbb{R}^m \to \mathbb{R}^n$ is the projection $\pi(x, y) = x$, then $\pi(S) \subset \mathbb{R}^n$ is semialgebraic.*

The above theorem implies that sentences given by a finite Boolean combination of polynomial inequalities where certain variables are quantified (using quantifiers $\exists$ or $\forall$), such as

$$\exists x \in \mathbb{R} : ax^2 + bx + c = 0,\ a > 0,$$

admit an equivalent description as a finite Boolean combination of *quantifier-free* polynomial inequalities in the remaining variables, in the above case,

$$b^2 - 4ac \geq 0.$$

This procedure is known as *quantifier elimination*, and (although very slow) it can be implemented algorithmically; e.g., on Mathematica, using `CylindricalDecomposition`, see `lecture21.nb` for more examples. Note that, if all variables are quantified, then the output of quantifier elimination is simply `True` (equivalently, $0 = 0$) or `False` (equivalently, $0 = 1$).

Exercise 2. Show that a linear projection of a basic closed semialgebraic set need not be closed.

Solution to Exercise 2. Let $S' = \left\{(x, y) \in \mathbb{R}^2 : \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} \succeq 0\right\}$ and $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ be the projection $\pi(x, y) = x$. Then $S' = \{(x, y) \in \mathbb{R}^2 : x\, y \geq 1,\ x \geq 0,\ y \geq 0\}$ is closed (basic) semialgebraic but $S = \pi(S') = (0, +\infty)$ is not closed.

Using the above, we can show that:[2]

Proposition 1. *A spectrahedron is a basic closed semialgebraic set. A spectrahedral shadow is a closed semialgebraic set, but not necessarily basic.*

Proof. Let $S = \{x \in \mathbb{R}^n : M(x) \succeq 0\}$ be a spectrahedron, where $M\colon \mathbb{R}^n \to \mathrm{Sym}^2(\mathbb{R}^d)$ is affine-linear, and recall that $M(x) \succeq 0$ if and only if all d eigenvalues of $M(x)$ are nonnegative. These eigenvalues are the d roots of the characteristic polynomial $p_{M(x)}(t) = \det(t\,\mathrm{Id} - M(x))$, so $x \in S$ if and only if $p_{M(x)}(-t)$ has no positive roots. Equivalently,[3] $x \in S$ if and only if all the coefficients of $(-1)^d p_{M(x)}(-t)$ are ≥ 0, so we can take $p_1, \dots, p_r \in \mathbb{R}[x]$ to be those coefficients. Thus, S is a basic closed semialgebraic set.

A spectrahedral shadow is a closed semialgebraic set as a consequence of the above and the Tarski–Seidenberg theorem. As the example in Exercise 1 above shows, it need not be basic. $\square$

[2]This proof is taken from "Geometry of Linear Matrix Inequalities" by T. Netzer and D. Plaumann.

[3]If a polynomial $p \in \mathbb{R}[x]_d$ has roots $-\lambda_i$, with $\lambda_i \geq 0$ for all $i = 1, \dots, d$ then $p(x) = (x + \lambda_1) \cdots (x + \lambda_d)$, i.e., all its coefficients are nonnegative. The converse is obvious.