Special Relativity



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IV: Relativistic Kinematics



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Transformation of velocities

Let a particle move (in our reference frame S)

Its velocity is $u = \frac{dx}{dt}$

a distance dx in time dt

In a reference frame S-bar which moves with v,
it has only moved a distance ☛ $d\bar{x} = \gamma(dx - vdt)$
this distance has been moved in a time $\bar{t}$: $d\bar{t} = \gamma\left(dt - \frac{v}{c^2}dx\right)$

it's velocity in the moving reference frame is therefore

$$\bar{u}_x = \frac{d\bar{x}}{d\bar{t}} = \frac{\gamma(dx - vdt)}{\gamma(dt - dxv/c^2)} = \frac{dx - vdt}{dt - dxv/c^2} = \frac{dx/dt - v}{1 - (dx/dt)v/c^2} = \frac{u_x - v}{1 - u_x v/c^2}$$

This is it: the velocity u-bar is not simply smaller by v but must be corrected by the denominator which admittedly ~1 when both the particle or the moving frame is much slower than the light velocity

For the components of velocity transverse to the motion of S'

$$\bar{u}_y = \frac{d\bar{y}}{d\bar{t}} = \frac{dy}{\gamma(dt - dxv/c^2)} = \frac{u_y}{\gamma(1 - u_x v/c^2)}$$

$$\bar{u}_z = \frac{d\bar{z}}{d\bar{z}} = \frac{dz}{\gamma(dt - dxv/c^2)} = \frac{u_z}{\gamma(1 - u_x v/c^2)}$$

In invariant terms (i.e. independent of the coordinate system)

$u_\parallel$ component of velocity parallel to v

Take

$u_\perp$ component of velocity perpendicular to v

then

$$\bar{u}_\parallel = \frac{u_\parallel - v}{1 - u_\parallel v/c^2} \qquad \bar{u}_\perp = \frac{u_\perp}{\gamma(1 - u_\parallel\, v/c^2)}$$

Example: A bullet is shot from a spaceship that speeds with 1/2c relative to our reference frame. The velocity of that bullet is 3/4c in the space-ship's reference frame in the direction of motion

What is the velocity of that bullet in our reference frame?

We define the space-ship's reference frame to be the resting frame then $\bar{u}$ is the velocity in our reference frame.

We move away from the space-ship with v=-1/2c and the bullet is shot with u=+3/4c

$$\bar{u} = \frac{u - v}{1 - \frac{uv}{c^2}} = \frac{3/4c + 1/2c}{1 + \frac{3/4c \cdot 1/2c}{c^2}} = \frac{5/4c}{1 + 3/8} = \frac{5/4}{11/8} c = 0.91c$$

so, the bullet has only 90% of the light velocity in our reference frame (with Galileo's velocity addition rule, it would have (3/4 + 1/2)c = 5/4 c)

Now, the space-ship emits a light pulse

$$\bar{u} = \frac{u - v}{1 - \frac{uv}{c^2}} = \frac{c + 1/2c}{1 + \frac{c \cdot 1/2c}{c^2}} = \frac{3/2c}{1 + 1/2} = \frac{3/2}{3/2} c = c$$

also in our reference frame the light pulse has velocity c !!!

Drag effect

Flowing air drags sound along with it ➤ To what extent a flowing transparent liquid will drag light along with it?

On the basis of an ether theory, it would be conceivable that there is no drag at all, since light is a disturbance of the ether and not of the liquid

yet experiments indicated that there was a drag: the liquid seemed to force the ether along with it but only partially

If the speed of light in the liquid at rest is u', and the liquid is set to move with velocity v, then the speed of light relative to the outside was found to be of the form

