Rotational Dynamics

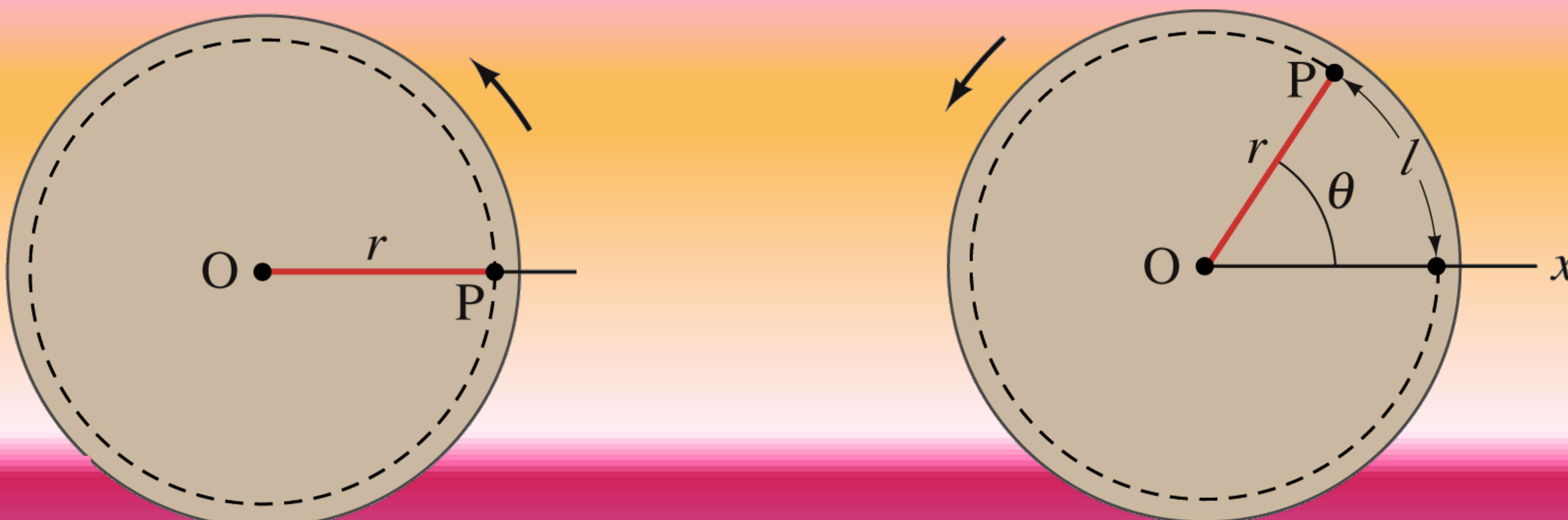
Luis Anchordoqui

Angular Quantities

In purely rotational motion, all points on the object move in circles around the axis of rotation (“*O* ”).
The radius of the circle is *r*. All points on a straight line drawn through the axis move through the same angle in the same time. The angle *θ* in radians is defined:

$$\theta = \frac{\ell}{r}$$

ℓ → arc length.



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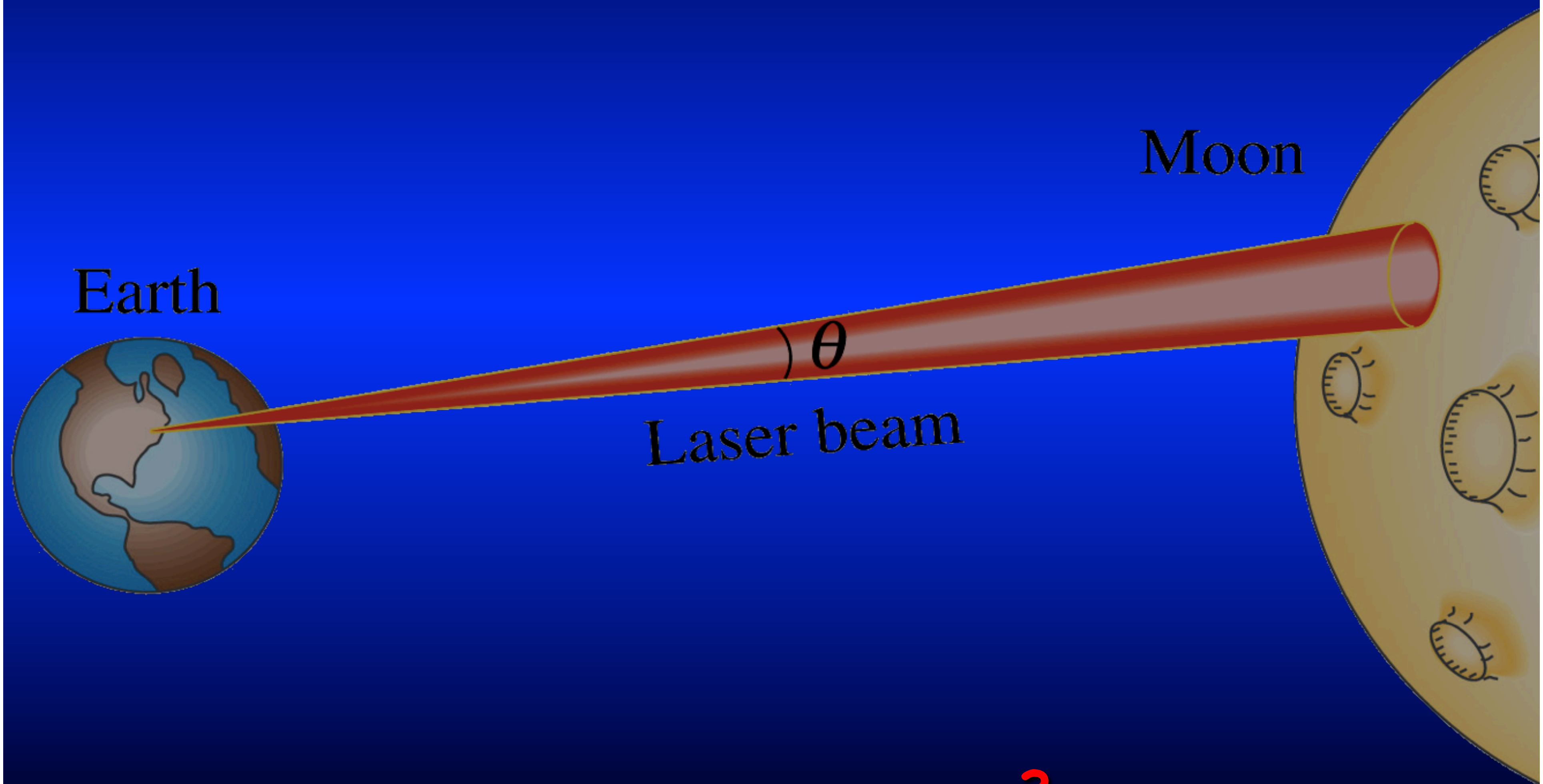
A laser jet is directed at the Moon, 380,000 km from Earth.

The beam diverges at an angle $\theta = 1.4 \times 10^{-5}$

What diameter spot will it make on the Moon?



diameter = 5.3×10^{3} m

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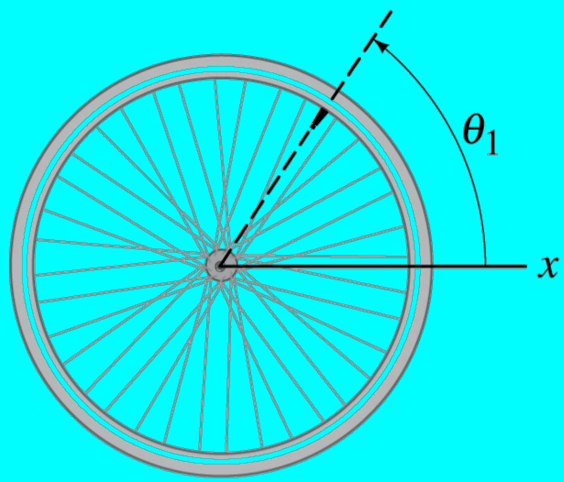
Angular Quantities (cont'd)

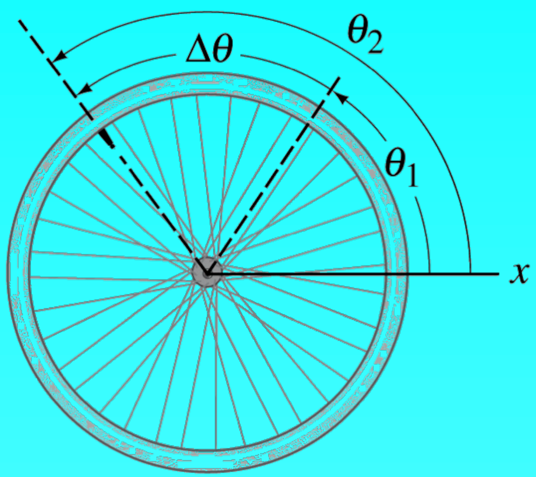
Angular displacement



$$\Delta\Theta = \Theta_2 - \Theta_1$$

The average angular velocity is defined as the total angular displacement divided by time:

$$\overline{\omega} = \frac{\Delta\Theta}{\Delta t}$$



The instantaneous angular velocity:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\Theta}{\Delta t}$$

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Angular Quantities (cont'd)

The angular acceleration is the rate at which the angular velocity changes with time:

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{\Delta t} = \frac{\Delta \omega}{\Delta t}$$

The instantaneous acceleration:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t}$$

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A CD Player

A compact disk rotates from rest to 500 rev/min in 5.5 s.

(a) What is the angular acceleration, assuming that it is constant?

(b) How many revolutions does the disk make in 5.5 s?

(c) How far does a point on the rim 6 cm from the center of the disk travel during the 5.5. s it takes to get 500 rev/min?

$\alpha = 9.5\ 1/s^2$

$\theta - \theta_0 = 144 \rightarrow 23$ rev

$\Delta s = 8.7$ m

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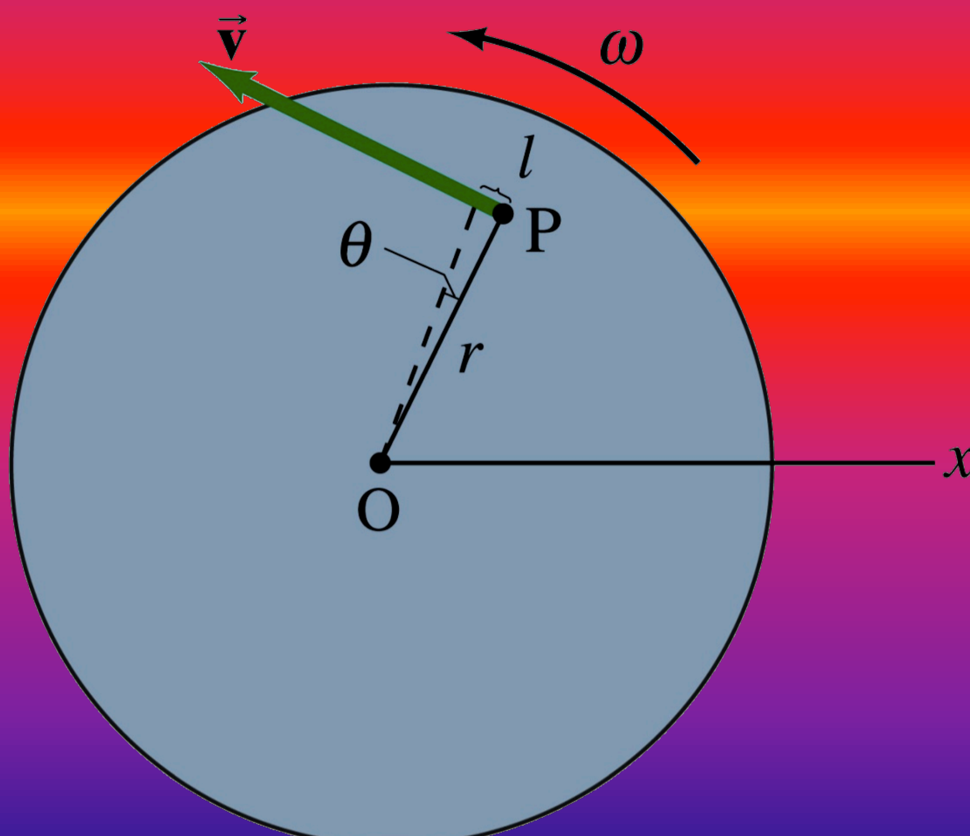
Angular Quantities (cont'd)

Every point on a rotating body has an angular velocity ω and a linear velocity v.

