Lesson 8

Physics 168

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Rotational Dynamics

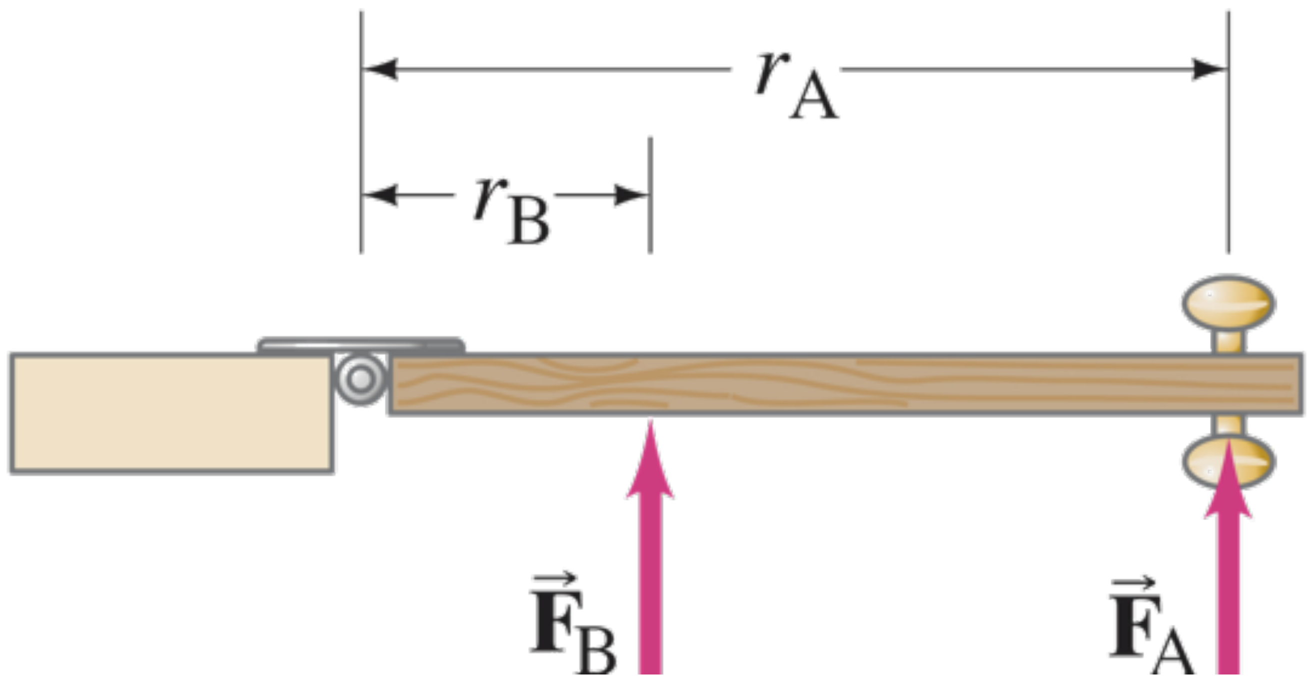
Torque

To make an object start rotating a force is needed

Position and direction of force matter as well

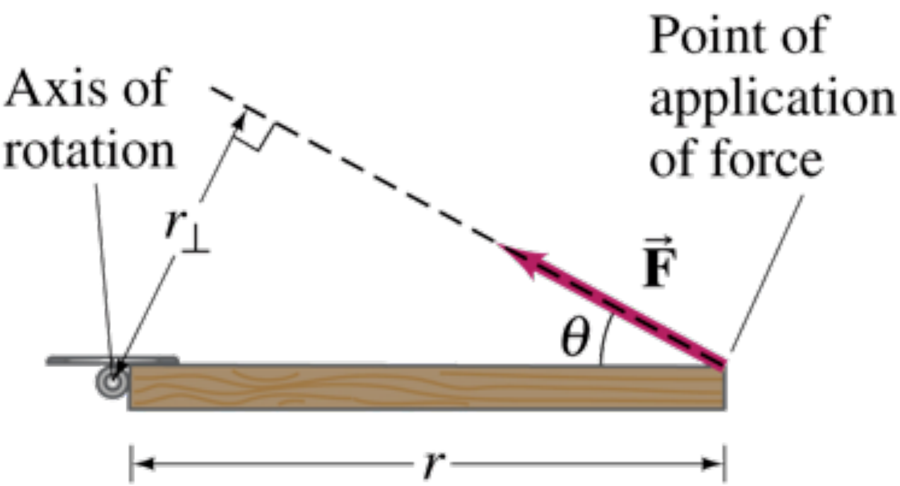
Perpendicular distance from axis of rotation

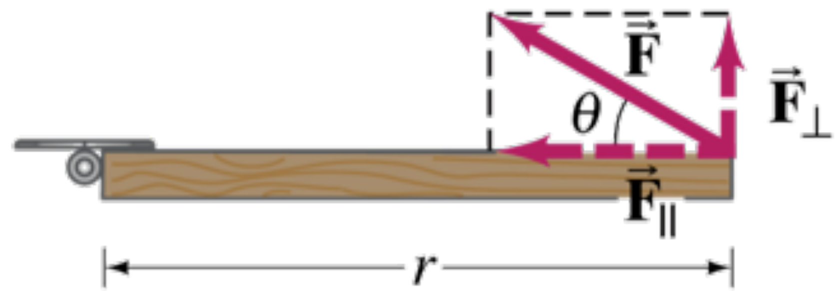
to line along which force acts is called lever arm



Torque (Cont'd)

Torque is defined as ☛ $\tau = r_\perp F$



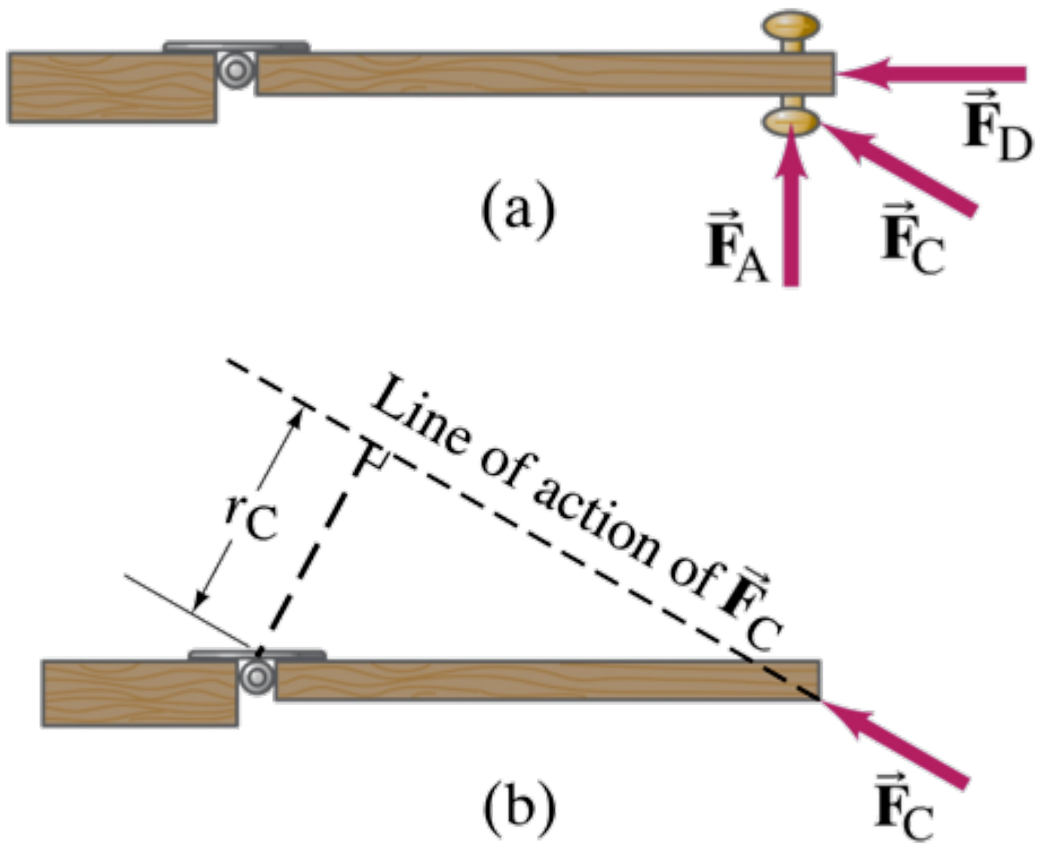


Torque (Cont'd)