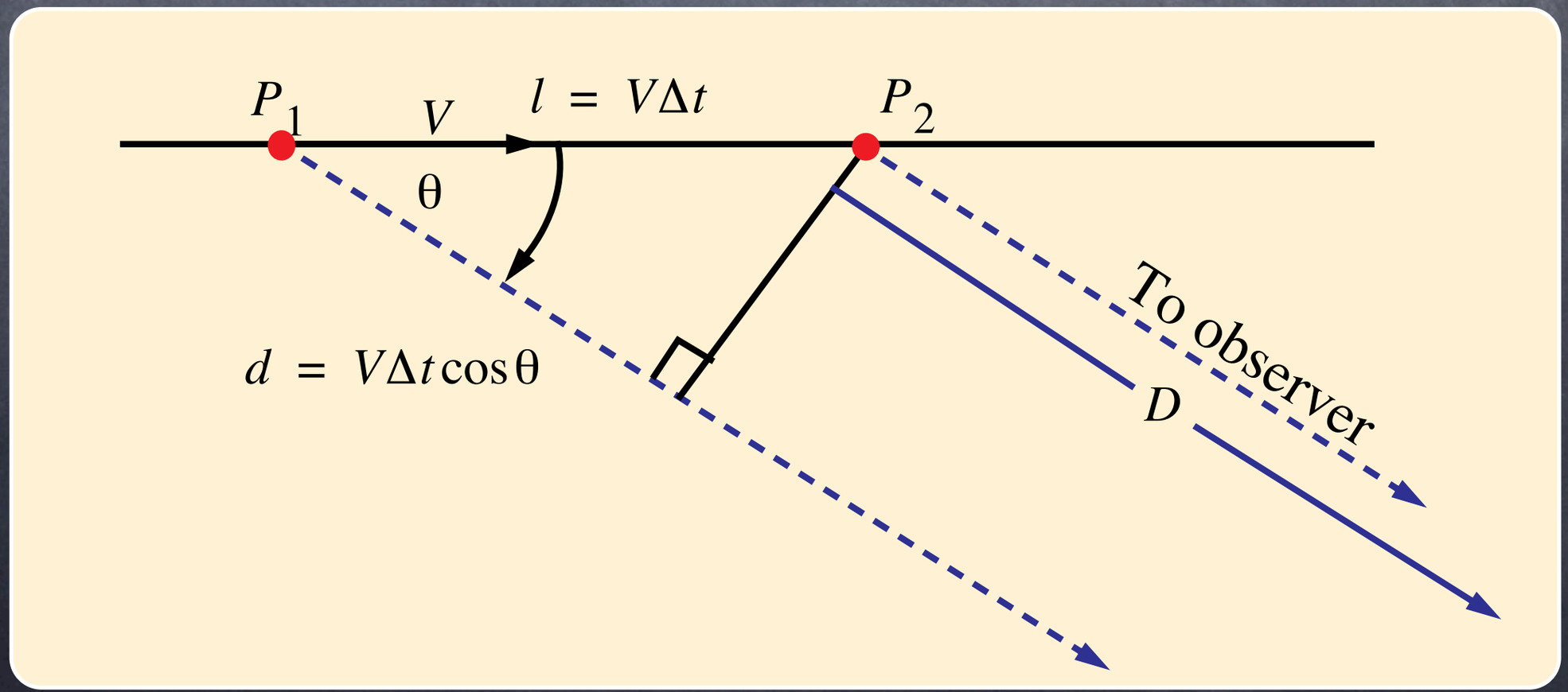
$$u = u' + kv \qquad k = 1 - 1/n^2 \qquad n = c/u'$$

Neglecting terms $\mathcal{O}(v^2/c^2)$ the velocity addition formula yields

$$u = \frac{u' + v}{1 + u'v/c^2} \approx (u' + v)\left(1 - \frac{u'v}{c^2}\right) \approx u' + v\left(1 - \frac{u'^2}{c^2}\right) = u' + kv$$

Doppler effect

- The Doppler effect is very important when describing the effects of relativistic motion in astrophysics
- The effect is the combination of both relativistic time dilation and time retardation
- Consider a source which emits a period of radiation over a time Δt it takes to move from P_1 to P_2



If ω_{em} is emitted circular frequency of the radiation

in the rest frame, then $\Delta t' = \dfrac{2\pi}{\omega_{em}}$

and the time between the two events in the observer's frame is:

$$\Delta t = \Gamma \Delta t' = \Gamma \frac{2\pi}{\omega_{em}} \qquad (\gamma \equiv \Gamma)$$

However, this is not the observed time between the events because there is a time difference involved in radiation emitted from P_1 and P_2