They are related: $v = r\omega$



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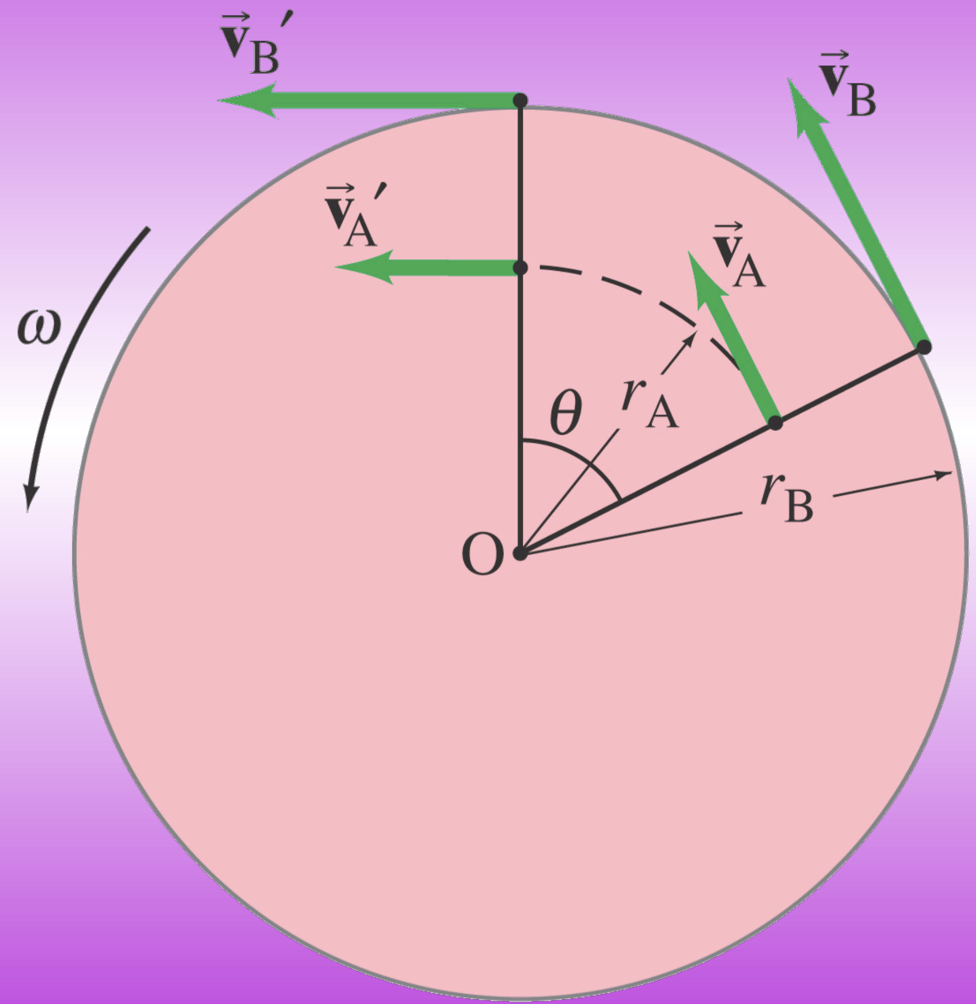
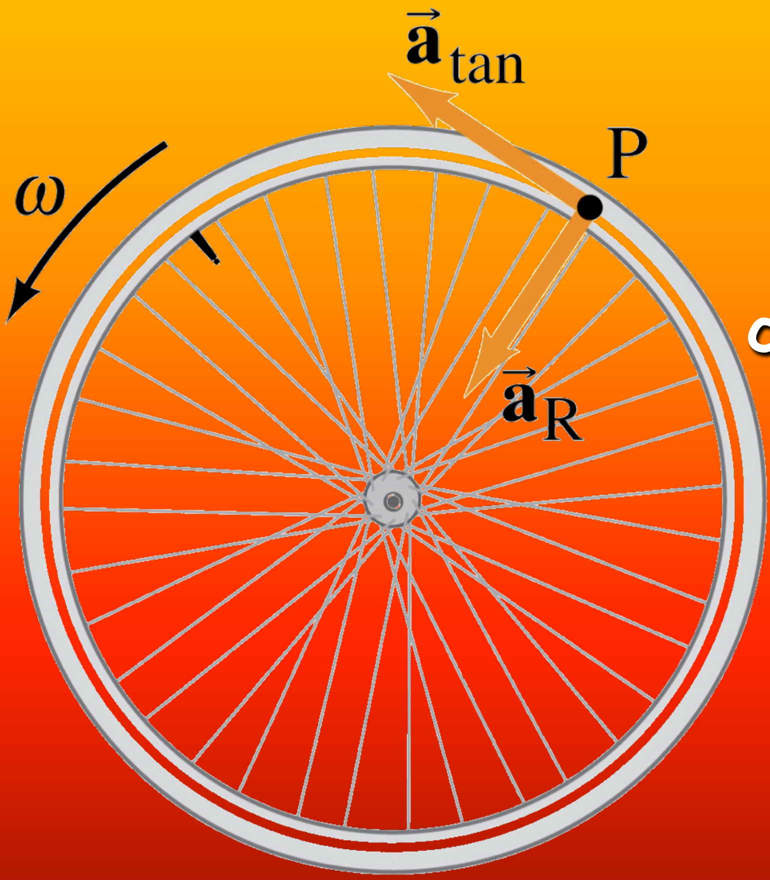
Angular Quantities (cont'd)

Therefore, objects farther from the axis of rotation will move faster.



Angular Quantities (cont'd)

If the angular velocity of a rotating object changes, it has a tangential acceleration:



$$a_{\tan} = r\alpha$$

Even if the angular velocity is constant, each point on the object has a centripetal acceleration:

$$a_R = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = \omega^2 r$$

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Angular Quantities (cont'd)

Here is the correspondence between linear and rotational quantities:

Linear	Type	Rotational	Relation
x	Displacement	θ	$x = r\theta$
v	Velocity	ω	$v = r\omega$
a_{tan}	Acceleration	α	$a_{tan} = r\alpha$

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Angular Quantities (cont'd)

The frequency is the number of complete revolutions per second:

$$f = \frac{\omega}{2\pi}$$

Frequencies are measured in hertz.

$$1 \text{ Hz} = 1 \text{ s}^{-1}$$

The period is the time one revolution takes:

$$T = \frac{1}{f}$$

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Constant Angular Acceleration

The equations of motion for constant angular acceleration are the same as those for linear motion, with the substitution of the angular quantities for the linear ones.

Angular	Linear
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$
$\overline{\omega} = \frac{\omega + \omega_0}{2}$	$\overline{v} = \frac{v + v_0}{2}$

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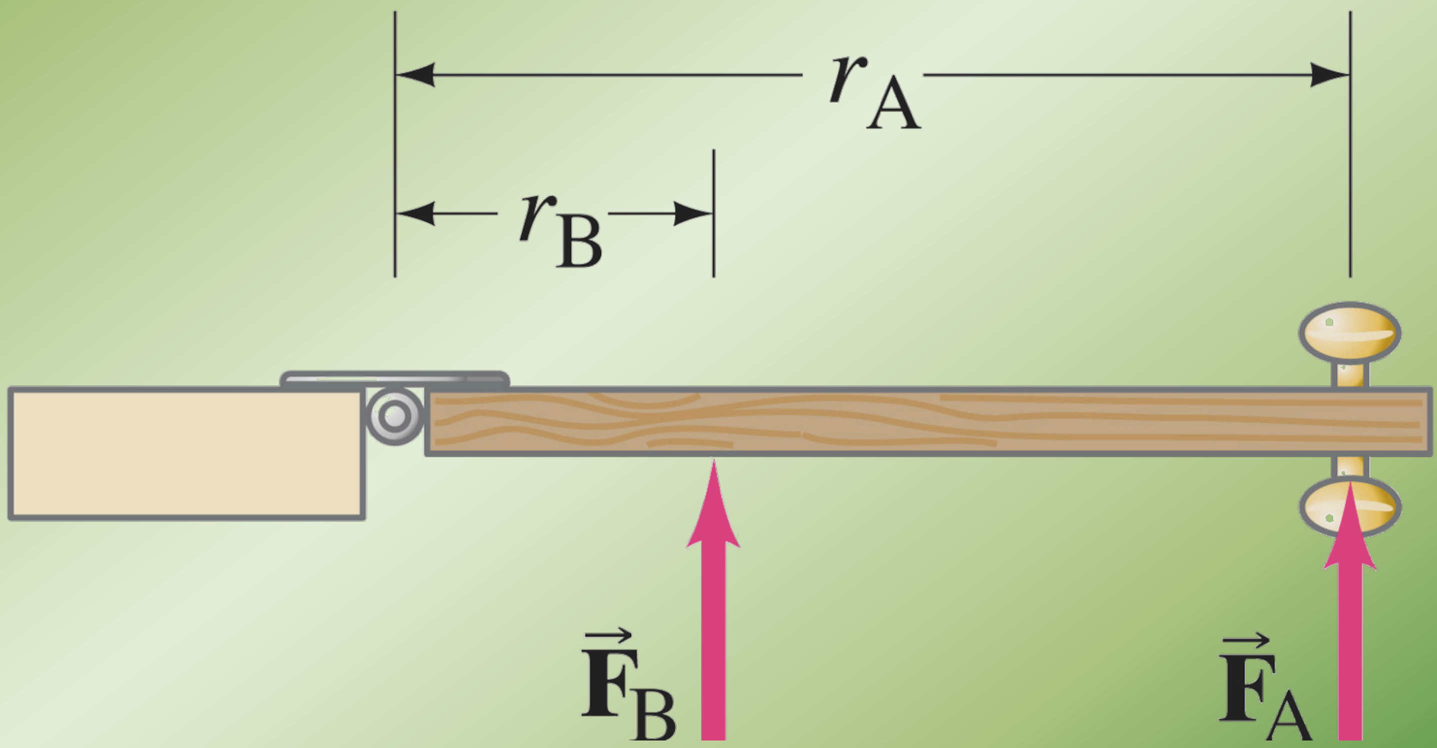
Torque

To make an object start rotating, a force is needed
The position and direction of the force matter as well.

The perpendicular distance from the axis of rotation to the line along which the force acts is called the lever arm.



