Kinematics of rotations

In this Lecture we call „solid objects“ such extended objects that are rigid (nondeformable) and thus retain their shape. In contrast to point objects that are characterized by their position in the space, extended solid objects are characterized by both position and orientation. These objects can move as the whole and change their orientation (that is, rotate) at the same time.

The simplest case of rotational motion is rotation around a fixed axis, like rotation of a door around a hinge. In this case each point of the object performs circular motion studied in Lecture 5. If the object does not have a fixed axis of rotation (like an arbitrary object thrown into the air) its rotation can be very complicated, with the rotation axis permanently changing its direction with time. It can be shown that very small (infinitesimal) rotations can be considered as rotations around some fixed axis.

We will mostly speak about rotations around fixed axis here, leaving general rotations for more advanced physics courses.
Rotations around a fixed axis

Rotational orientation is described by the angle \( \theta \), relative to some orientation for which we have chosen \( \theta = 0 \). In other words, \( \theta \) is defined up to a constant.

Rotational displacement is the angle of rotation, \( \Delta \theta = \theta_2 - \theta_1 \). Rotations counterclockwise are positive (\( \Delta \theta > 0 \)) while rotations clockwise are negative (\( \Delta \theta < 0 \)).

Radian is such an angle, for which the length of the arc is equal to the radius. In other words, the angle in radians is given by \( \frac{L}{R} \) and it is dimensionless. Revolution corresponds to \( L = 2\pi R \), thus \( 360^\circ = 2\pi \) radians, thus

\[
1 \text{ radian} = \frac{360^\circ}{(2\pi)} = 57.3^\circ
\]
**Rotational velocity and acceleration**

Rotational velocity (angular velocity):

\[ \omega = \frac{\Delta \theta}{\Delta t} \]

Rotational acceleration (angular acceleration):

\[ \alpha = \frac{\Delta \omega}{\Delta t} \]

(\(\Delta t\) is the time elapsed. For finite \(\Delta t\) this is the average angular velocity, for very small \(\Delta t\) this is the instantaneous angular velocity, as usual)

**Motion with constant angular acceleration**

\[ \omega = \omega_0 + \alpha t \]

\[ \theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \]

(Similar to the translational motion with constant acceleration)

**Similarities between translational and rotational motion**

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<tr>
<th>Position: (x)</th>
<th>Orientation: (\theta)</th>
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<td>Rotational displacement: (\Delta \theta)</td>
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<tr>
<td>Velocity: (v)</td>
<td>Angular velocity: (\omega)</td>
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<td>Acceleration: (a)</td>
<td>Angular acceleration: (\alpha)</td>
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For objects rotating around an axis, the linear velocity $v$ increases with the distance from the axis:

$$v = \omega r$$

That is, the object cannot be characterized by a unique linear velocity but it can be characterized by a unique angular velocity

Similarly, the relation between the linear and rotational accelerations has the form

$$a = \alpha r$$
Statics of rotations

In the mechanics of translational motion, two forces balance each other if their sum is zero.

In the mechanics of rotational motion, two forces balance each other if the sum of their torques is zero, in other words, if the two torques balance each other.

Definition of torque: Product of the force and the lever arm

$$\tau = F \cdot r = Fr \sin \phi$$

- Force with a greater lever arm has a stronger effect!

One should assign a sign to the torque: Torques rotating counterclockwise are positive and torques rotating clockwise are negative. Having different signs, torques can balance each other.
Work-energy principle for the rotational motion

The concept of the torque follows from the work-energy principle: The solid object is balanced if its displacements and rotations result in a zero work of applied forces. Indeed, if the work is nonzero, then displacements or rotations accompanied by the work will result in the change of its kinetic energy, that is, the object will accelerate or decelerate and thus it is unbalanced.

For translational displacement it obviously follows from this principle that the sum of all forces should be zero. For rotations it follows that the sum of all torques should be zero. The proof is the following:

For a small rotation $\Delta \theta$, the application point of the force makes a linear displacement $\Delta r = r \Delta \theta$ that is perpendicular to $r$, so that the work done is

$$\Delta W = F_\perp \Delta r = F \Delta r \sin \varphi = Fr \Delta \theta \sin \varphi = \tau \Delta \theta$$