Lever arm for F_A is distance from knob to hinge

Lever arm for F_D is zero

Lever arm for F_C is as shown

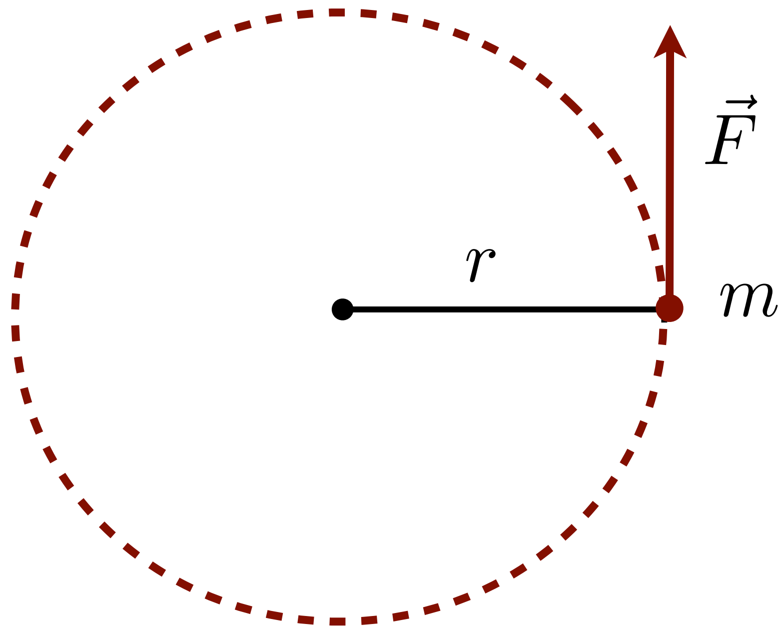


Rotational Dynamics ☛ Torque and Rotational Inertia

Knowing that $F = ma$ ☛ $\tau = mr^2\,\alpha$

This is for a single point mass

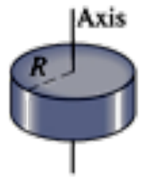
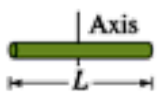
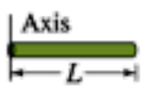
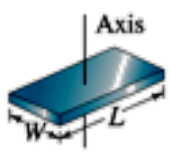
What about an extended object?



As angular acceleration is same for whole object we can write

$$\sum \tau_{\mathrm{i,\,net}} = \left(\sum m_{\mathrm{i}}\, r_{\mathrm{i}}^2\right) \alpha$$

Various Moment of Inertia

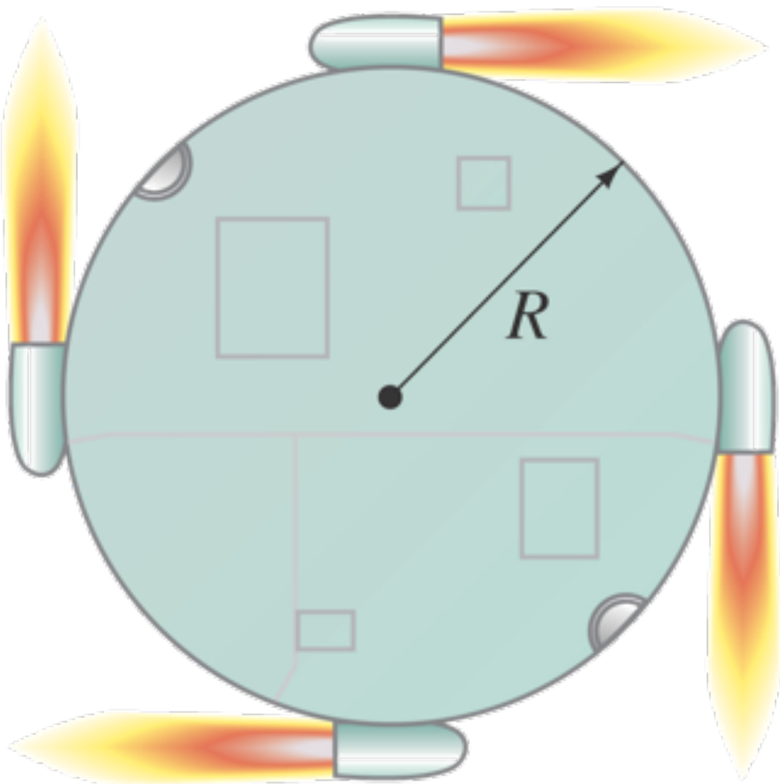
	Object	Location of axis		Moment of inertia
(a)	**Thin hoop,** radius R	Through center		MR^2
(b)	**Thin hoop,** radius R width W	Through central diameter		$\frac{1}{2}MR^2 + \frac{1}{12}MW^2$
(c)	**Solid cylinder,** radius R	Through center		$\frac{1}{2}MR^2$
(d)	**Hollow cylinder,** inner radius R_1 outer radius R_2	Through center		$\frac{1}{2}M(R_1^2 + R_2^2)$
(e)	**Uniform sphere,** radius R	Through center		$\frac{2}{5}MR^2$
(f)	**Long uniform rod,** length L	Through center		$\frac{1}{12}ML^2$
(g)	**Long uniform rod,** length L	Through end		$\frac{1}{3}ML^2$
(h)	**Rectangular thin plate,** length L, width W	Through center		$\frac{1}{12}M(L^2 + W^2)$

Rotational inertia of an object depends not only on its mass distribution but also location of axis of rotation – compare (f) and (g), for example -

Spinning Cylindrical Satellite

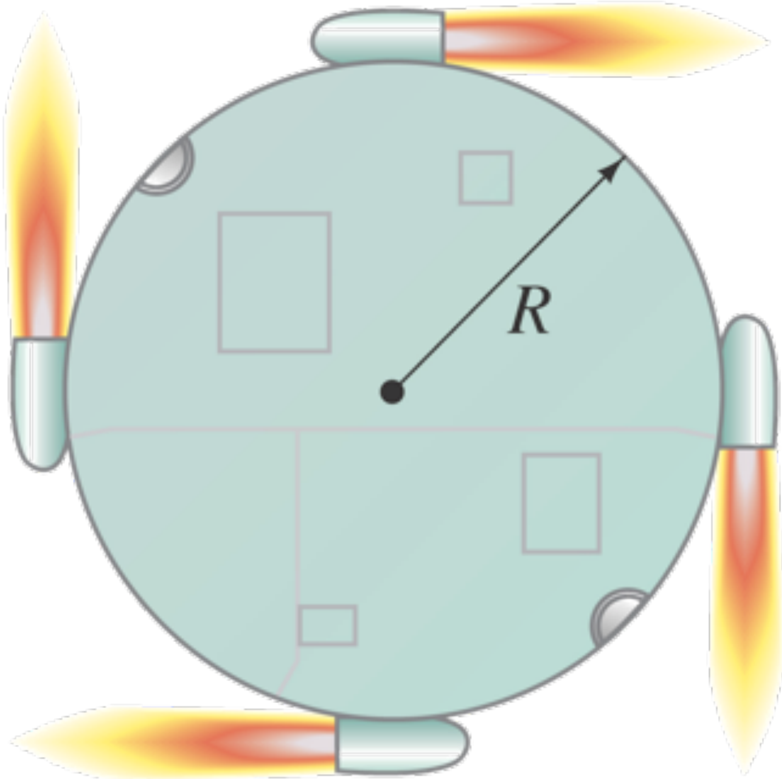
To get a flat, uniform cylindrical satellite spinning at correct rate, engineers fire four tangential rockets as shown in figure

If satellite has a mass of 3600 kg and a radius of 4 m, what is required steady force of each rocket if satellite is to reach 32 rpm in 5 min?



