7 – Impuls and Momentum; Conservation

(Notions useful in the description of collisions, in particular)

Newton’s second law

\[ F = ma \]

Impuls - Momentum relation

\[ F \Delta t = m \Delta v \equiv \Delta P \]

Momentum conservation in isolated systems

\[ \Delta P = 0 \]

Definitions:

\[ \Delta t = t_2 - t_1 \text{ etc.} \]

\[ P = \text{const} \]

For a system of interacting objects, adding up Newton’s second laws for all objects:

\[ F = F_1 + F_2 + \ldots = m_1 a_1 + m_2 a_2 + \ldots \]

In isolated systems, \( F = 0 \)

\[ \text{total momentum is conserved: } P = \text{const} \]

(Sum only external forces. Internal forces make no contribution according to Newton’s third law \( F_{21} = -F_{12} \))

Note that \( P \) is vector!
Collisions and recoil

Two limiting cases of collisions:

- **Inelastic** (objects glue together after collision; Some part of the mechanical energy is lost)
  \[ P_{A,1} + P_{B,1} = P_2 \]
  \[ m_A v_{A,1} + m_B v_{B,1} = (m_A + m_B)v_2 \]

- **Elastic** (Objects rebound; Mechanical energy is conserved)
  \[ P_{A,1} + P_{B,1} = P_{A,2} + P_{B,2} \]
  \[ E_{A,1} + E_{B,1} = E_{A,2} + E_{B,2} \]
  \[ m_A v_{A,1}^2 + m_B v_{B,1}^2 = m_A v_{A,2}^2 + m_B v_{B,2}^2 \]

Conservation of momentum for **inelastic** collisions and conservation of momentum and energy for **elastic** collisions allow to solve a multitude of important problems.

**Recoil** is a phenomenon inverse to the inelastic collision: Forces acting between two objects cause them to move in different directions.

Example: The gun and the bullet (the gun recoils back and hits the shoulder)

Whereas in the inelastic collision the energy is absorbed (mechanical energy is transformed into heat), in the recoil the energy is released (chemical energy is transformed into mechanical energy)
Transform

\[ m_A v_{A,1} + m_B v_{B,1} = m_A v_{A,2} + m_B v_{B,2} \]
\[ m_A v_{A,1}^2 + m_B v_{B,1}^2 = m_A v_{A,2}^2 + m_B v_{B,2}^2 \]

as

\[ m_A (v_{A,1} - v_{A,2}) = m_B (v_{B,2} - v_{B,1}) \]
\[ m_A (v_{A,1}^2 - v_{A,2}^2) = m_B (v_{B,2}^2 - v_{B,1}^2) \]

and divide second equation over the first

\[ v_{A,1} + v_{A,2} = v_{B,1} + v_{B,2} \]

or

\[ v_{A,1} - v_{B,1} = -(v_{B,2} - v_{A,2}) \]

(the relative velocity changes the sign after collision)

For equal masses, \( m_A = m_B \), the equation above can be combined with

\[ v_{A,1} + v_{B,1} = v_{B,2} + v_{A,2} \]

(momentum conservation)

Leading to

\[ v_{A,2} = v_{B,1}, \quad v_{B,2} = v_{A,1} \]

(colliding bodies exchange velocities)
Center of mass (CM) of an extended system

Definition: \[ \mathbf{r} = \frac{1}{M} \sum_i m_i \mathbf{r}_i \] where \( M = \sum_i m_i \) total mass of a system

Acceleration of CM:

\[ M \mathbf{a} = \sum_i m_i \mathbf{a}_i = \sum_i \mathbf{F}_i \equiv \mathbf{F} \] - the net force (total external force)

Center of mass of an extended system moves in the same way under the influence of the net force \( \mathbf{F} \) as the point mass \( M \) would move in this case. If we are interested in the motion of an object as a whole we can consider only the motion of the CM.

If the net force is zero, CM moves with a constant velocity, whereas the object, in general, rotates around its CM.

The force of gravity can be considered as applied to CM.
A golf club exerts an average force of 500 N on a 0.1 kg golf ball, but the club is in contact with the ball for only a hundredth of a second. a) What is the magnitude of the impulse delivered by the club? b) What is the velocity acquired by the golf ball?

