1. Write down the Lagrangian for a projectile (subject to no air resistance) in terms of its Cartesian coordinates \((x, y, z)\), with \(z\) measured vertically upward. Find the 3 Lagrange equations and show that they are exactly what you would expect for the equations of motion.

2. Write down the Lagrangian for a 1-dimensional particle moving along the \(x\)-axis and subject to a force \(F = -kx\) (with \(k\) positive). Find the Lagrange equation of motion and solve it.

3. Consider a mass \(m\) moving in two dimensions with potential energy \(U(x, y) = kr^2/2\), where \(r^2 = x^2 + y^2\). Write down the Lagrangian, using coordinates \(x\) and \(y\), and find the two Lagrange equations of motion. Describe their solutions. [This is the potential energy of an ion in an “ion trap,” which can be used to study the properties of individual ions.]

4. Consider a bead that is threaded on a rigid circular hoop of radius \(R\) lying in the \(xy\) plane with its center at \(O\), and use the angle \(\phi\) of 2-dimensional polar coordinates as the one generalized coordinate to describe the bead’s position. Write down the equations that give the Cartesian coordinates \((x, y)\) in terms of \(\phi\) and the equation that gives the generalized coordinate \(\phi\) in terms of \((x, y)\).

5. A particle is confined to move on the surface of a circular cone with its axis on the \(z\) axis, vertex at the origin (pointing down), and half-angle \(\alpha\). The particle’s position can be specified by 2 generalized coordinates, which you can choose to be the coordinates \((\rho, \phi)\) of cylindrical polar coordinates. Write down the equations that give the 3 Cartesian coordinates of the particle in terms of the generalized coordinates \((\rho, \phi)\) and vice versa.

6. Use the Lagrangian method to find the acceleration of the Atwood machine, including the effect of the pulley’s having moment of inertia \(I\). [Hint: the kinetic energy of the pulley is \(I\omega^2/2\), where \(\omega\) is its angular velocity.

7. Using the usual angle \(\phi\) as generalized coordinate, write down the Lagrangian for a simple pendulum of length \(l\) suspended from the ceiling of an elevator that is accelerating upward with constant acceleration \(a\). (Be careful when writing \(T\); it is probably safest to write the bob’s velocity in component form.) Find the Lagrange equation of motion and show that it is the same as that for a normal, nonaccelerating pendulum, except that \(g\) has been replaced by \(g + a\). In particular the angular frequency of small oscillations is \(\sqrt{(g + a)/l}\). A simple pendulum (mass \(m\) and length \(l\))

8. Prove that the potential energy of a central force \(\vec{F} = -kr^n\vec{r}\) (with \(n \neq -1\)) is

\[ U = \frac{k\rho^{n+1}}{n+1}. \]

In particular if \(n = 1\) then \(\vec{F} = -kr\vec{r}\) and \(U = k r^2/2\)

9. A small cart (mass \(m\)) is mounted on rails inside a large cart. The two are attached by a spring (force constant \(k\)) in such a way that the small cart is in equilibrium at the midpoint of the large. The distance from the small cart from its equilibrium is denoted \(x\) and that of the large one from a fixed point on the ground is \(X\), as shown in the figure. The large cart is now force to
oscillate such that $X = A \cos \omega t$, with both $A$ and $\omega$ fixed. Set up the Lagrangian for the motion of the small cart and show that the Lagrange equation has the form

$$\ddot{x} + \omega_0^2 x = B \cos \omega t,$$

where $\omega_0$ is the natural frequency $\omega_0 = \sqrt{k/m}$ and $B$ is a constant.

10. Consider a bed of mass $m$ sliding without friction on a wire that is bent in the shape of a parabola and is being spun with constant angular velocity $\omega$ about its vertical axis, as shown in the figure. Use cylindrical polar coordinates and let the equation of the parabola be $z = k \rho^2$. Write down the Lagrangian in terms of $\rho$ as the generalized coordinate. Find the equation of motion of the bed and determine whether there are positions of equilibrium, that is values of $\rho$ at which the bed can remain fixed, without sliding up or down the spinning wire. Discuss the stability of any equilibrium positions you find.

11. A simple pendulum (mass $M$ and length $L$) is supended from a cart (mass $m$) that can oscillate on the end of a spring of force constant $k$ as shown in the figure. (i) Write the Lagrangian in terms of the 2 generalized coordinates $x$ and $\phi$, where $x$ is the extension of the spring from its equilibrium length. [Hint: Be careful writing down the kinetic energy $T$. A safe way to get the velocity right is to write down the position of the bob at time $t$ and then differentiate.] Find the 2 Lagrange equations. (ii) Simplify the equations to the case that both $x$ and $\phi$ are small.
12. Consider a particle of mass $m$ constrained to move on the surface of a paraboloid whose equation (in cylindrical coordinates) is $r^2 = 4az$. If the particle is subject to a gravitational force, show that the frequency of small oscillations about a circular orbit with radius $\rho = \sqrt{a z_0}$ is

$$\omega = \sqrt{\frac{2g}{a + z_0}}.$$