Let ☛ D = distance to observer from P_2

and ☛ t_1 = time of emission of radiation from P_1

☛ t_2 = time of emission of radiation from P_2

Then, the times of reception $t_1^{\rm rec}$ and $t_2^{\rm rec}$ are

$$t_1^{\rm rec} = t_1 + \frac{D + V\,\Delta t\,\cos\theta}{c}$$

$$t_2^{\rm rec} = t_2 + \frac{D}{c}$$

Hence the period of the pulse received in the observer's frame is

$$\begin{aligned} t_2^{\rm rec} - t_1^{\rm rec} &= \left(t_2 + \frac{D}{c}\right) - \left(t_1 + \frac{D + V\Delta t\cos\theta}{c}\right) \\ &= (t_2 - t_1) - \frac{V}{c}\Delta t\cos\theta \\ &= \Delta t\left(1 - \frac{V}{c}\cos\theta\right) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{2\pi}{\omega_{\rm obs}} &= \Gamma\frac{2\pi}{\omega_{\rm em}}\left(1 - \frac{V}{c}\cos\theta\right) \\ \Rightarrow \omega_{\rm obs} &= \frac{\omega_{\rm em}}{\Gamma\left(1 - \frac{V}{c}\cos\theta\right)} = \frac{\omega_{\rm em}}{\Gamma(1 - \beta\cos\theta)} \end{aligned}$$

- the factor Γ is a pure relativistic effect
- the factor $(1-\beta\cos\theta)$ is the result of time retardation
- In terms of linear frequency

$$\nu_{obs} = \frac{\nu_{em}}{\Gamma(1-\beta\cos\theta)}$$

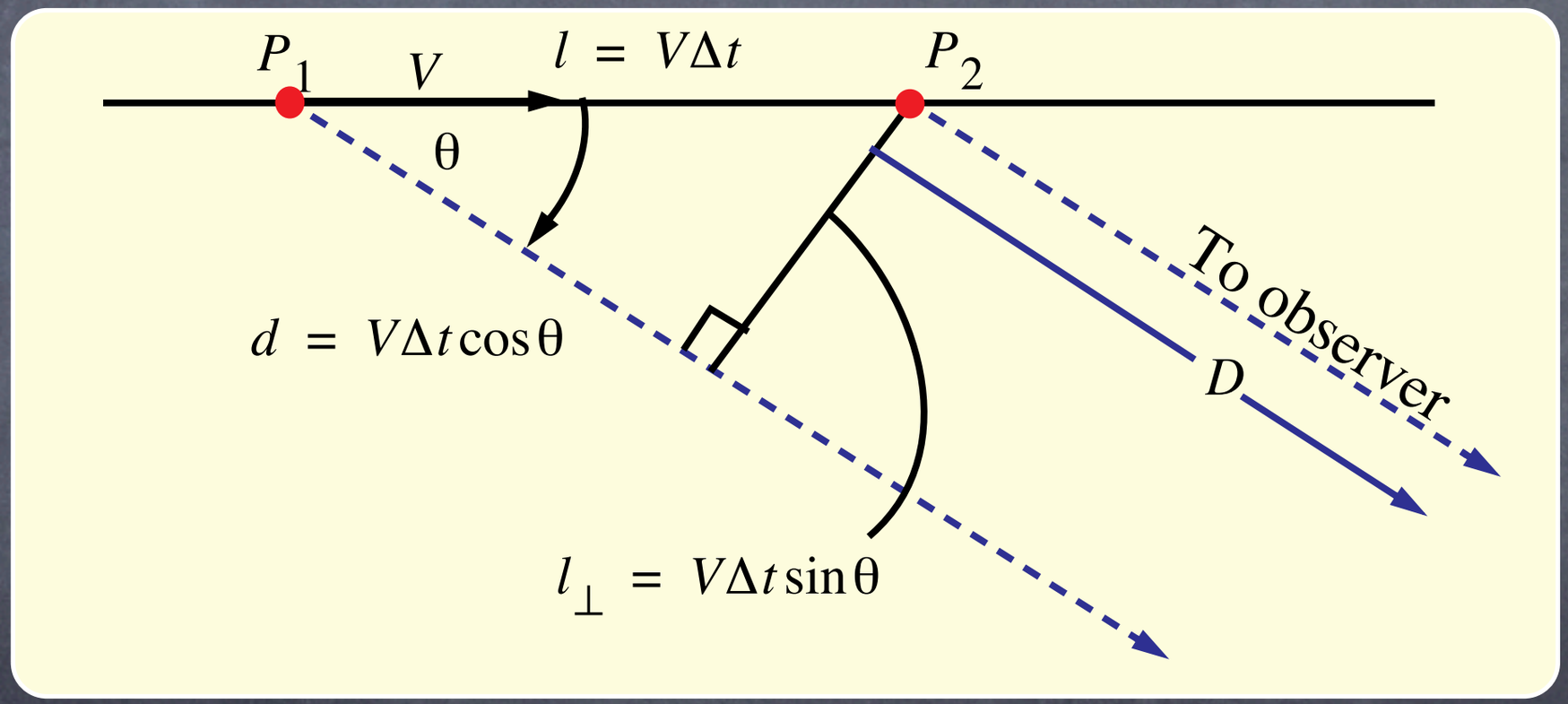
The factor

$$\delta = \frac{1}{\Gamma(1-\beta\cos\theta)}$$

is known as the Doppler factor and figures prominently in the theory of relativistically beamed emission