End view of cylindrical satellite

The ring of rockets will create a torque with zero net force



$$\tau_{\rm net} = 4FR$$

$$\tau_{\rm net} = I\alpha$$

$$I = \frac{1}{2}MR^2$$

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

$$4FR = I\alpha = \frac{1}{2}MR^2\frac{\Delta\omega}{\Delta t} \Rightarrow F = \frac{1}{8}MR\frac{\Delta\omega}{\Delta t} \approx 20\ {\rm N}$$

Rolling

Intuitive Question

"Why is it that when a body is rolling on a plane without slipping

the point of contact with the plane does not move?"

A simple answer to this question is quite simply: "because the body does not slip"

Why? Because "slipping" implies 2 bodies in contact moving relative to each other

Here ☛ there is no slipping

Therefore ☛ the point of contact and the plane don't move relative to each other

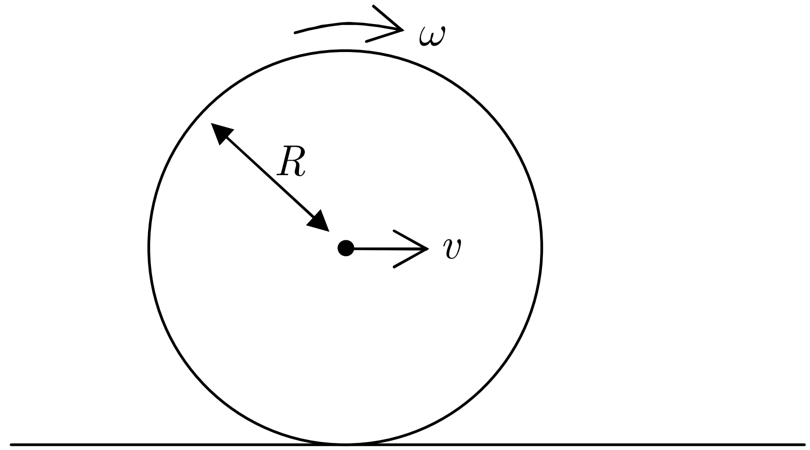
Therefore ☛ the point of contact does not move

However for those still unconvinced

We'll work out a mathematical argument which should help

Nonslip conditions

Imagine a cylinder moving forward at speed v on a plane without slipping



As indicated in diagram ☛ cylinder must also be rotating about its axis because it is rolling at an angular speed ω

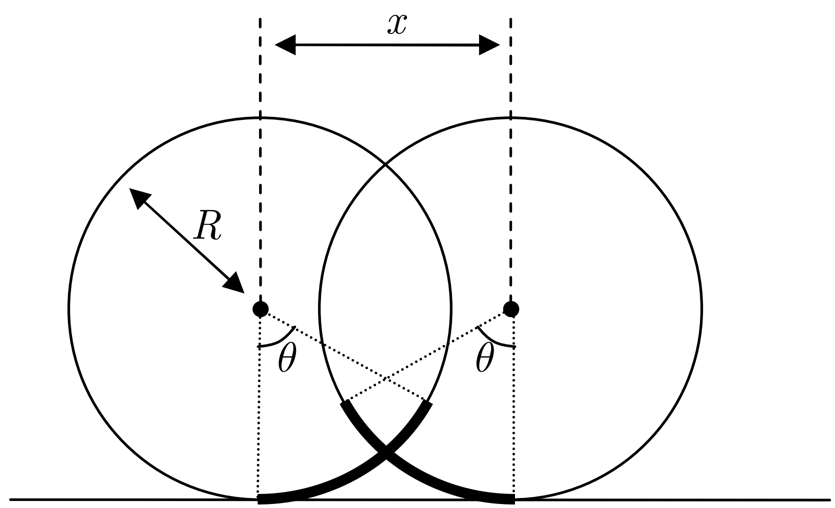
It is clear that ω is completely determined by v and vice-versa

because if we change v (say ☛ make the cylinder move faster) ω must also increase (the cylinder must roll faster)

Our aim is to find the relation between v and ω

Nonslip conditions

To do this ☛ consider cylinder moving forward a distance x



As a result ☛ it will rotate through angle θ

Now ☛ consider following argument

On purely mathematical grounds ☛ length of bold arc of the circle is $R\theta$

However ☛ this is equal to x

because the distance the circle has move forward is equal to the arc length

Nonslip conditions

We have found thus ☛ $x = R\theta$

Consider differentiating each side of this equation

$$\frac{dx}{dt} = R\frac{d\theta}{dt}$$

BUT ☛ $\frac{dx}{dt}$ is the velocity and $\frac{d\theta}{dt}$ is the angular velocity

As such ☛ $v = R\omega$

These two equations are the non-slip conditions

$v = R\omega$ is the same relationship that we had obtained for circular motion

However ☛ it's important to realize that the v in each equation is different

In circular motion ☛ v refers to a point on the rim of the circle

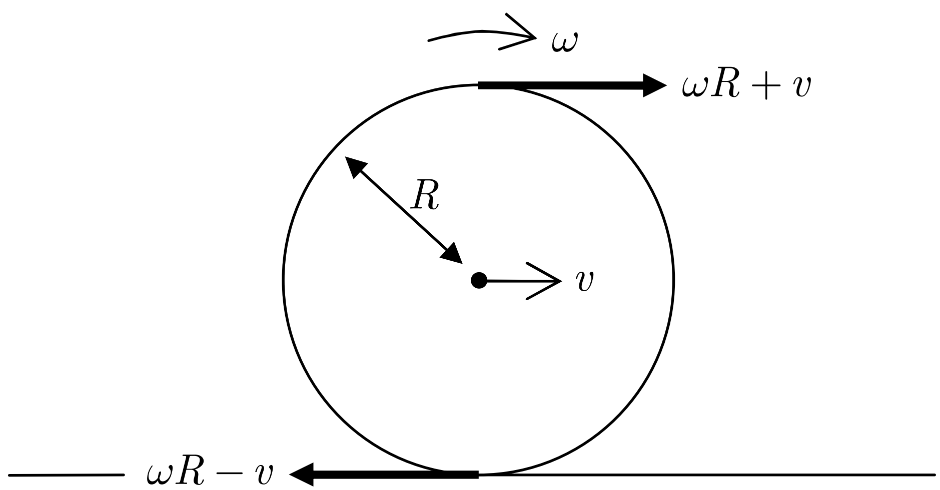
Here ☛ v refers to the velocity of the centre of mass

Let's convince ourselves they have to be the same as predicted by these equations

Point of contact

Consider cylinder rolling on a plane again

What are the velocities of points at very top and very bottom of circle?