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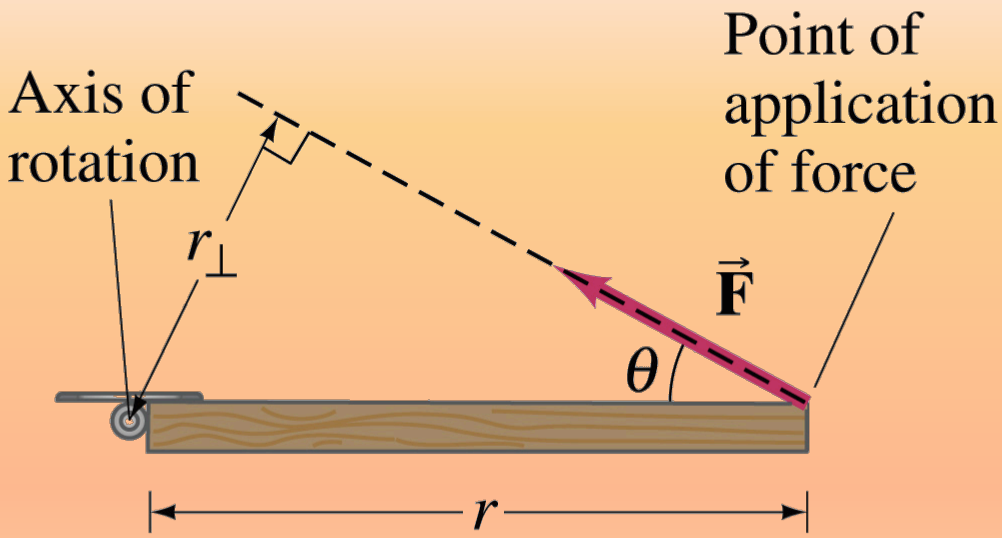
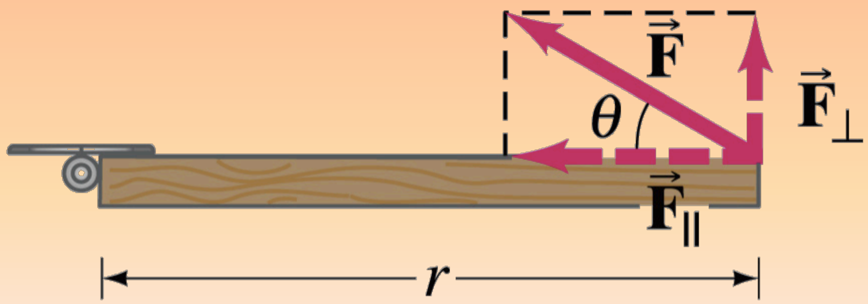
Torque (cont'd)

The torque is defined as:

$$\tau = r_{\perp} F$$





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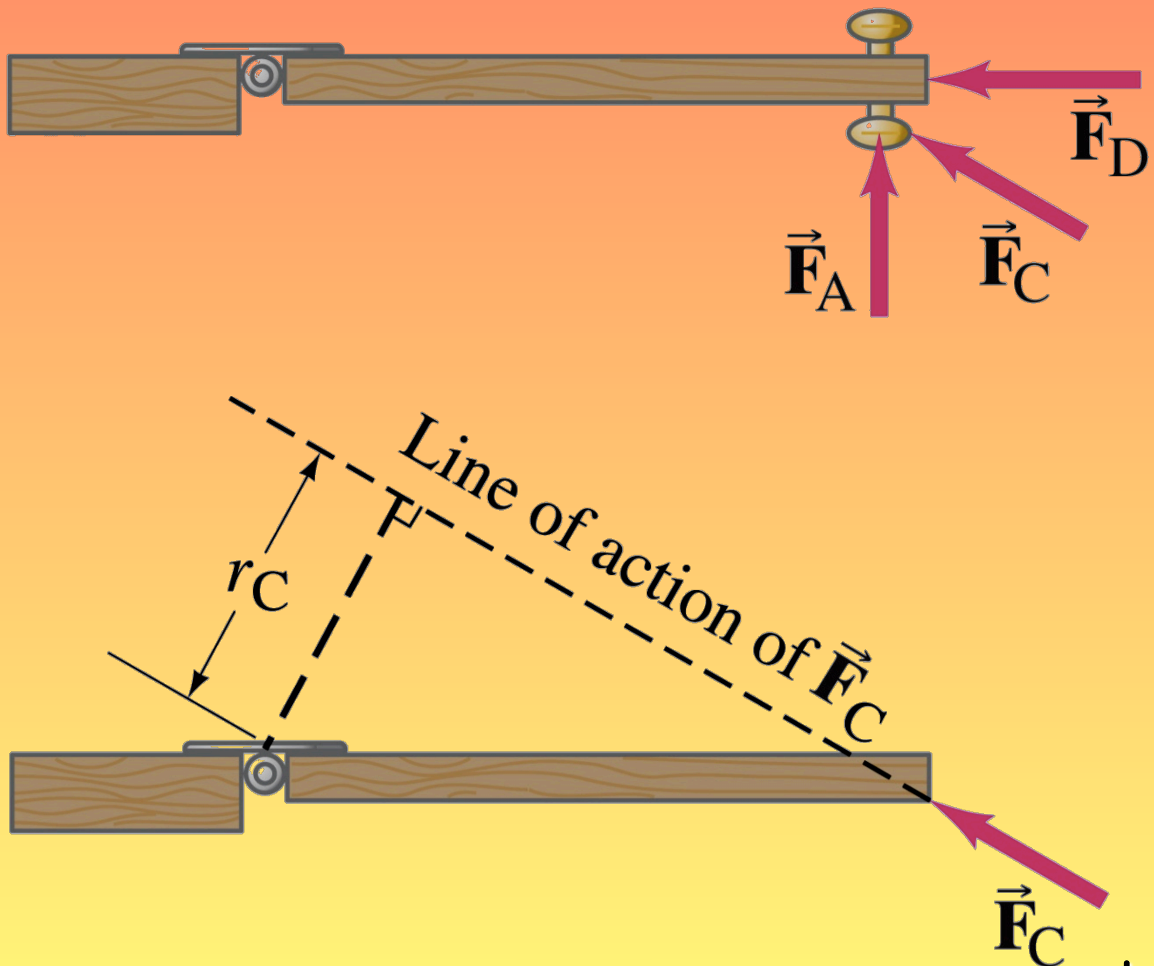
Torque (cont'd)

The lever arm for F_A is the distance from the knob to the hinge; the lever arm for F_D is zero; and the lever arm for F_C is as shown.



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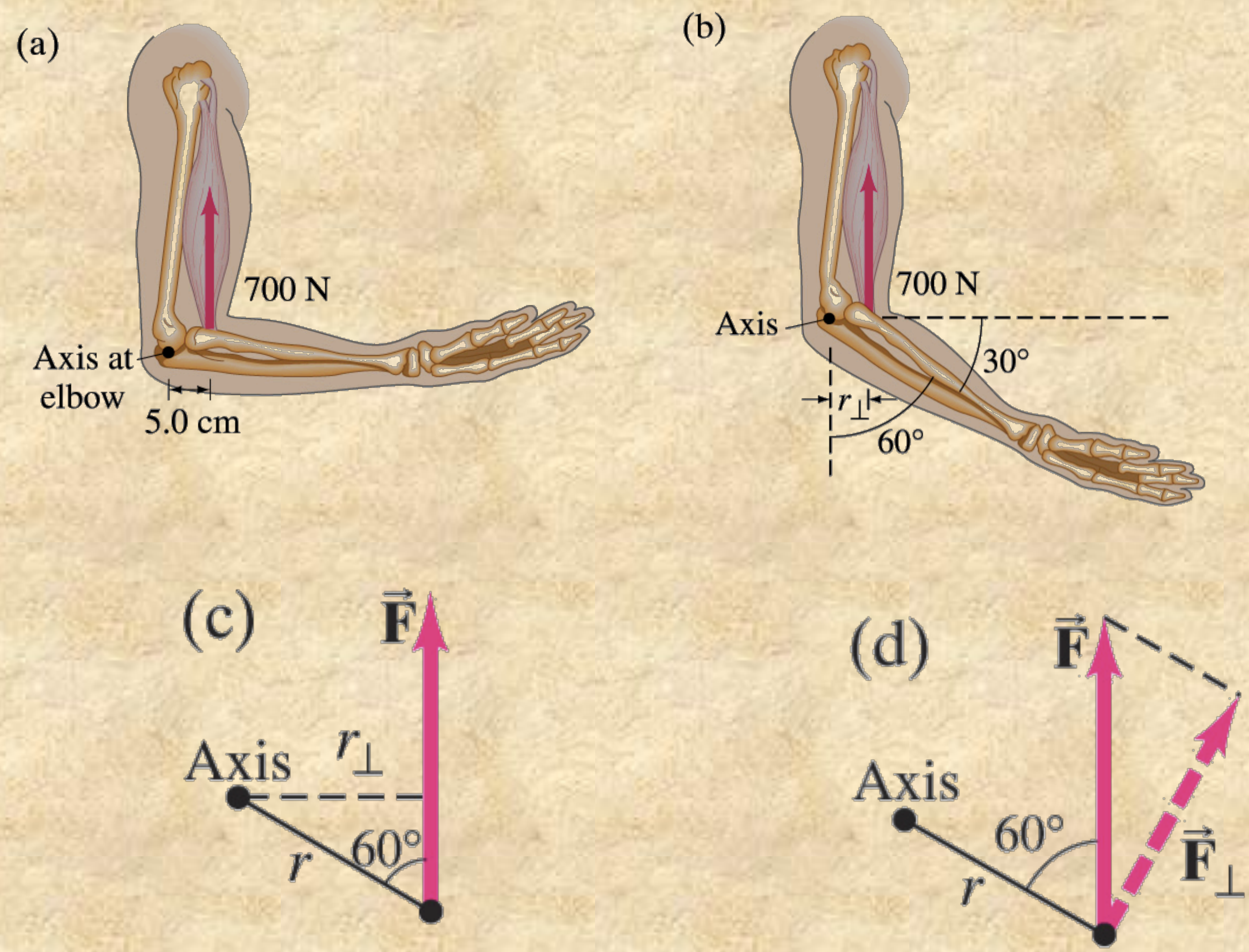
Biceps torque

The biceps muscle exerts a vertical force on the lower arm bent as shown in the figures.

For each case, calculate the torque about the axis of rotation through the elbow joint, assuming the muscle is attached 5 cm from the elbow.



τ_1 = 35 N m

τ_2 = 30 N r

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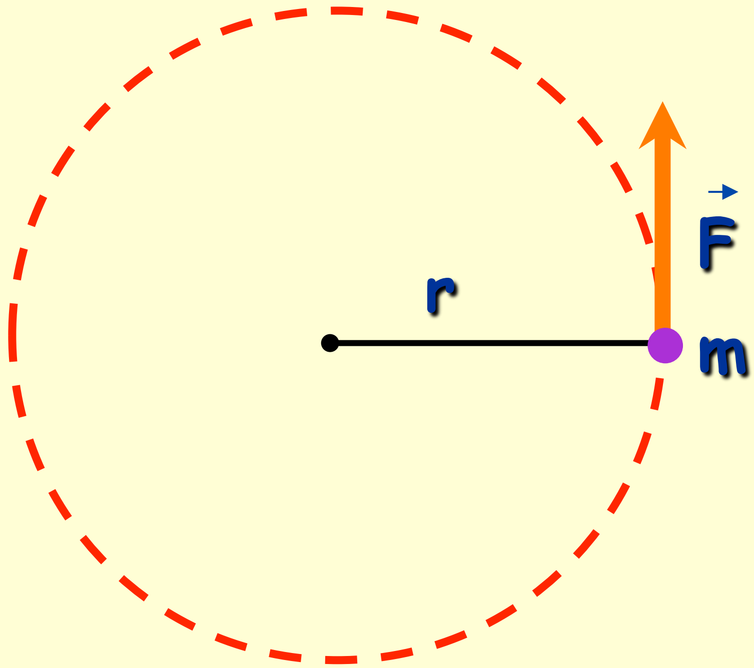
Rotational Dynamics; Torque and Rotational Inertia

Knowing that $F = ma$, we see that $\tau = mr^2\alpha$

This is for a single point mass.

What about an extended object?