If several forces are applied, then the work is

$$\Delta W = (\tau_1 + \tau_2 + \ldots) \Delta \theta$$

since $\Delta \theta$ is the same for all forces. The work will be zero if

$$\tau_1 + \tau_2 + \ldots = 0$$

- the balance condition

$$r_1 \cong r_2 \cong r$$

$$\Delta r \cong r \Delta \theta$$
Center of mass and stability

Center of mass (CM), also called center of gravity, is a point about which gravitational forces applied to different parts of the object produce no torque. That is, if we choose an axis going through or a pivot point in CM, the object will be balanced. One can consider gravitational force as applied to CM, so that its lever arm is zero and the torque vanishes. The object does not rotationally accelerate around its own CM under the influence of gravitational force and it only can have linear (translational) acceleration. If we now choose another pivot point that does not coincide with CM, then the torque will be nonzero, the lever arm being defined by the distance between CM and the pivot point. The position of CM is given by

\[ \mathbf{r} = \frac{1}{M} \sum_i m_i \mathbf{r}_i \quad \text{where} \quad M = \sum_i m_i \quad \text{total mass of a system} \]

For symmetric objects such as a cube, CM is in the geometric center.

CM is important, in particular, for the analysis of the stability of solid objects with respect to tipping. If, as the result of a small rotation away from the initial position, CM goes up, the object is stable. Indeed, such a rotation leads to the increase of the potential energy and thus it is impossible without violation of the energy conservation law. On the contrary, if a small rotation leads to CM going down, then the object is unstable. In this case the decrease of the potential energy is compensated for by the increase of the kinetic energy, and the object will rotationally accelerate away from the initial position. See illustration on the next page.
**Pisa tower**

CM goes up, stable

Torque of the gravity force is compensated for by that of the normal force

CM goes down, unstable

Torque of the gravity force is not compensated for by that of the normal force: The two torques have the same sign!
Rotational dynamics

If the torques acting on a body are unbalanced, that is, the total torque is nonzero, the body will rotationally accelerate. This is governed by the second Newton’s law for rotations. The latter can be obtained from Newton’s second law for translational motion. Consider at first the simplest case of a point mass $m$ on a weightless rigid rod of length $r$, fixed to an axis at the other side:

The point mass accelerates in the direction of the force according to the Newton’s second law

$$ F = ma $$

One can rewrite it in terms of the torque $\tau = Fr$ and rotational acceleration $\alpha = a/r$ as follows

$$ \tau = Fr = mar = mr^2\alpha $$

that is,

$$ \tau = I\alpha \quad \text{with} \quad I = mr^2 \quad \text{- rotational inertia} $$

This is Newton’s second law for rotational motion
As each solid object can be represented as a collection of point masses $m_i$, the simplest case above can be generalized for arbitrary objects as

$$\tau = I\alpha$$

where $\tau$ is the net torque acting on the body and rotational inertia $I$ is given by

$$I = \sum_i m_i r_{i,\perp}^2$$

$r_{i,\perp}$ – distance from the axis

Expressions for $I$ for typical symmetric bodies can be obtained with calculus:

- Rod with axis through end
  $$I = \frac{1}{3} ml^2$$

- Rod with axis through center
  $$I = \frac{1}{12} ml^2$$

- Disc or cylinder, around the symmetry axis
  $$I = \frac{1}{2} mR^2$$

- Ring, around the symmetry axis
  $$I = mR^2$$

- Uniform sphere
  $$I = \frac{2}{5} mR^2$$
Angular momentum and its conservation

From \( \tau = I \alpha \) follows that if \( \tau = 0 \), then

the angular momentum \( L = I \omega \) is conserved:

\[
L = \text{const}
\]

For instance, the gravitational force acting upon the planets from the sun has zero lever arm, so that the torque on the planets is zero and their angular momentum is conserved. This leads to Kepler’s second law.

An interesting difference between translational and rotational motions is that, whereas the mass is constant, the rotational inertia \( I \) can be changed by changing the distances of masses from the axis. If the torque is zero and \( L \) is conserved, changing \( I \) results in the change of the angular velocity \( \omega \).

Rotational kinetic energy

It can be shown to have the form

\[
E_{k,\text{rot}} = \frac{1}{2} I \omega^2
\]

In general, the total kinetic energy is the sum of translational and rotational kinetic energies

\[
E_k = E_{k,\text{tr}} + E_{k,\text{rot}} = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2
\]
### Corresponding concepts of translational (linear) and rotational (angular) motion

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<th>Rotational motion</th>
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<td>$m$</td>
<td>$I$</td>
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<td>Newton’s second law</td>
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<td>$E_k = \frac{1}{2}mv^2$</td>
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