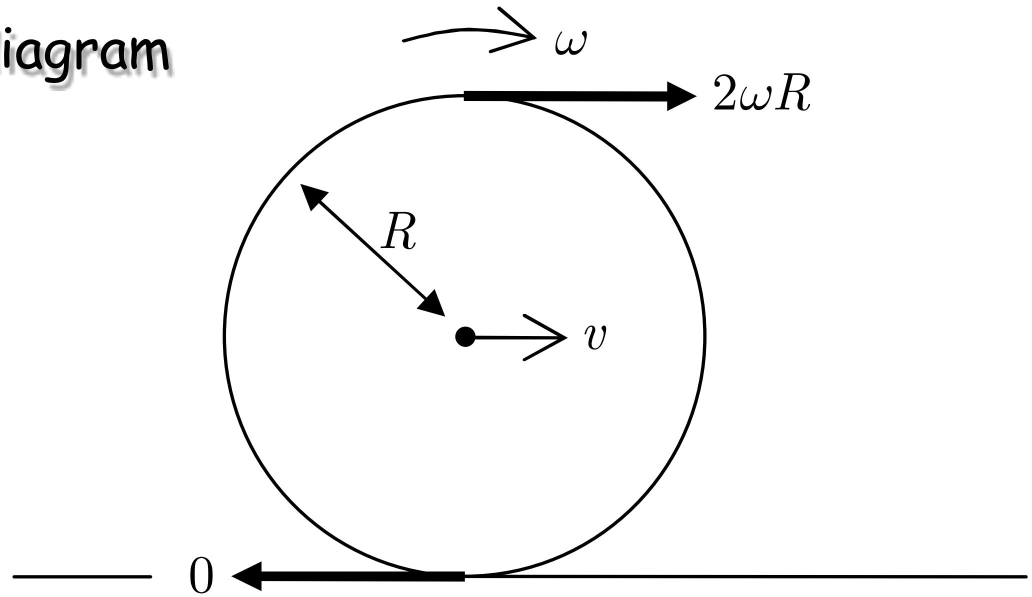
• Top point – moving forward at v (because the cylinder is moving forward at v) but also has an extra forward speed ωR because of the rotation

• Bottom point – moving forward at v (for the same reason) but is now moving backwards at ωR because of the rotation (because the circle is rotating clockwise ☛ it's moving back at its bottom point)

Point of contact (cont'd)

However ☛ we have found that $v = \omega\ R$

Feeding this into our diagram



As indeed expected

we have found that the velocity of the point of contact is zero

It does not move!!!

Differentiating on both sides the non-slip condition with respect to time

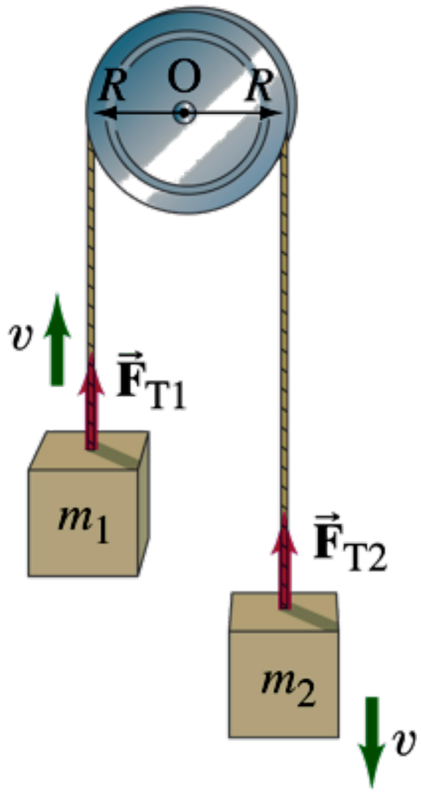
tangential acceleration ☛ $a_T = R\alpha$

(e.g. for string not to slip on a pulley wheel)

Atwood's machine

An Atwood's machine consists of two masses, m_1 and m_2 which are connected by a mass less inelastic cord that passes over a pulley

If pulley has radius R and moment of inertia I about its axle determine acceleration of masses m_1 and m_2



Force pulling pulley so it can rotate is

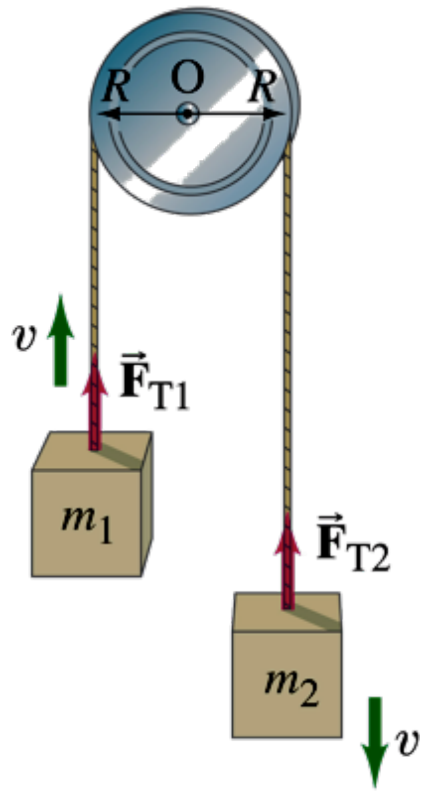
$$F_{T_2} - F_{T_1}$$

and it is at displacement R away from axis of rotation

Assume $m_2 > m_1$ and so pulley will accelerate clockwise

Align coordinate system with acceleration

clockwise ☛ positive



$$\sum F_{y_1} = m_1 a \Rightarrow F_{T_1} - m_1 g = m_1 a$$

$$\sum F_{y_2} = m_2 a \Rightarrow m_2 g - F_{T_2} = m_2 a$$

$$\sum \tau = I\alpha \Rightarrow F_{T_2} R - F_{T_1} R = I\alpha = I \frac{a}{R}$$

$$F_{T_1} = m_1 g + m_1 a$$

$$F_{T_2} = m_2 g - m_2 a$$

Substituting the force relation in the torque equation

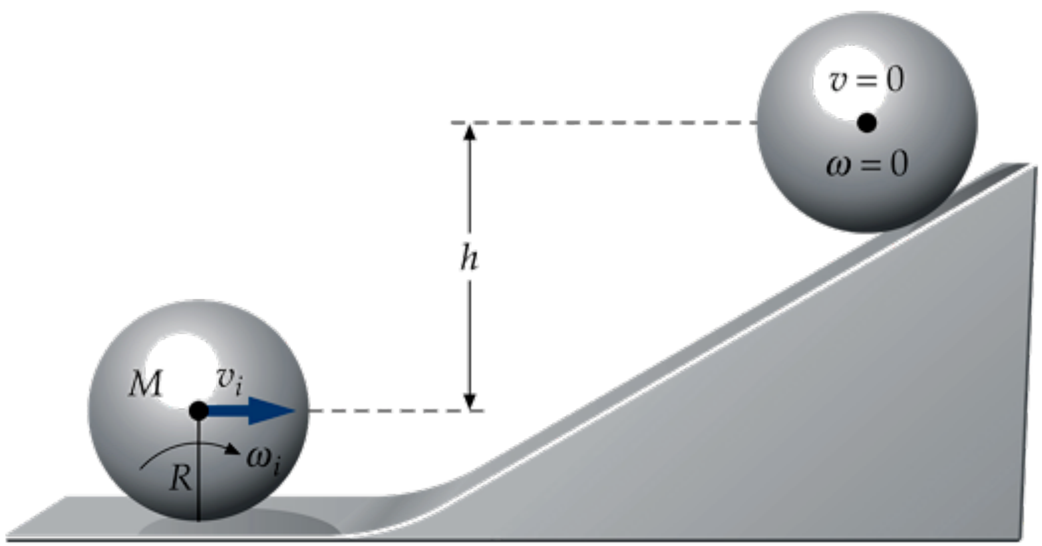
$$a = \frac{(m_2 - m_1)}{(m_1 + m_2 + I/R^2)} g$$