As the angular acceleration is the same for the whole object, we can write:

$$\sum \tau_{i,\,net} = \left(\sum m_i r_i^2\right)\alpha$$

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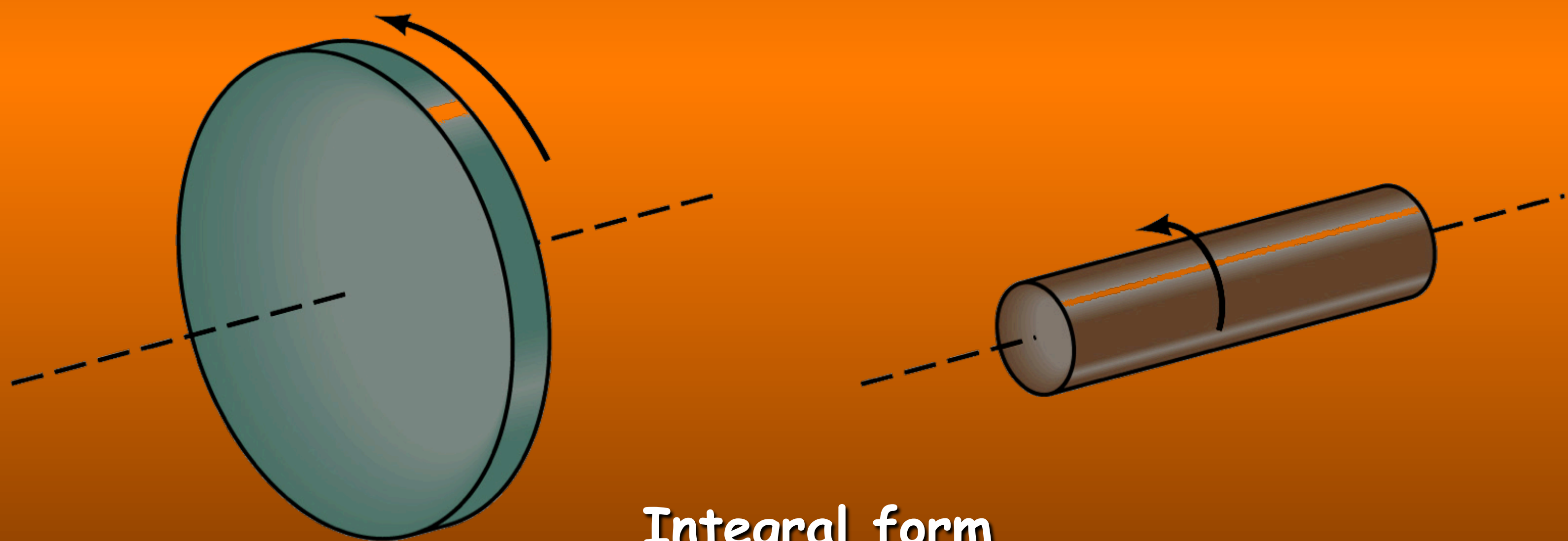
Rotational Dynamics; Torque and Rotational Inertia (cont'd)

The quantity $I = \sum m_i r_i^2$ is called the rotational inertia of an object.

The distribution of mass matters here – these two objects have the same mass, but the one on the left has a greater rotational inertia, as so much of its mass is far from the axis of rotation.



Integral form

$$I = \int r^2 dm$$

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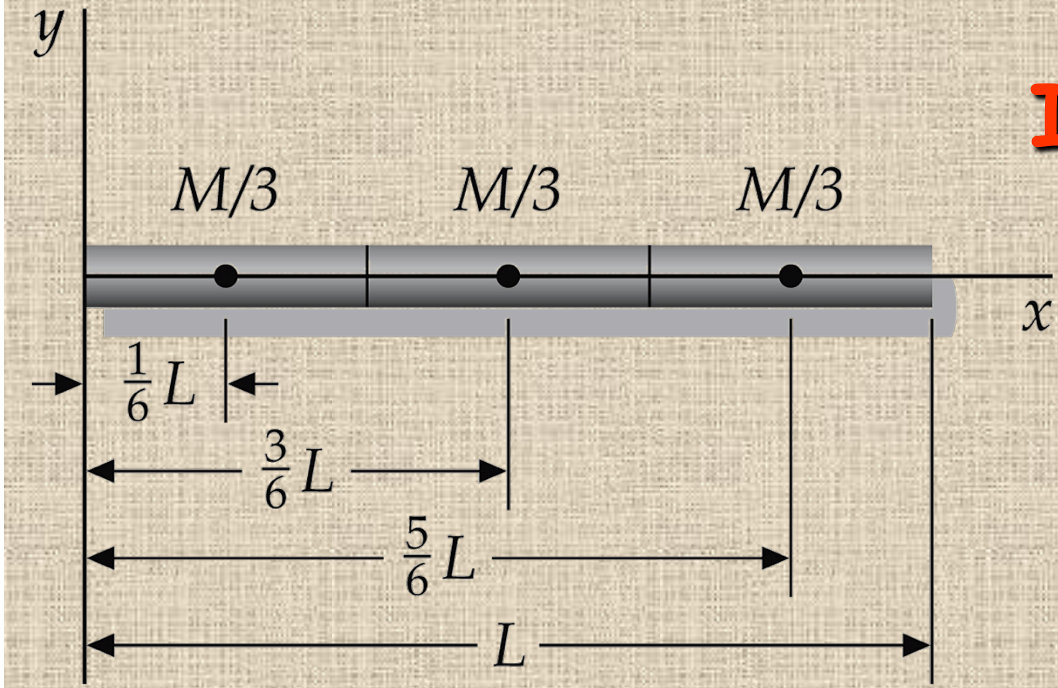
Estimating the moment of inertia

Estimate the moment of inertia of a thin uniform rod of length L and mass M about an axis perpendicular to the rod and through one end.

Execute this estimation by modelling the rod as three point masses, each point mass representing 1/3 of the rod.



$I = 35\ M\ L^2/108$

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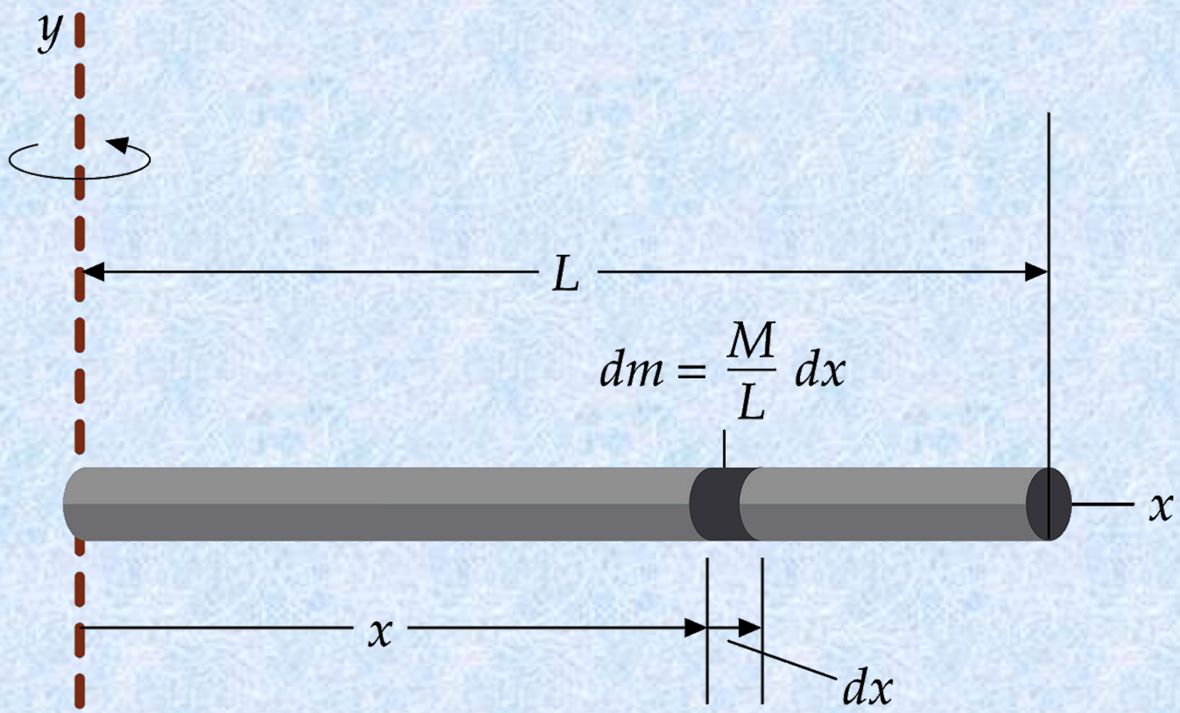
Moment of inertia of a thin uniform road

Find the moment of inertia of a thin uniform rod of length L and mass M about an axis perpendicular to the rod and through one end.



$$I = \frac{ML^2}{3}$$

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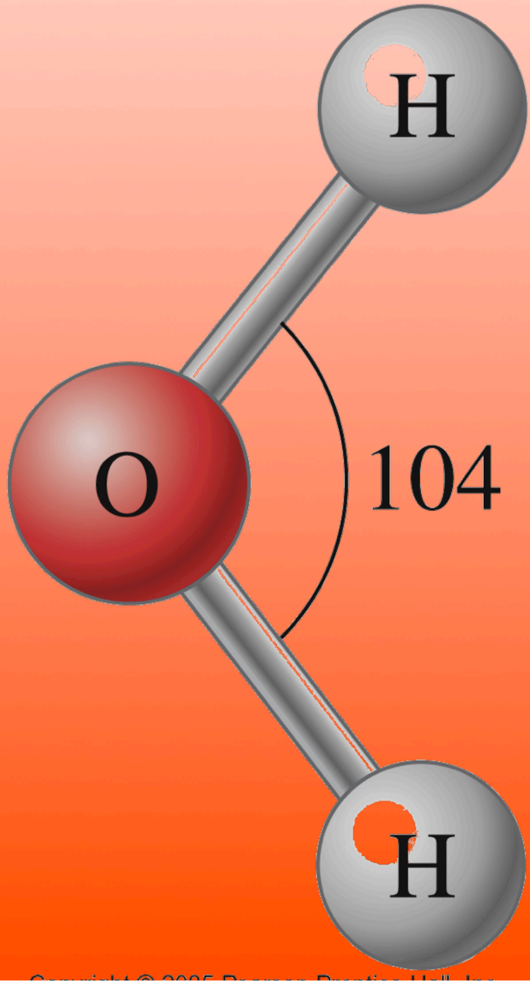
Calculate the moment of inertia for H_2O molecule about an axis passing through the center of the oxygen atom

