Lecture 6

1. Slack variables and equational formulation of LP

As in the past lecture, consider an LP of the form

$$\min \quad c^T x \quad \text{s.t.} \quad A\,x \leq b, \tag{1}$$

where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$, corresponding to the description of its feasible set as the polyhedron

$$S = \{x \in \mathbb{R}^n : Ax \leq b\}. \tag{2}$$

Recall that, up to changing A, b, and c, we can restate any LP in this form, even if it is a maximization problem, or if some of its original constraints were in terms of $=$, or $\geq$.

In order to solve general LPs, it will be convenient to rewrite (1) in the *equational form*

$$\begin{aligned} \min \quad c^T x \quad \text{s.t.} \quad & A\,x = b, \\ & x \geq 0. \end{aligned} \tag{3}$$

This can be achieved by introducing *slack variables.* Namely, $a_i^T \cdot x \leq b_i$ if and only if there exists $s_i \geq 0$ such that $a_i^T x + s_i = b_i$. In other words, we can augment A and b to turn an LP in standard form (1) into an LP in equational form (3).

Exercise 1. Add slack variables to the following LPs in standard form, transforming them in equational form (3). What are the corresponding x, A, b, c?

(i) $$\begin{aligned} \min \quad x_1 - 2x_2 \quad \text{s.t.} \quad & x_1 + x_2 \leq 1, \\ & x \geq 0, \end{aligned}$$

(ii) $$\begin{aligned} \min \quad 2x_1 - 3x_2 + x_3 \quad \text{s.t.} \quad & x_1 + 3x_2 + x_3 \leq 5, \\ & 4x_1 - x_2 - x_3 \leq 2 \\ & 9x_1 + 2x_2 + 7x_3 \leq 3 \\ & x \geq 0, \end{aligned}$$

Exercise 2. Describe the general procedure to find x, A, b, c that transform an LP in standard form into an LP in equational form.

After adding slack variables, if some variables x_i in the original LP in standard form were not constrained to be nonnegative, we can replace them by $x_i = y_i - z_i$ where $y_i \geq 0$ and $z_i \geq 0$. This second step further augments x, A, b, c from the original ones.

Exercise 3. Make the appropriate changes to the following LP to transform it into equational form (3).

$$\begin{aligned} \min \quad 2x_1 + x_2 \quad \text{s.t.} \quad & x_1 + x_2 \leq 1, \\ & x_1 \leq 1, \end{aligned}$$

After performing the above steps (adding slack variables to replace $\leq$ with $=$) and, if needed, exchanging variables so that all of them are constrained to be nonnegative, we arrive at an LP in equational form (3). In other words, the feasible set (2) is now written as the intersection of an affine space with the nonnegative orthant in $\mathbb{R}^n$, i.e.,

$$S = \{x \in \mathbb{R}^n : Ax = b\} \cap \{x \in \mathbb{R}^n : x \geq 0\}. \tag{4}$$

We shall further assume that the rows a_i of A are linearly independent, i.e., rank $A = m$, since we may discard redundant (linearly dependent) constraints.

Exercise 4. Suppose that the feasible region of an LP is $Ax = b$ where

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 3 & 3 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 8 \\ 15 \end{pmatrix}$$

appears in the formulation of an LP in equational form. Remove the appropriate row(s) so that the remaining rows are linearly independent. What are the new A and b?

To simplify notation, we will continue using the convention that the matrix A in an LP has dimensions $m \times n$, after possibly being augmented and having certain rows removed, according to the procedures described above.

2. Basic feasible solutions

Adding slack variables $s = (s_1, s_2, s_3)$ to the LP in Exercise (ii), we arrive at the equational formulation of the feasible set

$$\begin{aligned} &x_1 + 3x_2 + x_3 + s_1 = 5, \\ &4x_1 - x_2 - x_3 + s_2 = 2 \\ &9x_1 + 2x_2 + 7x_3 + s_3 = 3 \\ &x \geq 0, \ s \geq 0 \end{aligned}$$

Note that this system of equations has an *obvious* solution $x = 0$, $s = (5, 2, 3)$, obtained by “zeroing” out the original variables x and “leaving” everything (from the right-hand side) in the slack variables s. This is an example of a *basic feasible solution*, corresponding to choosing s_1, s_2, s_3 as *basic variables*, i.e., choosing the last three columns of the augmented matrix

$$\begin{pmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 4 & -1 & -1 & 0 & 1 & 0 \\ 9 & 2 & 7 & 0 & 0 & 1 \end{pmatrix}$$

Of course, the same can be done every time we have an LP with one slack variable for each constraint, i.e., every time the augmented matrix has an $m \times m$ identity block appended on the right, as the one above.

More on this next time!