Question

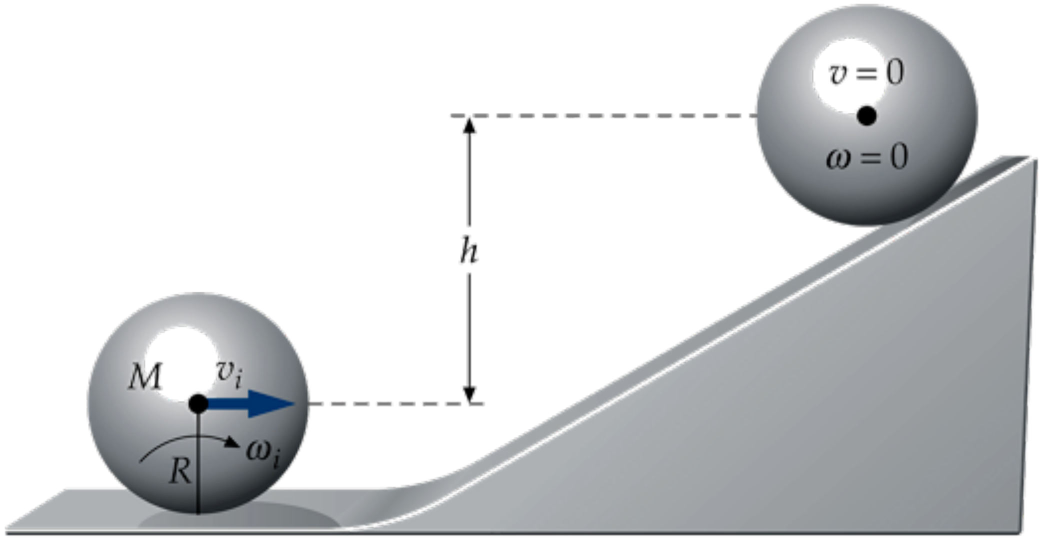
A bowling ball that has $11\,cm$ radius and $7.2\,kg$ mass is rolling without slipping at $2\,m/s$ on a horizontal ball return

It continues to roll without slipping up a hill to a height h before momentarily coming to rest and then rolling back down hill

Model ball as a uniform sphere and find h



Answer



$$\Delta E_{\rm mech} = 0$$

$$U_f + K_f = U_i + K_i \Rightarrow Mgh = \frac{1}{2} M v_{{\rm CM}_i}^2 + \frac{1}{2} I_{\rm CM} \omega_i^2$$

$$I_{\rm CM} = \frac{2}{5} MR^2 \Rightarrow Mgh = \frac{1}{2} M v_{{\rm CM}_i}^2 + \frac{1}{2} \left(\frac{2}{5} MR^2 \right) \frac{v_{{\rm CM}_i}^2}{R^2}$$

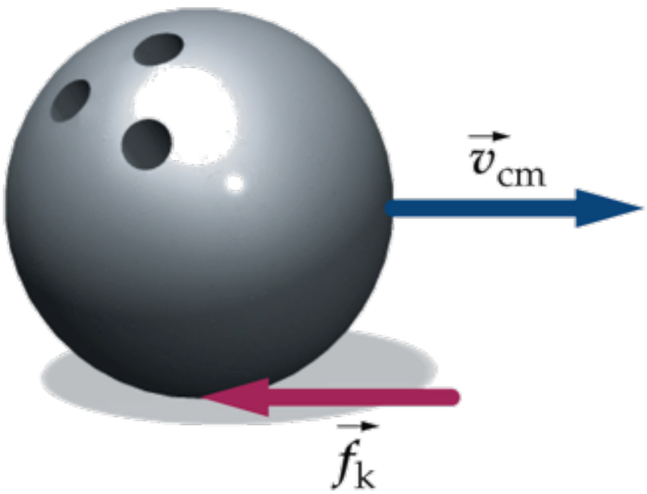
$$Mgh = \frac{7}{10} M v_{{\rm CM}_i}^2 \Rightarrow h = 29 \text{ cm}$$

Rolling with slipping

When an object slips (skids) as it rolls nonslip condition $v_{\rm cm} = R\omega$ does not hold

Suppose a bowler releases a ball with no initial rotation $(\omega_0 = 0)$

as ball skids along bowling lane $v_{\rm cm} > R\omega$



Kinetic frictional force will both reduce its linear speed $v_{\rm cm}$

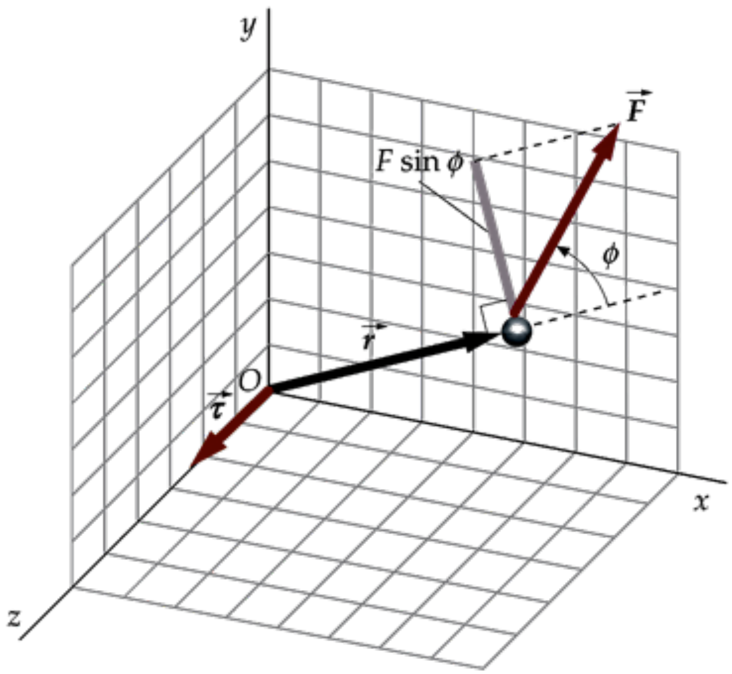
and increase its angular speed ω until nonslip condition $v_{\rm cm} = R\omega$ is reached

after which balls rolls without slipping

Conservation Theorems: Angular Momentum

Vector Nature of Rotation

Torque is expressed mathematically as a vector product of $\vec{r}$ and $\vec{F}$



$$\vec{\tau} = \vec{r} \times \vec{F}$$

If $\vec{F}$ and $\vec{r}$ are both perpendicular to z axis ☛ $\vec{\tau}$ is parallel to z axis