(a) perpendicular to the plane of the molecule

(b) in the plane of the molecule, bisecting the H-O-H bonds



$$I_{perp} = 3.1 \times 10^{-45} \text{ kg m}^2$$

$$I_{plane} = 1.9 \times 10^{-45} \text{ kg m}^2$$

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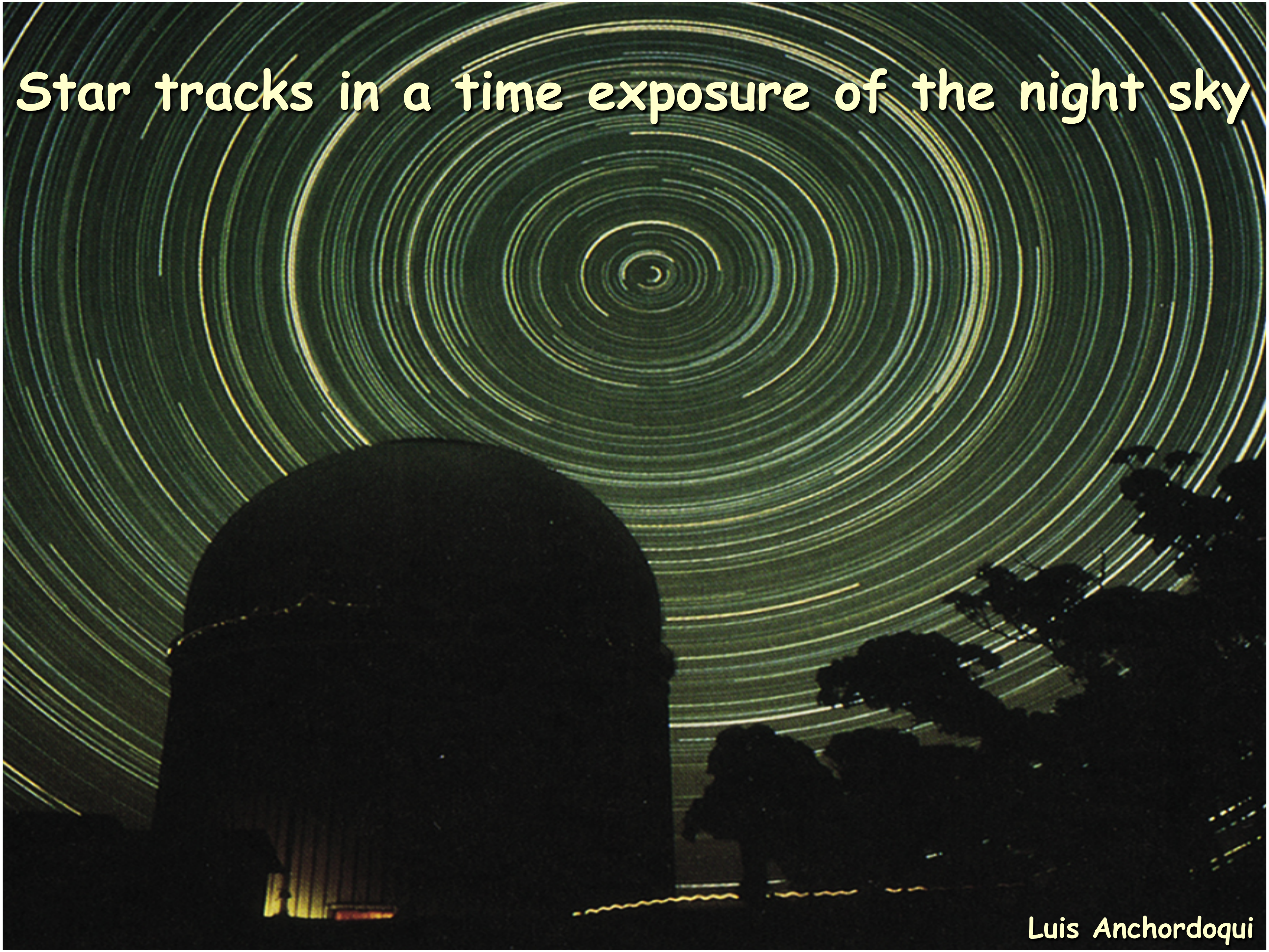
Star tracks in a time exposure of the night sky
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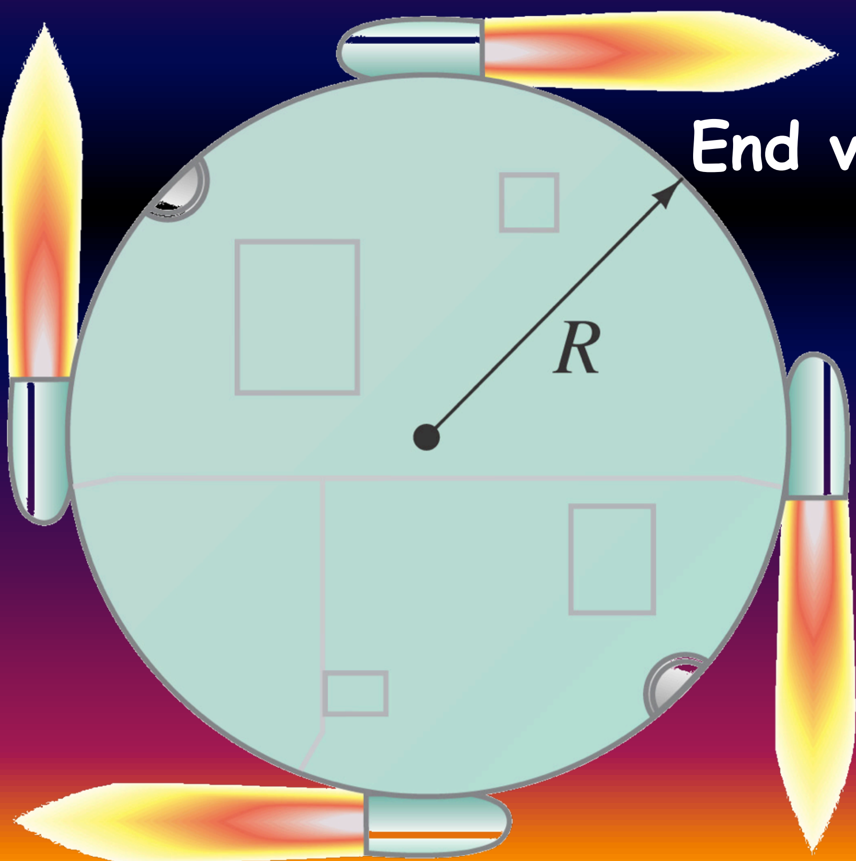
To get a flat, uniform cylindrical satellite spinning at the correct rate, engineers fire four tangential rockets as shown in the figure

If the satellite has a mass of 3600 kg and a radius of 4 m, what is the required steady force of each rocket if the satellite is to reach 32 rpm in 5 min?



End view of cylindrical satellite

$F = 20\ N$

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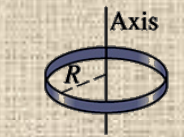
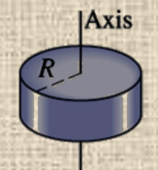
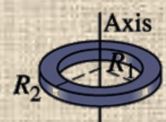
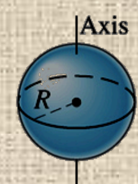
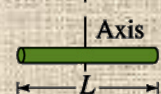
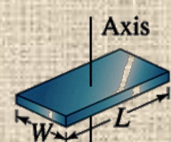
Rotational Dynamics; Torque and Rotational Inertia (cont'd)

	Object	Location of axis		Moment of inertia
(a)	**Thin hoop,** radius R	Through center		MR^2
(b)	**Thin hoop,** radius R width W	Through central diameter		$\frac{1}{2}MR^2 + \frac{1}{12}MW^2$
(c)	**Solid cylinder,** radius R	Through center		$\frac{1}{2}MR^2$
(d)	**Hollow cylinder,** inner radius R_1 outer radius R_2	Through center		$\frac{1}{2}M(R_1^2 + R_2^2)$
(e)	**Uniform sphere,** radius R	Through center		$\frac{2}{5}MR^2$
(f)	**Long uniform rod,** length L	Through center		$\frac{1}{12}ML^2$
(g)	**Long uniform rod,** length L	Through end		$\frac{1}{3}ML^2$
(h)	**Rectangular thin plate,** length L, width W	Through center		$\frac{1}{12}M(L^2 + W^2)$

The rotational inertia of an object depends not only on its mass distribution but also the location of the axis of rotation – compare (f) and (g), for example.

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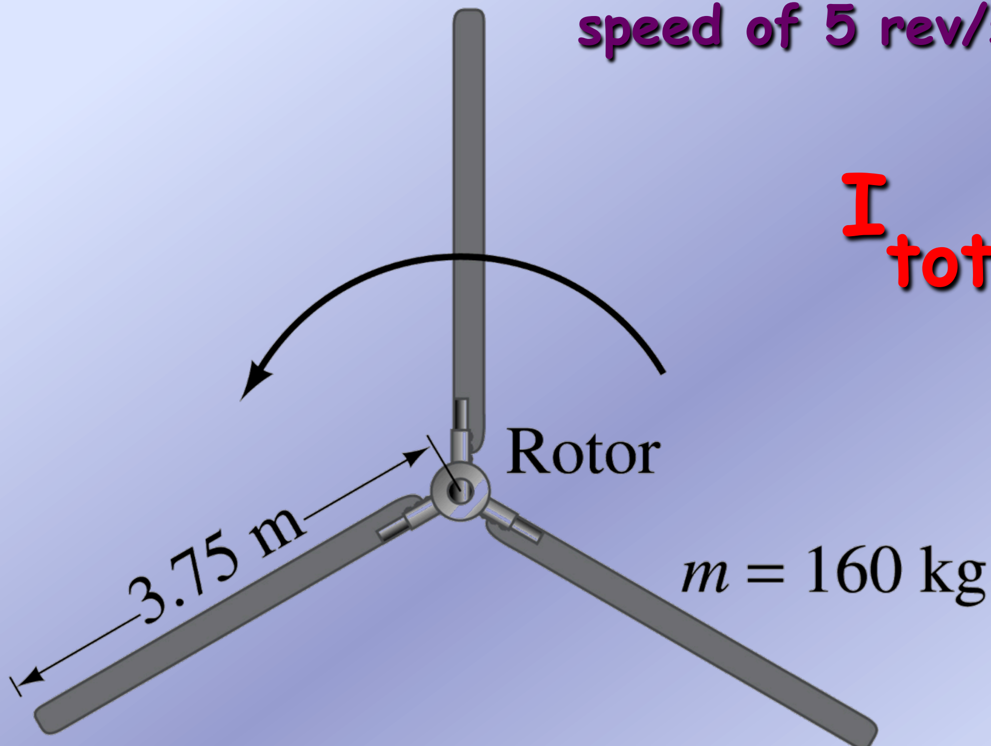
A helicopter rotor blade can be considered a long thin rod.

(a) If each of the three rotor helicopter blades is 3.75 m long and has a mass of 160 kg, calculate the moment of inertia of the three rotor blades about the axis of rotation.

(b) How much torque must the rotor apply to bring the blades up to a speed of 5 rev/s in 80 s?



$$I_{total} = 2250 \text{ kg m}^2$$

$$\tau = 8.8 \times 10^3 \text{ N m}$$

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Rotational Kinetic Energy

The kinetic energy of a rotating object is given by

$$KE = \Sigma \left(\tfrac{1}{2} m_i v_i^2\right)$$

By substituting the rotational quantities, we find that the rotational kinetic energy can be written:

$$\text{Rotational KE} = \tfrac{1}{2} I \omega^2$$

A object that has both translational and rotational motion also has both translational and rotational kinetic energy:

$$KE = \tfrac{1}{2} M v_{CM}^2 + \tfrac{1}{2} I_{CM} \omega^2$$

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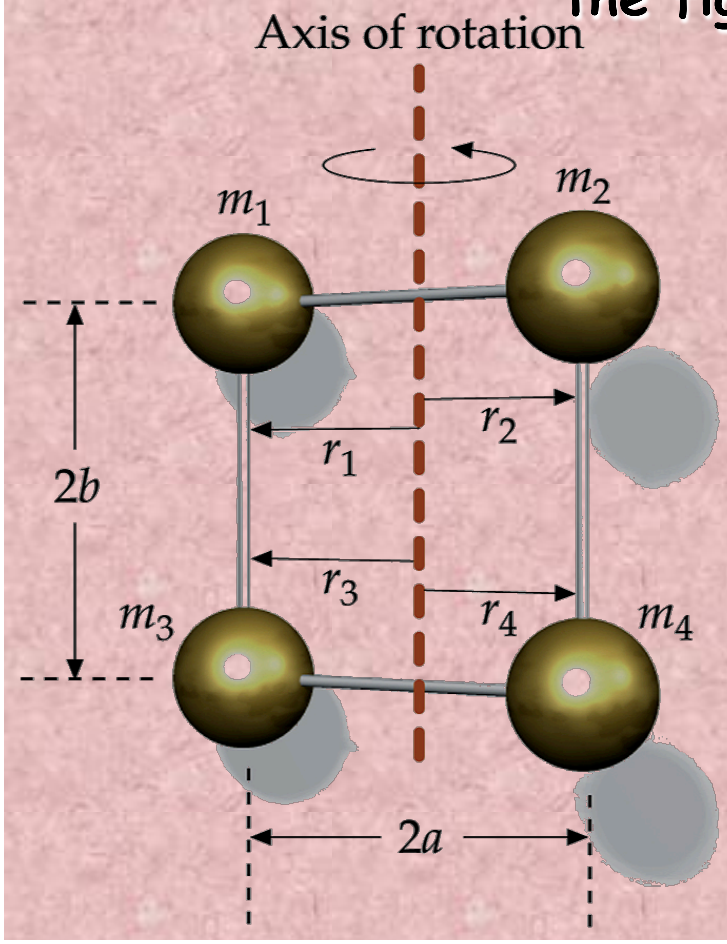
A rotating system of particles

An object consists of four point particles, each of mass m, connected by rigid mass less rods to form a rectangle of edge lengths 2a and 2b, as shown in the figure.

The system rotates with angular speed ω about an axis in the plane of the figure through the center.