Apparent transverse velocity

A relativistic effect which is extremely important in high energy astrophysics and which is analyzed in a very similar way to the Doppler effect, relates to the apparent transverse velocity of a relativistically moving object



Consider an object which moves from P_1 to P_2 in a time Δt in the observer's frame. In this case Δt need not be the time between the beginning and end of a periodic. Indeed, in practice, Δt is usually of order a year

As before, the time difference between the time of receptions of photons emitted at P_1 and P_2 are given by

$$\Delta t_{\rm rec} = \Delta t\left(1 - \frac{V}{c}\cos\theta\right)$$

The apparent distance moved by the object is $l_\perp = V\,\Delta t \sin\theta$

Hence, the apparent velocity of the object is:

$$V_{app} = \frac{V\,\Delta t\sin\theta}{\Delta t\left(1-\frac{V}{c}\cos\theta\right)} = \frac{V\sin\theta}{\left(1-\frac{V}{c}\cos\theta\right)} \quad \Rightarrow \quad \frac{V_{app}}{c} = \frac{\frac{V}{c}\sin\theta}{\left(1-\frac{V}{c}\cos\theta\right)}$$

In terms of $\beta_{app} = \dfrac{V_{app}}{c}$ $\Rightarrow$ $\beta = \dfrac{V}{c}$ $\Rightarrow$ $\beta_{app} = \dfrac{\beta\sin\theta}{1-\beta\cos\theta}$

Then non-relativistic limit is just $V_{app} = V\sin\theta$ as we would expect

However, note that this result is not a consequence of the Lorentz transformation, but a consequence of light travel time effects as a result of the finite speed of light

Consequences

For angles close to the line of sight the effect of this equation can be dramatic.

First determine the angle for which the apparent velocity is a maximum

$$\frac{d\beta_{app}}{d\theta} = \frac{(1-\beta\cos\theta)\beta\cos\theta - \beta\sin\theta\beta\sin\theta}{(1-\beta\cos\theta)^2} = \frac{\beta\cos\theta - \beta^2}{(1-\beta\cos\theta)^2}$$