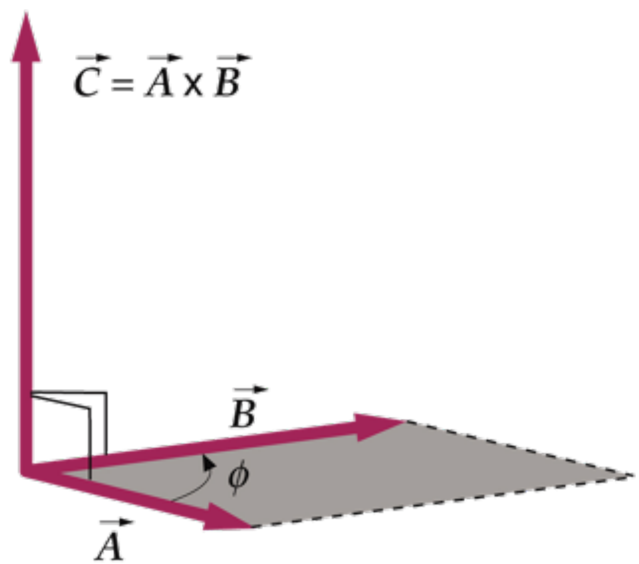
Vector product

Vector product of two vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C}$

that is perpendicular to both A and B

and has a magnitude $|C| = |A|\,|B| \sin\phi$

$|\vec{C}|$ equals area of parallelogram shown

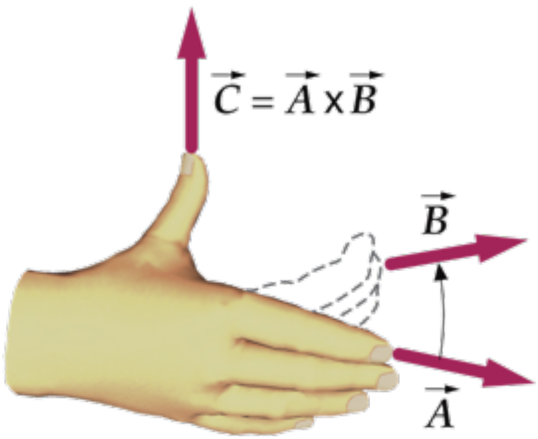


$$\vec{C} = \vec{A} \times \vec{B} = A\,B \sin\phi$$

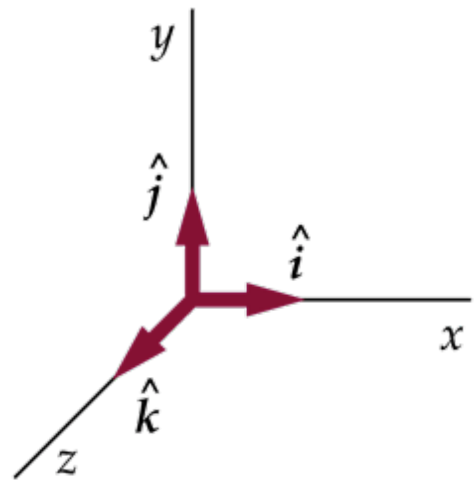
Vector product (cont'd)

Direction of $\vec{A} \times \vec{B}$ is given by right-hand rule

when fingers are rotated from direction of $\vec{A}$ toward $\vec{B}$ through angle ϕ

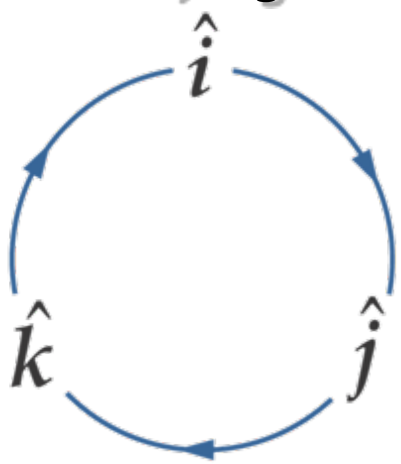


Defines a right-handed cartesian system



Vector product (cont'd)

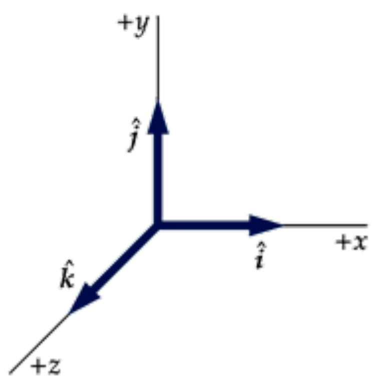
If we take vector product by going around figure in direction of arrows (clockwise) sign is positive



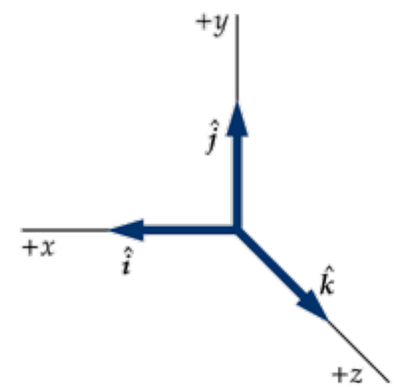
$$\vec{i} \times \vec{j} = k$$

Going around against arrows ☛ sign is negative

$$\hat{i} \times \hat{k} = -\hat{j}$$



Right-handed system ($\hat{i} \times \hat{j} = \hat{k}$)



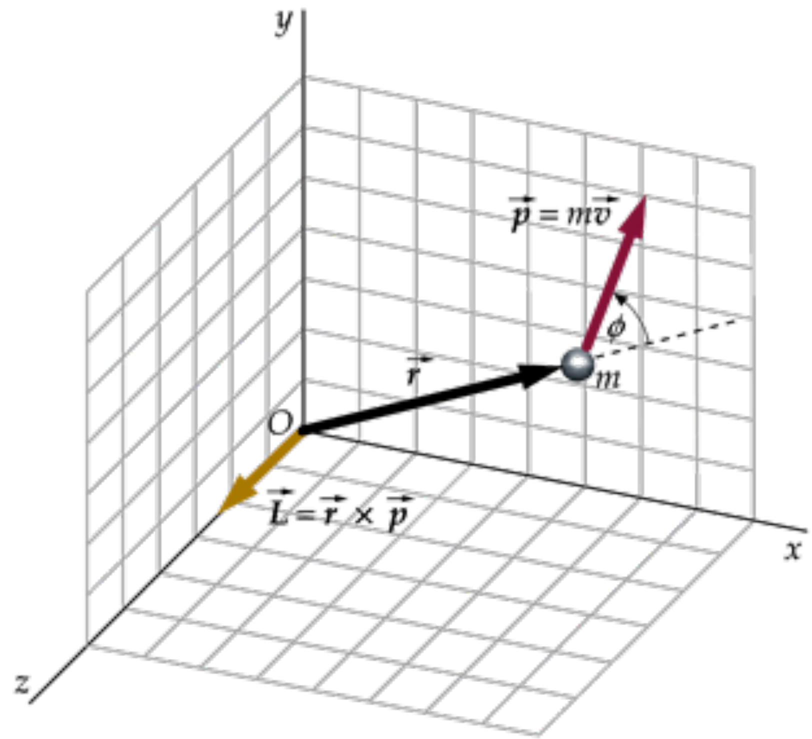
Left-handed system ($\hat{i} \times \hat{j} \neq \hat{k}$)

Throughout this course we adopt right handed coordinate systems

Angular momentum

Angular momentum $\vec{L}$ of particle relative to origin O

is defined to be vector product of $\vec{r}$ and $\vec{p}$

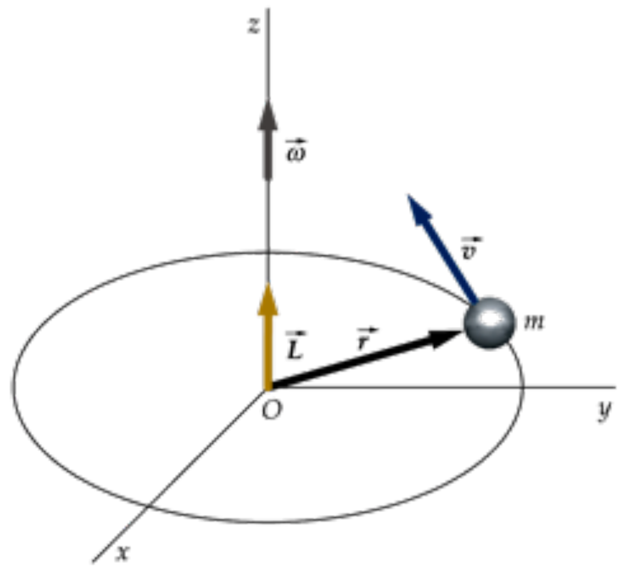


$$\vec{L} = \vec{r} \times \vec{p}$$

Angular momentum (cont'd)

Figure shows particle of mass m attached to circular disk of negligible mass moving in a circle in xy plane with its center at origin

disk is spinning about z-axis with angular speed ω



$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = r\,m\,v\,\hat{k} = m\,r^2\,\omega\,\hat{k} = m\,r^2\,\vec{\omega}$$