Find the kinetic energy of this object.

$$K = 2\, m\, a^2\, \omega^2$$

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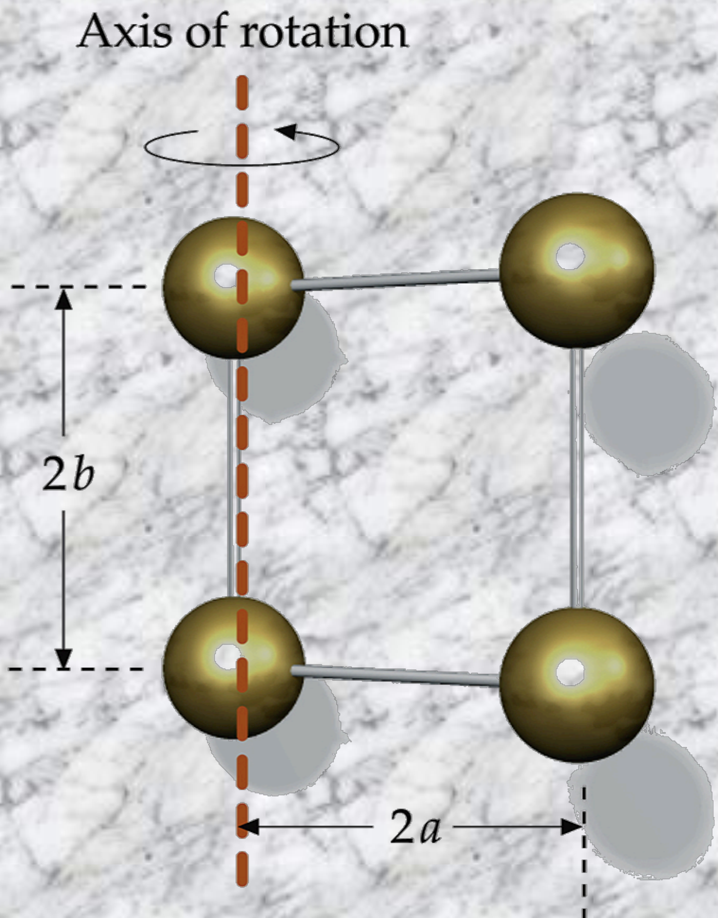
A rotating system of particles (cont'd)

Find the moment of inertia of the system for rotation about an axis parallel to the first axis but passing through two of the particles.



$$I = 8\ m\ a^2$$

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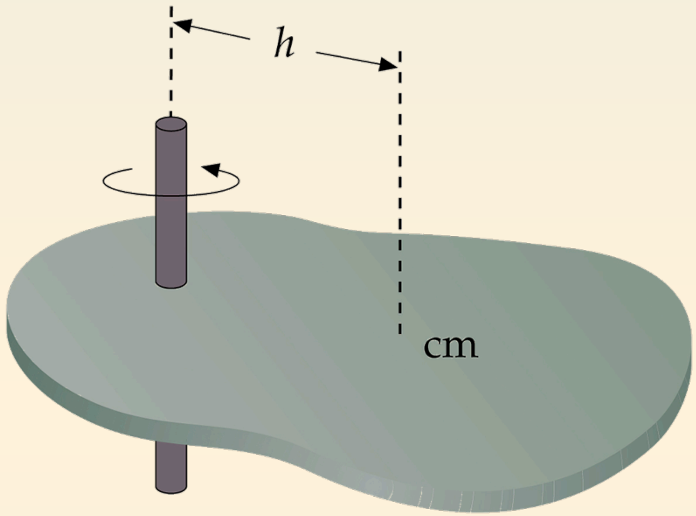
Steiner's Theorem

The parallel-axis theorem relates the moment of inertia about an axis through the center of mass to the moment of inertia about a second parallel axis



Let I be the moment of inertia and let I_{cm} be the moment of inertia about a parallel-axis through the center of mass. Let M be the total mass of the object and h the distance between the two axes. The parallel axis theorem states that

$$I = I_{cm} + Mh^2$$

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Steiner's Theorem (cont'd)

Consider an object rotating about a fixed axis that does not pass through the cm

The kinetic energy of such a system is

$$K = \tfrac{1}{2} I\omega^2$$

I: Moment of inertia about the fixed axis

The kinetic energy of a system can be written as the sum of its translational kinetic energy ($\tfrac{1}{2} mv^2_{cm}$) and the kinetic energy relative to its cm

For an object that is rotating the kinetic energy relative to its cm is

$$\tfrac{1}{2} I_{cm}\, \omega^2$$

I_{cm}: Moment of inertia about the axis through the cm

The total kinetic energy of the object is

$$K = \tfrac{1}{2} Mv^2_{cm} + \tfrac{1}{2} I_{cm}\, \omega^2$$

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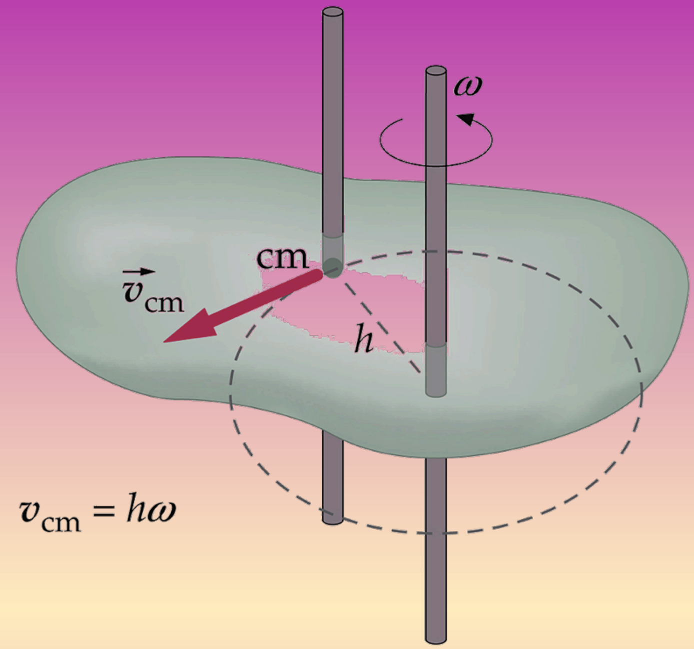
Steiner's Theorem (cont'd)



The cm moves along a circular path of radius h ➔ $v_{cm} = h\omega$

Substituting

$$\frac{1}{2} I\omega^2 = \frac{1}{2} Mh^2\omega^2 + \frac{1}{2} I_{cm}\,\omega^2$$

Multiplying through this equation by $2/\omega^2$ leads to

$$I = Mh^2 + I_{cm}$$

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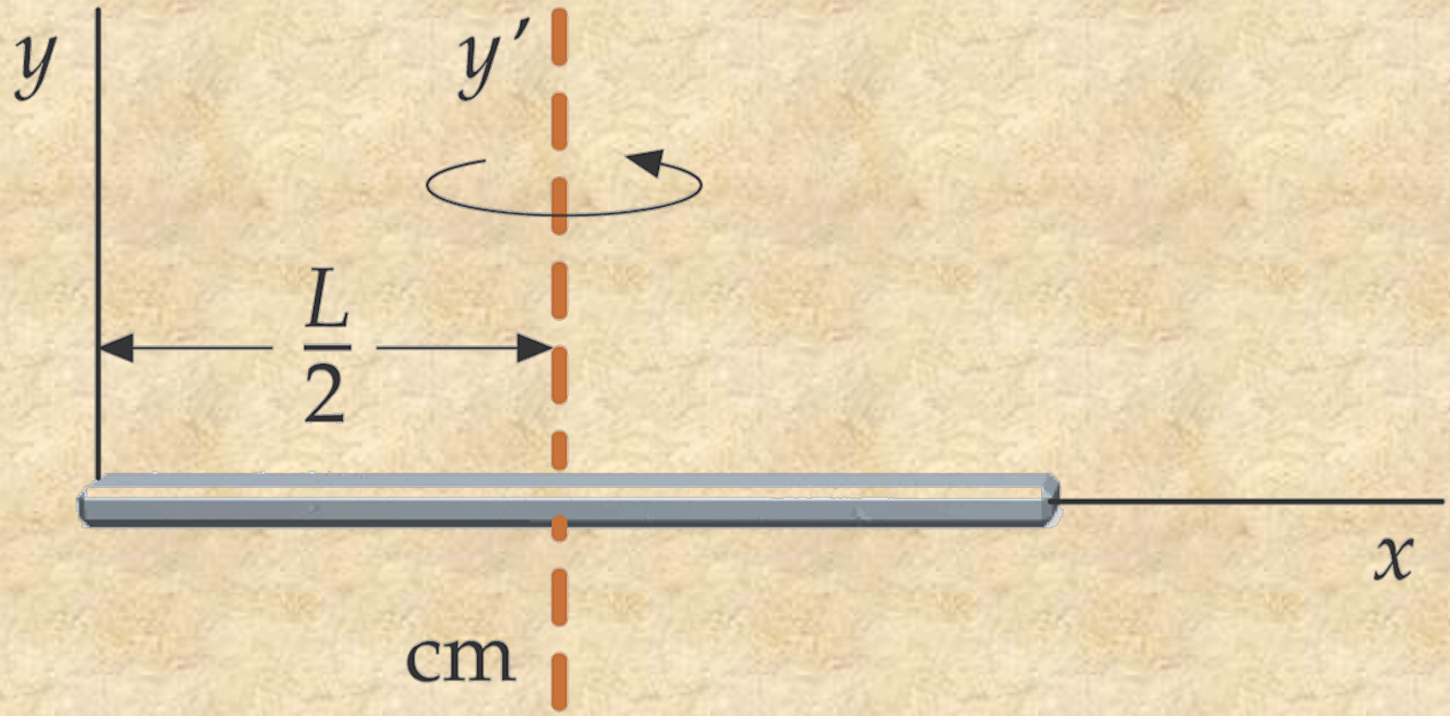
A thin uniform rod of mass M and length L on the x axis has one end at the origin. Using the parallel-axis theorem, find the moment of inertia about the y' axis, which is parallel to the y axis, and through the center of the rod.



$$I_{cm} = \frac{ML^2}{12}$$

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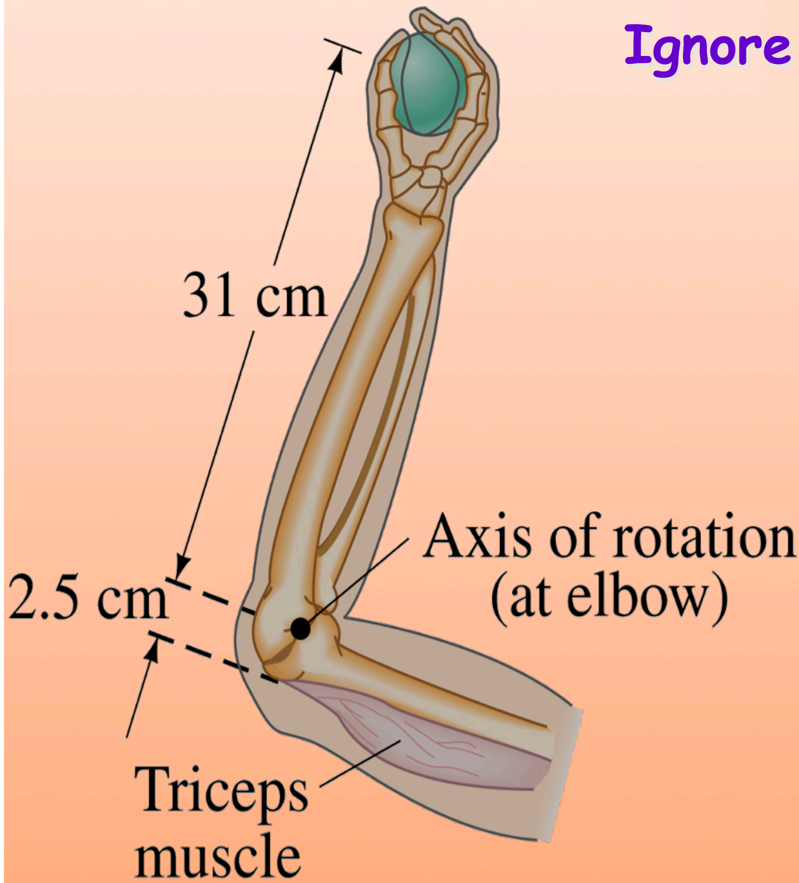
The forearm shown in the figure accelerates a 3.6 kg ball at 7 m/s^2 by means of the triceps muscle.

Calculate

(a) the torque

(b) the force that must be exerted by the triceps muscle.