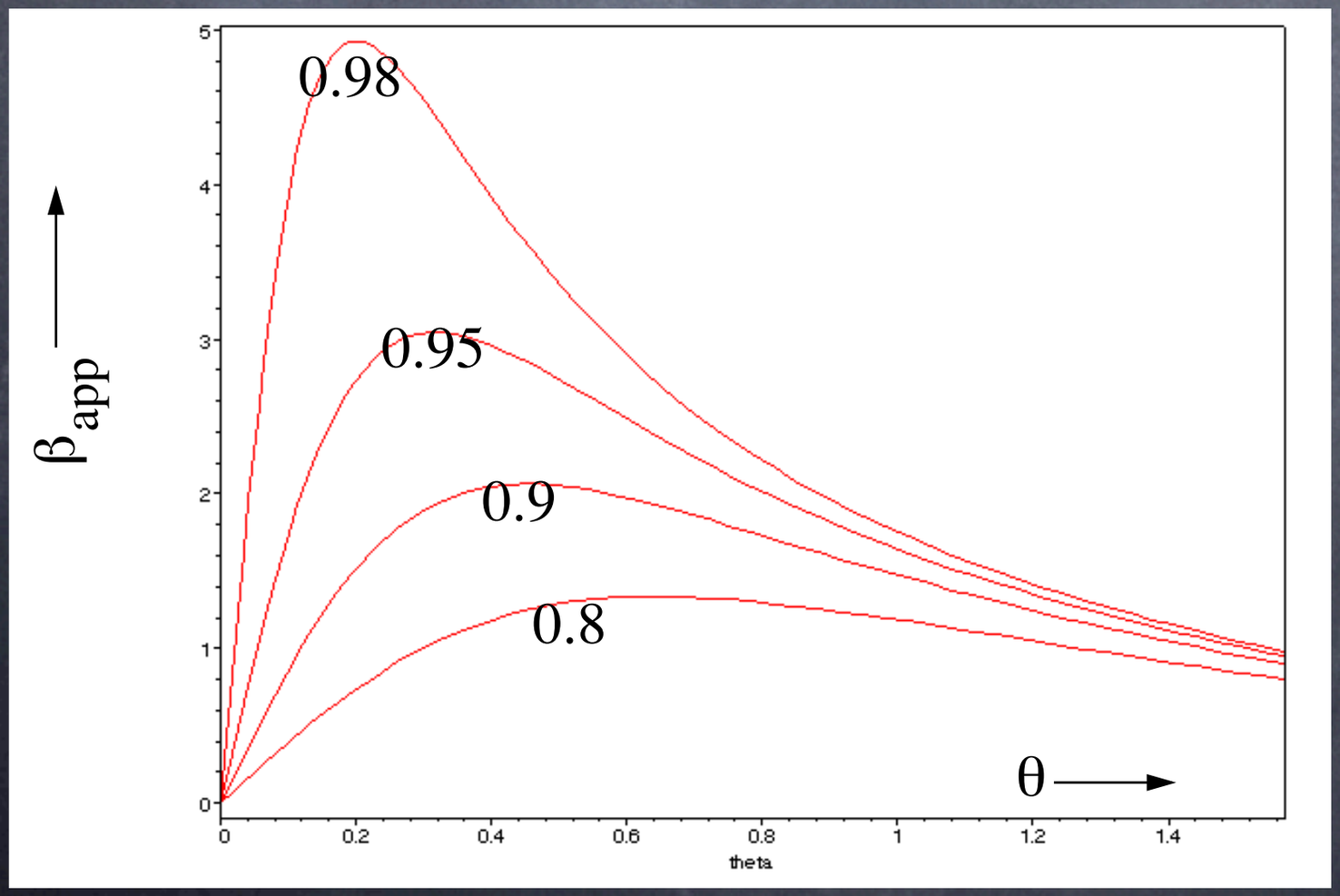
This derivative is zero when $\cos\theta = \beta$

At the maximum

$$\beta_{\rm app} = \frac{\beta\sin\theta}{1-\beta\cos\theta} = \frac{\beta\sqrt{1-\beta^2}}{1-\beta^2} = \frac{\beta}{\sqrt{1-\beta^2}} = \Gamma\beta$$

If $\Gamma \gg 1$ then $\beta \approx 1$ and the apparent velocity of an object can be larger then the speed of light

We actually see such effects in AGN.
Features in jets apparently move at faster than light speed (after conversion of the angular motion to a linear speed using the redshift of the source)
This was originally used to argue against the cosmological interpretation of quasar redshifts.
However, as you can see such large apparent velocities are an easily derived feature of large apparent velocities