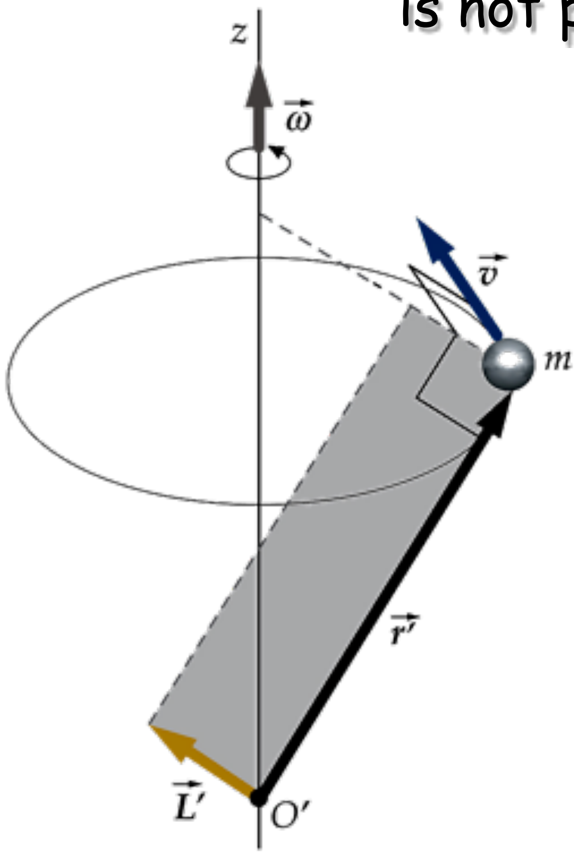
Angular momentum is in same direction as angular velocity vector

Because mr^2 is moment of inertia for a single particle we have

$$\vec{L} = I\vec{\omega}$$

Angular momentum (cont'd)

Angular momentum of this particle about a general point on z axis is not parallel to angular velocity vector



Angular momentum $\vec{L}'$ for same particle attached to same disk but with $\vec{L}'$ computed about a point on z axis that is not at center of circle

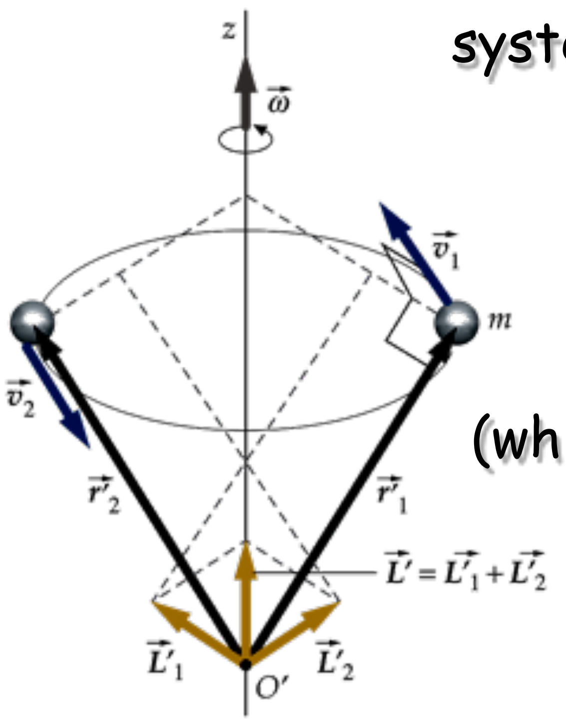
Angular momentum (cont'd)

For any system of particles that rotates about a symmetry axis

We now attach a second particle of equal mass to spinning disk at a point diametrically opposite to first particle

Total angular momentum $\vec{L}' = \vec{L}'_1 + \vec{L}'_2$ is again parallel to angular velocity vector $\vec{\omega}$

In this case axis of rotation passes through center of mass of two-particle system and mass distribution is symmetric about this axis



Such an axis is called a symmetry axis

For any system of particles that rotates about a symmetry axis total angular momentum (which is sum of angular momenta of individual particles) is parallel to angular velocity

$$\vec{L} = I\,\vec{\omega}$$

Conservation of angular momentum

Angular momentum of a particle ☛ $\vec{L} = \vec{r} \times \vec{p}$

(with respect to origin from which position vector $\vec{r}$ is measured)

Torque (or moment of force) with respect to same origin is ☛ $\vec{\tau} = \vec{r} \times \vec{F}$

Position vector from origin to point where force is applied ☛ $\vec{r}$

$$\vec{\tau} = \vec{r} \times \dot{\vec{p}}$$

$$\dot{\vec{L}} = \frac{d}{dt}(\vec{r} \times \vec{p}) = (\dot{\vec{r}} \times \vec{p}) + (\vec{r} \times \dot{\vec{p}})$$

But of course ➠ $\dot{\vec{r}} \times \vec{p} = \dot{\vec{r}} \times mv = m(\dot{\vec{r}} \times \dot{\vec{r}}) = 0$

$$\dot{\vec{L}} = \vec{r} \times \dot{\vec{p}} = \vec{\tau}$$

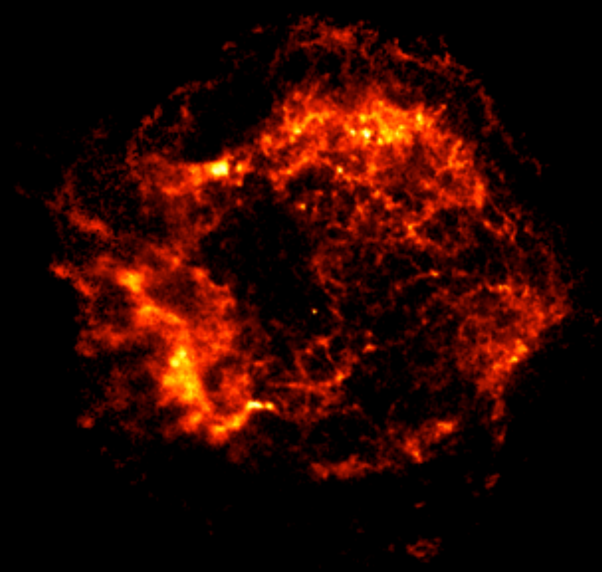
If $\vec{\tau} = 0 \Rightarrow \dot{\vec{L}} = 0 \Rightarrow L$ is a vector constant in time

If net external torque acting on a system about some point is zero

total angular momentum of system about that point remains constant

(a) Use conservation of angular momentum to estimate angular velocity of a neutron star which has collapsed to a diameter of $10\ \mathrm{km}$, from a star whose radius was equal to that of Sun $(7 \times 10^8\ m)$ of mass $1.5\, M_\odot$ and which rotated like our Sun once a month

(b) By what factor rotational kinetic energy change after collapse?