Ignore the mass of the arm.



$$\tau = 7.8 \text{ N m}$$

$$F = 310 \text{ N}$$

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Nonslip conditions

For the string not to slip on a pulley wheel the parts of the string and the wheel that are in contact with each other must share the same tangential velocity

$$v_t = R\omega$$

$R\omega$ → Tangential velocity of the perimeter of the pulley wheel

v_t: Tangential velocity of the string

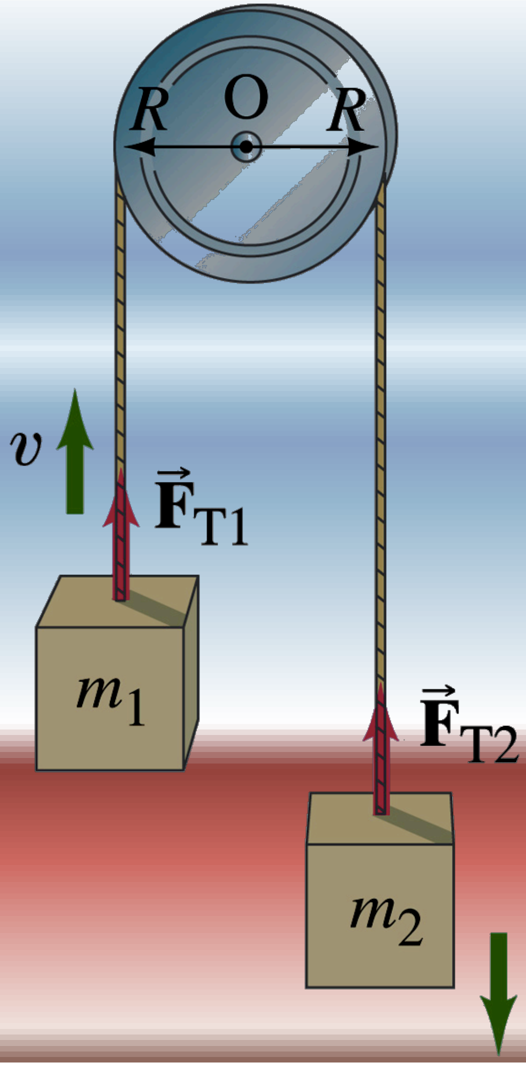
Differentiating both sides the nonslip condition with respect to time leads to

$$a_t = R\alpha$$

α → Angular acceleration of the wheel

a_t: Tangential acceleration of the string

An Atwood's machine consists of two masses, m_1 and m_2 which are connected by a mass less inelastic cord that passes over a pulley. If the pulley has radius R and moment of inertia I about its axle, determine the acceleration of the masses m_1 and m_2 .



$$a = \frac{(m_2 - m_1)}{(m_1 + m_2 + I/r^2)} g$$

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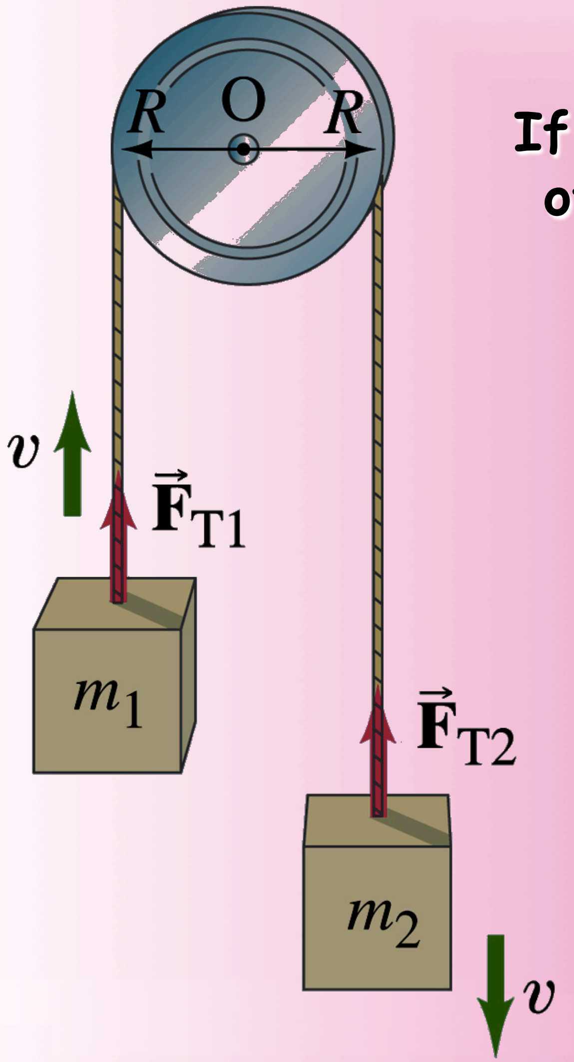
Two masses m_1 = 18 kg and m_2 = 26.5 kg are connected by a rope that hangs over a pulley.

The pulley is a uniform cylinder of radius 0.26 m and mass 7.5 kg. Initially, m_1 is on the ground and m_2 rests 3 m above the ground.



If the system is now released, use conservation of energy to determine the speed of m_2 just before it strikes the ground.

(Assume the pulley is frictionless)

v_f = 3.22 m/s

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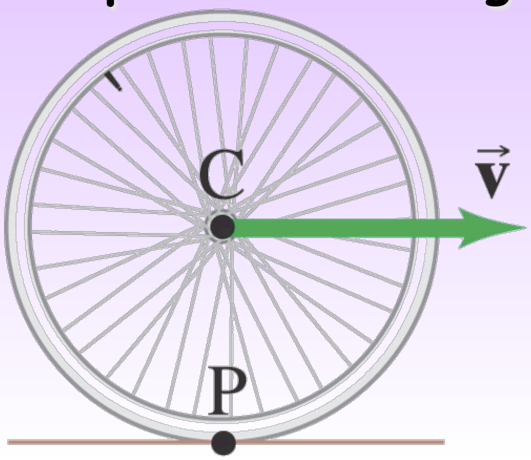
Rolling Motion (Without Slipping)

A wheel is rolling without slipping

The point P touching the ground is instantaneously at rest

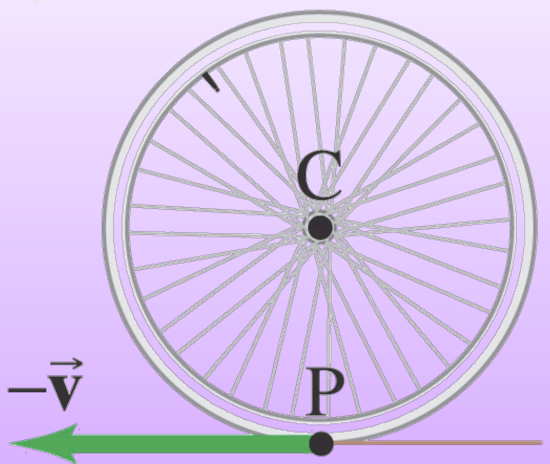
and the center moves with velocity v

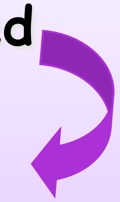


The same wheel is seen from a reference frame where C is at rest

Now point P is moving with velocity –v

The linear speed of the wheel is related to its angular speed





$$v = r\omega$$

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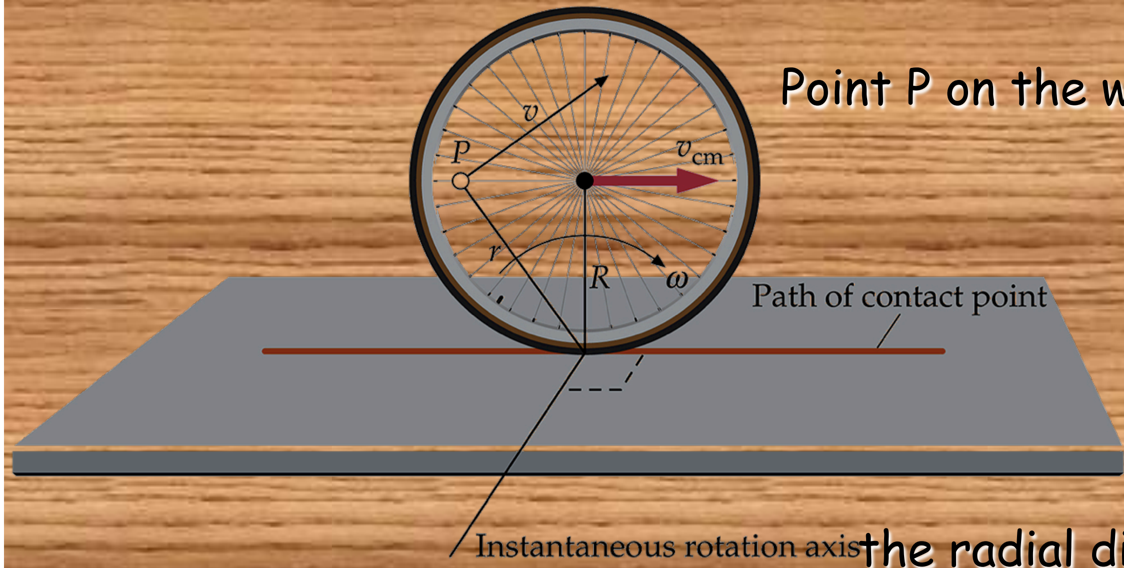
Rolling without slipping

Consider a wheel of radius R rolling without slipping along a flat surface



Point P on the wheel moves as shown with speed

$$v = r\omega$$

the radial distance from the rotation axis to P

The cm of the wheel moves with speed $v_{cm} = R\omega$

For a point on the very top of the wheel r = 2R

so the top of the wheel is moving at twice the speed of the center of the wheel

Differentiating on both sides $a_{cm} = R\alpha$

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A skateboarder accelerates from rest at a rate of 1 m/s². How fast will a point on the rim of the wheel (diameter = 75 mm) at the top be moving after 3 s.

$v_{top} = 6 m/s$

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Rolling without slipping (cont'd)

A wheel of radius R is rolling without slipping along a straight path.

As the wheel rotates through an angle Ø the point of contact between the wheel and the surface moves a distance s that is related to Ø by

$$s = R\,Ø$$

If the wheel is rolling on a flat surface the wheel's CM remains directly over the point of contact so it also moves through a distance RØ

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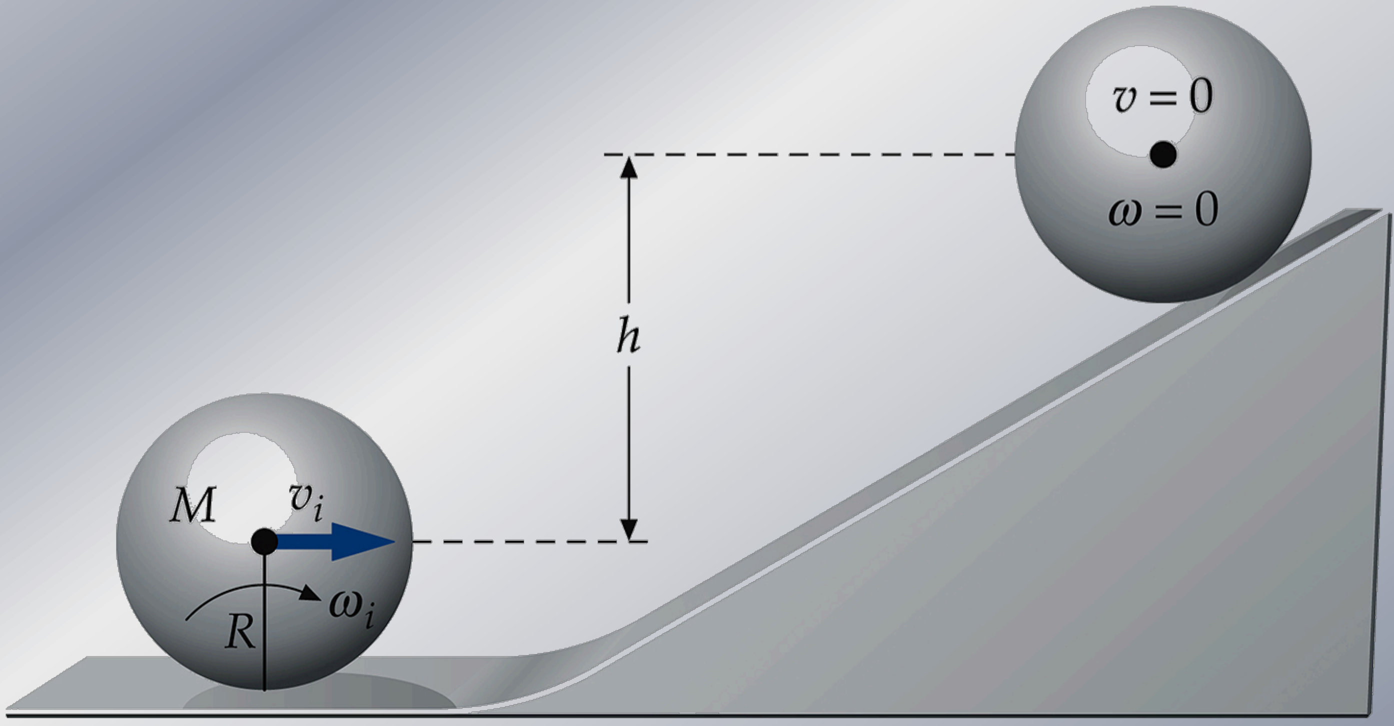
A bowling ball that has 11 cm radius and 7.2 kg mass is rolling without slipping at 2 m/s on a horizontal ball return.
It continues to roll without slipping up a hill to a height h before momentarily coming to rest and then rolling back down the hill.
Model the ball as a uniform sphere and find h.