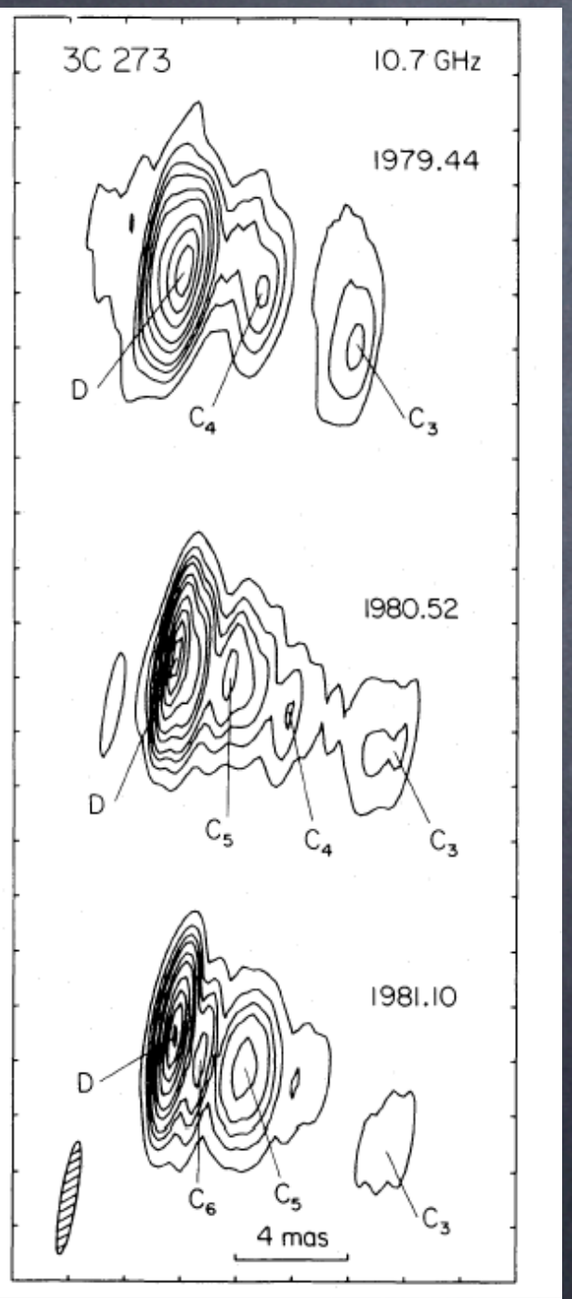


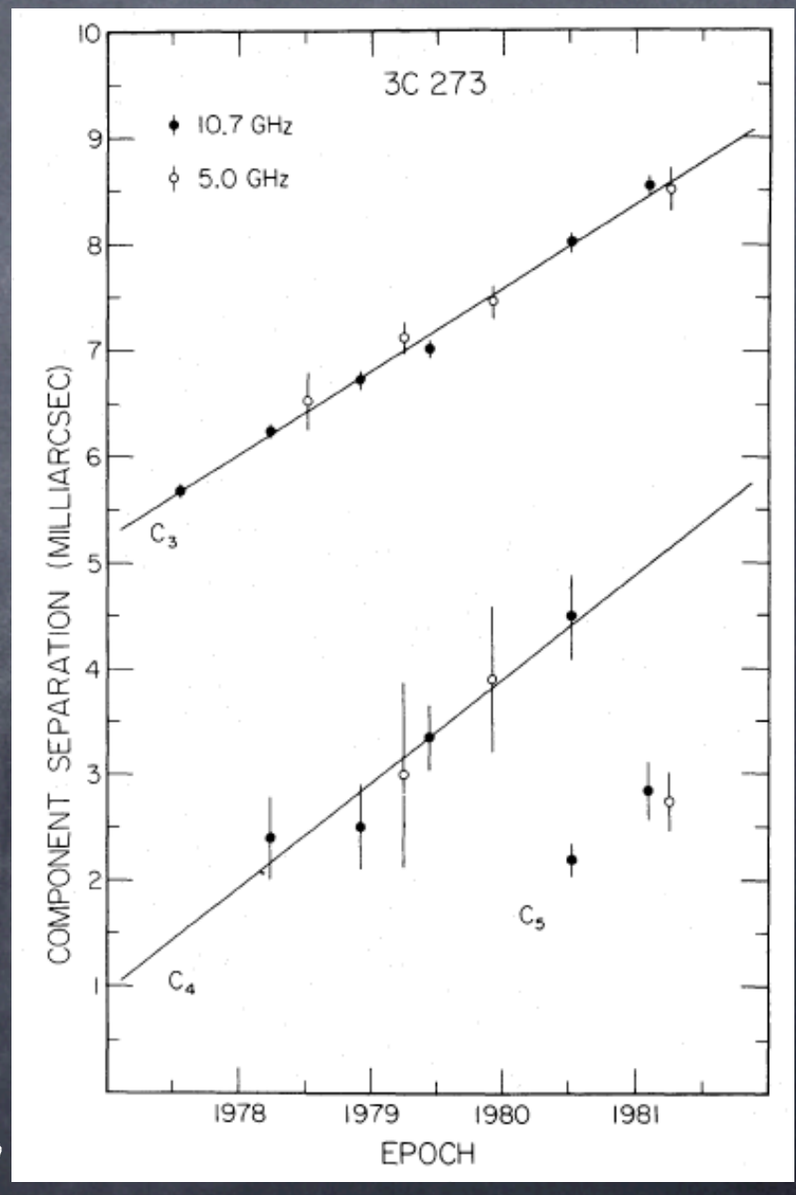
Plots of $\beta_{\rm app}$ for various indicated values of β as a function of θ

The following images are from observations of 3C 273 over a period of 5 years from 1977 to 1982

They show proper motions in the knots C_3 and C_4 of 0.79 ± 0.03 mas/yr and 0.99 ± 0.24 mas/yr respectively