Bright dot in middle is believed to be hot young neutron star result of a supernova explosion from about 300 years ago

ⓐ **Conservation of angular momentum**

$$(I\omega)_{\rm initial} = (I\omega)_{\rm final}$$

$$\omega_{\rm final} = \omega_{\rm initial}\left(\frac{I_{\rm initial}}{I_{\rm final}}\right) = \omega_{\rm initial}\left(\frac{\frac{2}{5}MR^2_{\rm initial}}{\frac{2}{5}MR^2_{\rm final}}\right)$$

$$\omega_{\rm final} = 1\ {\rm rev/month}\left(\frac{7\times 10^8\ {\rm m}}{1\times 10^4\ {\rm m}}\right)^2 = 4.9\times 10^9\ {\rm rev/month} = 1900\ {\rm rev/s}$$

ⓑ **Rotational kinetic energy** ☛ $K = \frac{1}{2}I\omega^2$

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}I_f\omega_f^2}{\frac{1}{2}I_i\omega_i^2} = \left(\frac{R_f\omega_f}{R_i\omega_i}\right)^2 = 4\times 10^9$$

Angular Momentum of a System of Particles

Newton's second law for angular motion

Net external torque about a fixed point acting on a system equals rate of change of angular momentum of system about same point

$$\vec{\tau}_{\rm net,ext} = \frac{d\vec{L}_{\rm sys}}{dt}$$

Angular impulse

$$\Delta\vec{L}_{\rm sys} = \int_{t_{\rm i}}^{t_{\rm f}} \vec{\tau}_{\rm net,\,ext}\,dt$$

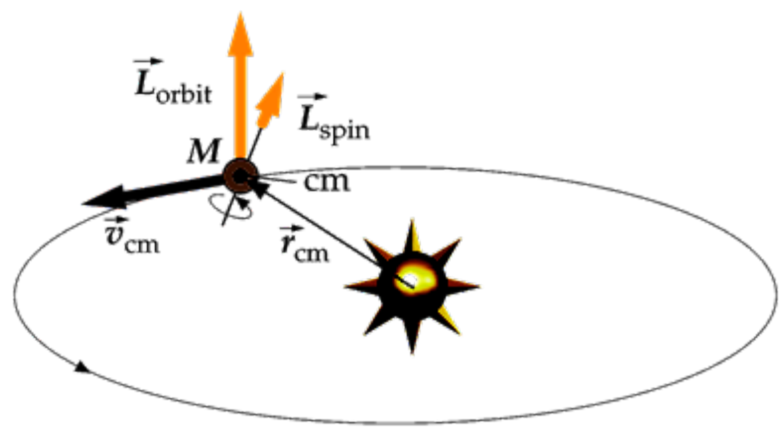
It is often useful to split total angular momentum of a system about an arbitrary point O into orbital angular momentum and spin angular momentum

$$\vec{L}_{\rm sys} = \vec{L}_{\rm orbit} + \vec{L}_{\rm spin}$$

Angular Momentum of a System of Particles

Newton's second law for angular motion (cont'd)

Earth has spin angular momentum due to its spinning motion about its rotational axis and it has orbital angular momentum about center of Sun due to its orbital motion around Sun



$$\vec{L}_{\rm orbit} = \vec{r}_{\rm cm} \times M_\oplus \vec{v}_{\rm cm} = M_\oplus r^2_{\rm cm} \omega_{\rm yearly} = 2.7 \times 10^{40} \, kg \, m^2/s$$

Pulling Through a Hole

A particle of mass m moves with speed v_0 in a circle with radius r_0 on a frictionless table top

Particle is attached to a string that passes through a hole in table

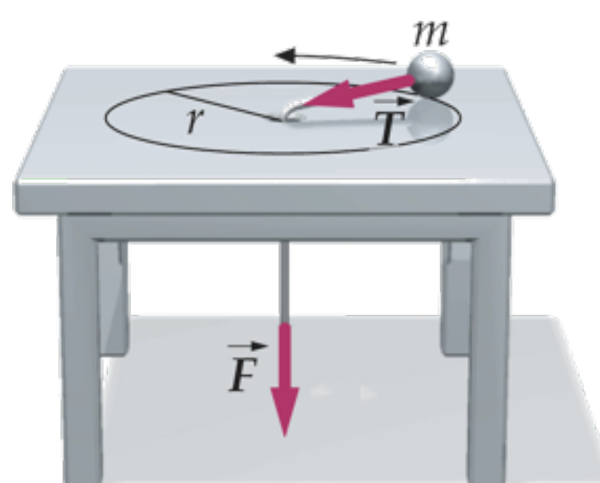
String is slowly pulled downward until particle is a distance r from hole after which particle moves in a circle of radius r

ⓐ Find final velocity in terms of r_0, v_0 and r

ⓑ Find tension when particle is moving in a circle of radius r in terms of r, m and angular momentum $\vec{L}$

ⓒ Calculate work done on particle by tension force $\vec{T}$ by integrating $\vec{T} \cdot d$

Express your answer in terms of r and L_0

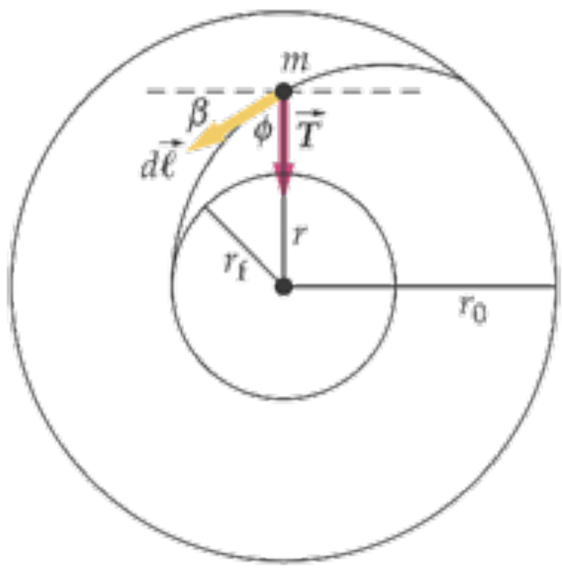


Pulling Through a Hole (cont'd)

$$L_\mathrm{f} = L_0 \Rightarrow v_\mathrm{f} = \frac{r_0 v_0}{r_\mathrm{f}}$$

Because particle is being pulled in slowly

acceleration is virtually same as if particle were moving in a circle



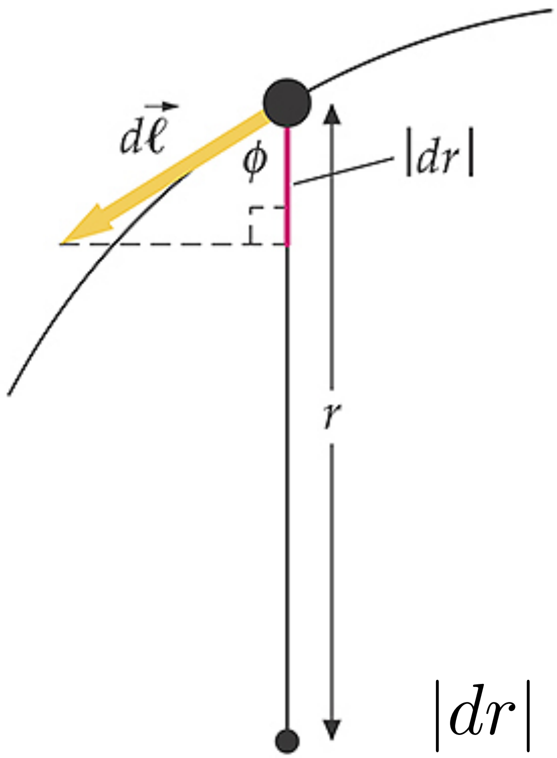
$$T \approx m \frac{v^2}{r}$$

$$\vec{L} = \vec{r} \times \vec{p} = r\, m\, v \cos\beta \approx r\, m\, v$$

$$\beta \ll 1 \to \cos\beta \approx 1$$

$$T = m\frac{v^2}{r} = \frac{m}{r}\left(\frac{L}{m\,r}\right)^2 = \frac{L^2}{m\,r^3}$$

Pulling Through a Hole (cont'd)



$$dr = -|dr|$$

$$dW = \vec{T} \cdot d\vec{\ell} = T d\ell \cos\phi$$

$$|dr| = d\ell \cos\phi \Rightarrow dW = T|dr| = -T dr$$

$$W = -\int_{r_0}^{r_\mathrm{f}} T\, dr = -\frac{L^2}{m}\int_{r_0}^{r_\mathrm{f}} r^{-3}\, dr = \frac{L^2}{2m}\left(\frac{1}{r_\mathrm{f}^2} - \frac{1}{r_0^2}\right)$$