h = 29 cm

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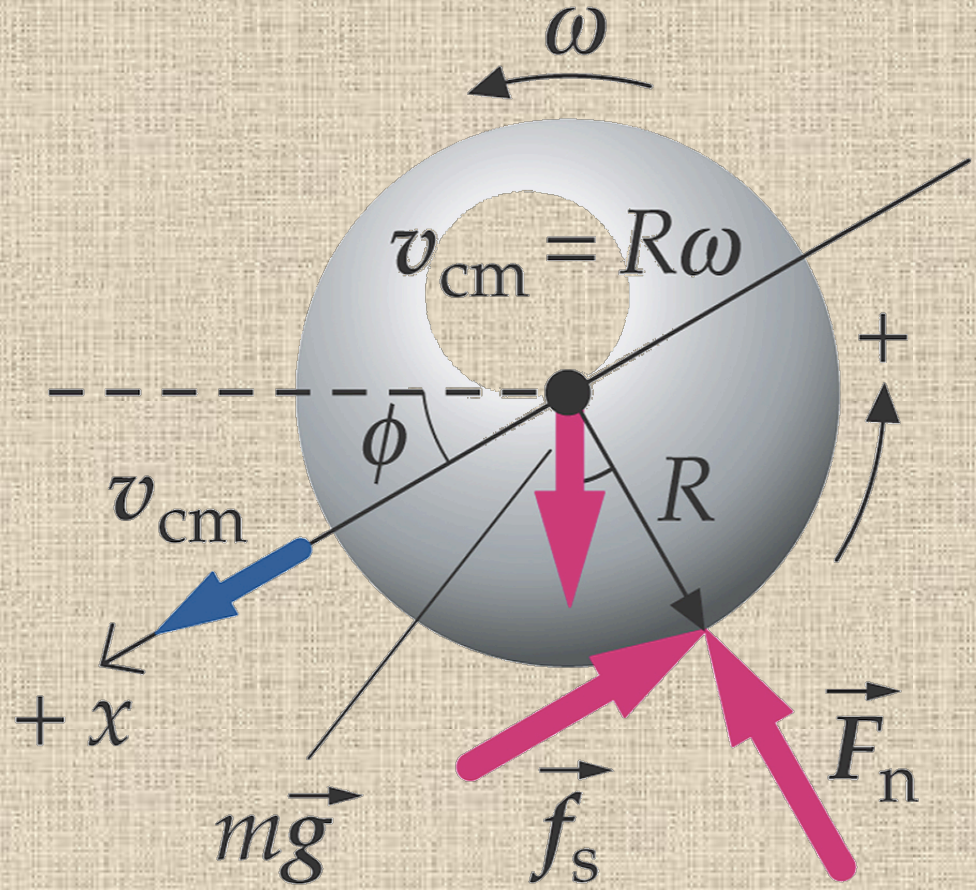
A uniform solid ball of mass m and radius R rolls without slipping down a plane inclined at an angle φ above the horizontal.

Find the frictional force and the acceleration of the center of the mass.



$a_{cm} = (5/7)\, g \sin\varphi$ $\quad f_s = (2/7)\, m\, g \sin\varphi$

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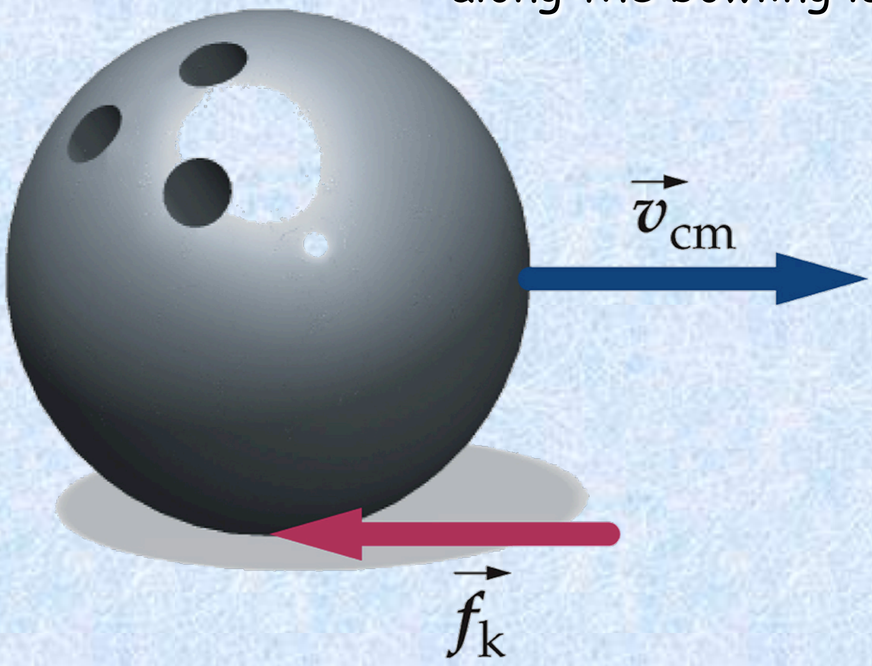
Rolling with slipping

When an object slips (skids) as it rolls the nonslip condition $v_{cm} = R\,\omega$ does not hold

Suppose a bowler releases a ball with no initial rotation ($\omega_0 = 0$) as the ball skids

along the bowling lane $v_{cm} > R\omega$



The kinetic frictional force will both reduce its linear speed v_{cm}
and increase its angular speed ω until the nonslip condition $v_{cm} = R\omega$ is reached,
after wich the balls rolls without slipping

Luis Anchordoqui

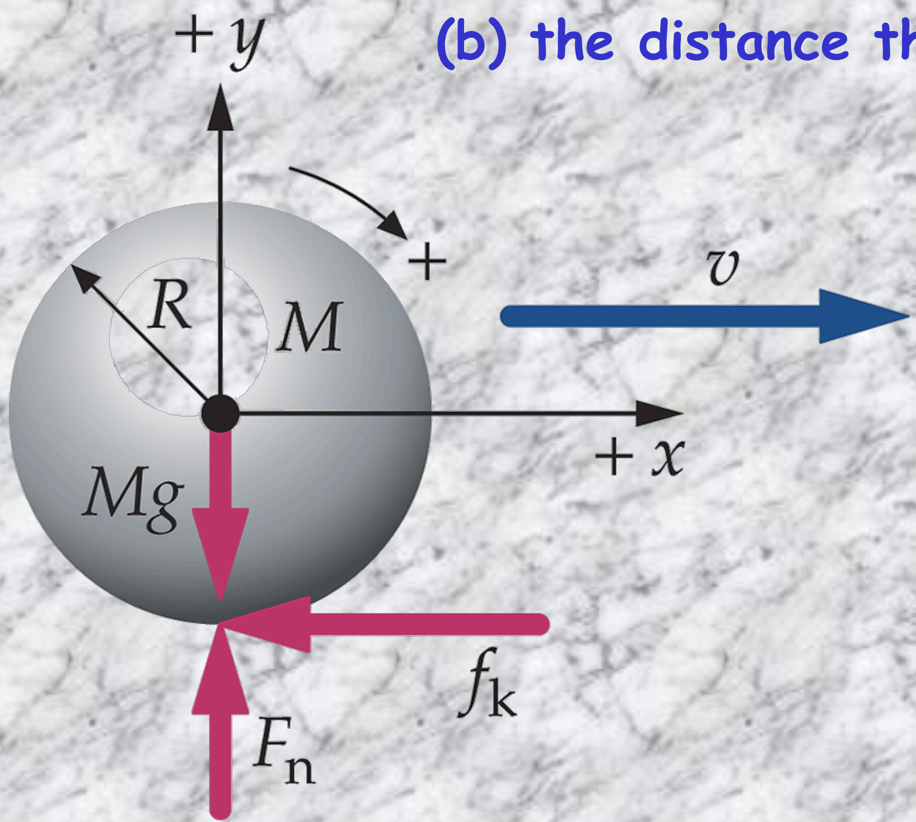
A bowling ball of mass M and radius R is released at floor level so that at release it is moving horizontally with speed $v_0 = 5$ m/s and is not rotating.

The coefficient of kinetic friction between the ball and the floor is $\mu_k = 0.08$.

Find

(a) the time the ball slides

(b) the distance the ball skids.



$t = 1.8$ s

$\Delta x = 7.8$ m

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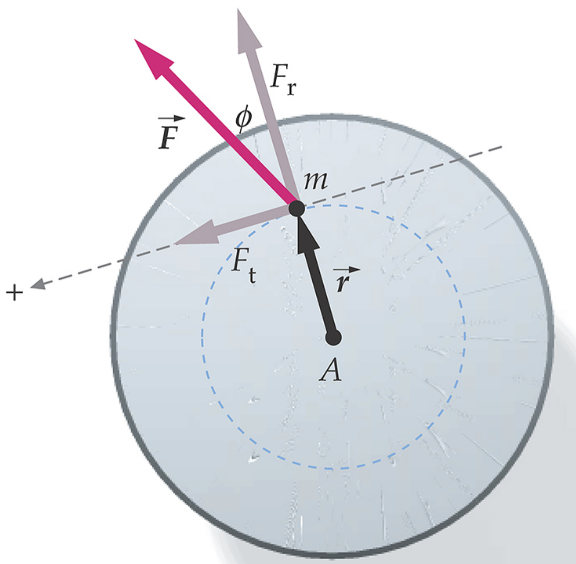
Power

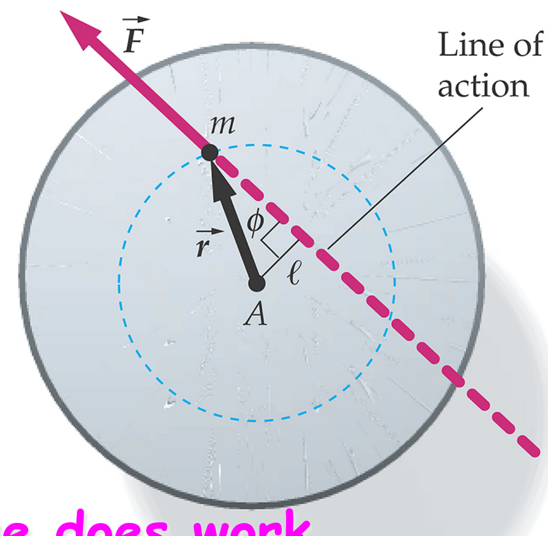
Consider a force $\vec{F}$ acting on a rotating object

As the object rotates through an angle $d\theta$ the point of application of the force moves a distance $ds = r d\theta$ and the force does work

$$dW = F_t \, ds = F_t \, r \, d\theta = \tau \, d\theta$$

$$\tau = F \, r \sin \phi = F l$$





The rate at which the torque does work

$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt}$$

The power input of the torque reads

$$P = \tau \, \omega$$

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