**Given:** F=500 N, Δt=0.01 s, m=0.1 kg

**To find:** a) $F\Delta t$ - ?  b) $v$ - ? (this is final velocity; initial velocity is zero: $v_1=0$, $v_2=v$)

**Solution**

a) $F\Delta t = 500 \text{ N} \times 0.01 \text{ s} = 5 \text{ N s} = 5 \text{ kg m / s}$

Check units: N s / kg = kg m / s^2 = m/s, OK
**Problem**

Four railroad cars, all with the same mass of 20000 kg sit on a track. A fifth car of identical mass approaches them with a velocity of 15 m/s. This car collides with the other four cars. a) what is the initial momentum of the system? b) what is the velocity of the five coupled cars after the collision?

**Solution**

**Given:** \( m_A = 4m, \quad m_B = m, \quad m = 20000 \text{ kg}, \quad v_{A,1} = 0, \quad v_{B,1} = 15 \text{ m/s} \)

**To find:** a) \( P_1 \) - ?  b) \( v_2 \) - ?

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a) \( P_1 = m_B v_{B,1} = 20000 \text{ kg} \times 15 \text{ m/s} = 300000 \text{ kg m / s} \)

b) \( m_A v_{A,1} + m_B v_{B,1} = (m_A + m_B)v_2, \quad v_{A,1} = 0 \)

\[
v_2 = \frac{m_B v_{B,1}}{m_A + m_B} = \frac{m v_{B,1}}{5m} = \frac{1}{5} v_{B,1} = \frac{1}{5} \times 15 \text{ m/s} = 3 \text{ m/s}
\]
Problem

Calculate the fraction of the energy lost in the railroad car collision of the problem above

Solution

\[
E_{\text{lost}} = E_1 - E_2 = \frac{m_B v_{B,1}^2}{2} - \frac{(m_A + m_B) v_2^2}{2} = \frac{m v_{B,1}^2}{2} - \frac{5 m (v_{B,1}/5)^2}{2}
\]

\[
= \frac{m v_{B,1}^2}{2} \left(1 - \frac{1}{5}\right) = \frac{4}{5} \frac{m v_{B,1}^2}{2} = \frac{4}{5} E_1
\]

Fraction of the energy lost:

\[
\frac{E_{\text{lost}}}{E_1} = \frac{4}{5}, \quad \text{that is, 80%}
\]
Problem

Billiard ball A moving with a speed 3 m/s in the +x direction strikes an equal-mass ball B initially at rest. The two balls are observed to move off at 45° to the x axis, ball A above the x axis and ball B below. What are the speeds of the two ball after the collision?

Solution

Given: \( m_A = m_B \), \( v_{A,1} = 3 \text{ m/s} \), \( v_{B,1} = 0 \), \( \theta_{A,1} = 45^\circ \), \( \theta_{B,1} = -45^\circ \)

To find: \( v_{2, A} \) - ? \( v_{2, B} \) - ?

Momentum of the isolated system of two balls is conserved:

\[
P_1 = P_{1, A} + P_{1, B} = P_2 = P_{2, A} + P_{2, B}
\]

Momentum conservation along the y axis:

\[
P_{1, y} = 0 = P_{2, y} = m(v_{2, A, y} + v_{2, B, y})
\]

\[
= m(v_{2, A} \sin(45^\circ) + v_{2, B} \sin(-45^\circ)) = \frac{m}{\sqrt{2}} (v_{2, A} - v_{2, B}) \Rightarrow v_{2, A} = v_{2, B}
\]

Momentum conservation along the x axis:

\[
P_{1, x} \equiv m v_{1, A} = P_{2, x} = m(v_{2, A, x} + v_{2, B, x})
\]

\[
= m(v_{2, A} \cos(45^\circ) + v_{2, B} \cos(-45^\circ)) = \frac{m}{\sqrt{2}} (v_{2, A} + v_{2, B}) = \sqrt{2} m v_{2, A} \Rightarrow v_{2, A} = v_{2, B} = \frac{v_{1, A}}{\sqrt{2}}
\]

\[
= 2.1 \text{ m/s}
\]
Homework:

Giancoli Chapter 7, Problems 7, 17, 23, 29, 51