These translate to proper motions of $5.5 \pm 0.2h^{-1}c$ and $6.9 \pm 1.7h^{-1}c$ respectively

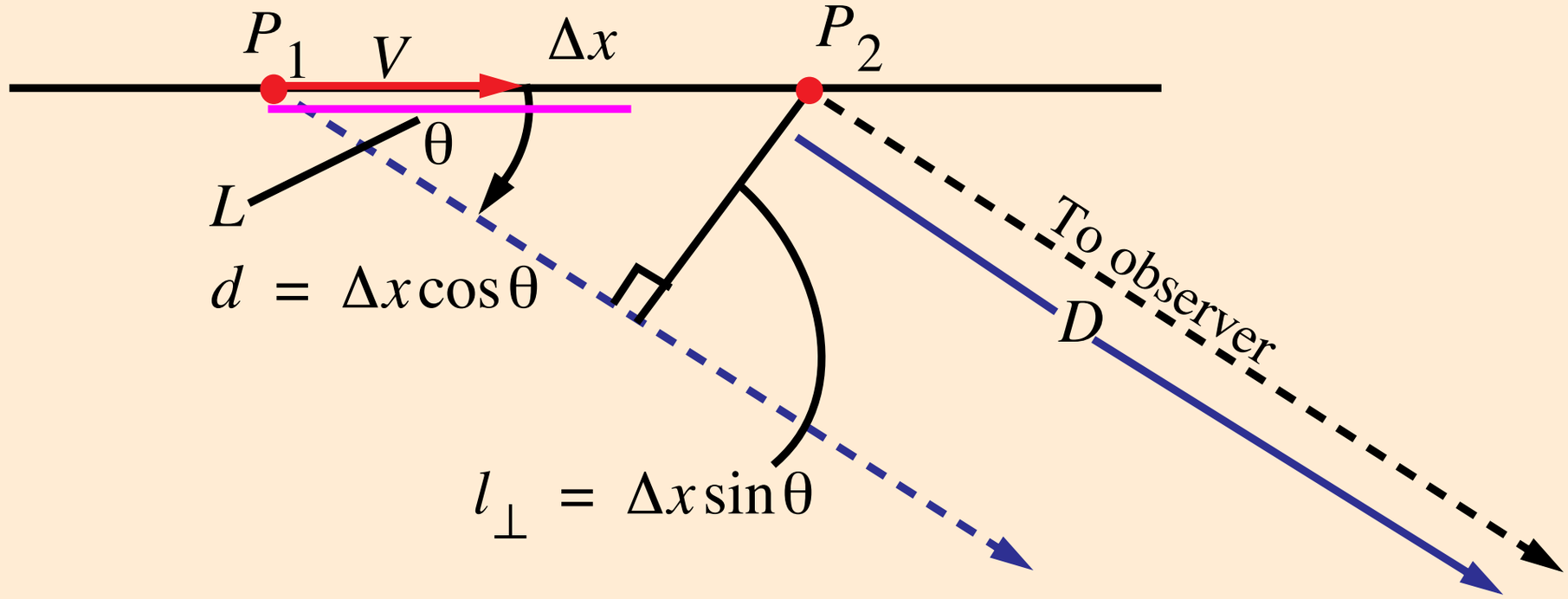




Unwin et al., ApJ 289 (1985) 109

Apparent length of a moving rod

The Lorentz-Fitzgerald contraction gives us the relationship between the proper lengths of moving rods An additional factor enters when we take into account time retardation



Consider a rod of length

$$L = \Gamma^{-1} L_0$$

in the observer's frame

Now the apparent length of the rod is affected by the fact that photons which arrive at the observer at the same time are emitted at different times

P_1 corresponds to when the trailing end of the rod passes at time t_1 and P_2 corresponds to when the leading end of the rod passes at time t_2

Equating the arrival times for photons emitted from P_1 and P_2 at times t_1 and t_2 respectively

$$t_1 + \frac{D + \Delta x \cos\theta}{c} = t_2 + \frac{D}{c} \Rightarrow t_2 - t_1 = \frac{\Delta x \cos\theta}{c}$$

When the trailing end of the rod reaches P_2 the leading end has to go a further distance $\Delta x - L$ which it does in $t_2 - t_1$ secs

Hence

$$\Delta x - L = \frac{V\,\Delta x \cos\theta}{c} \Rightarrow \Delta x = \frac{L}{1 - \frac{V}{c}\cos\theta}$$

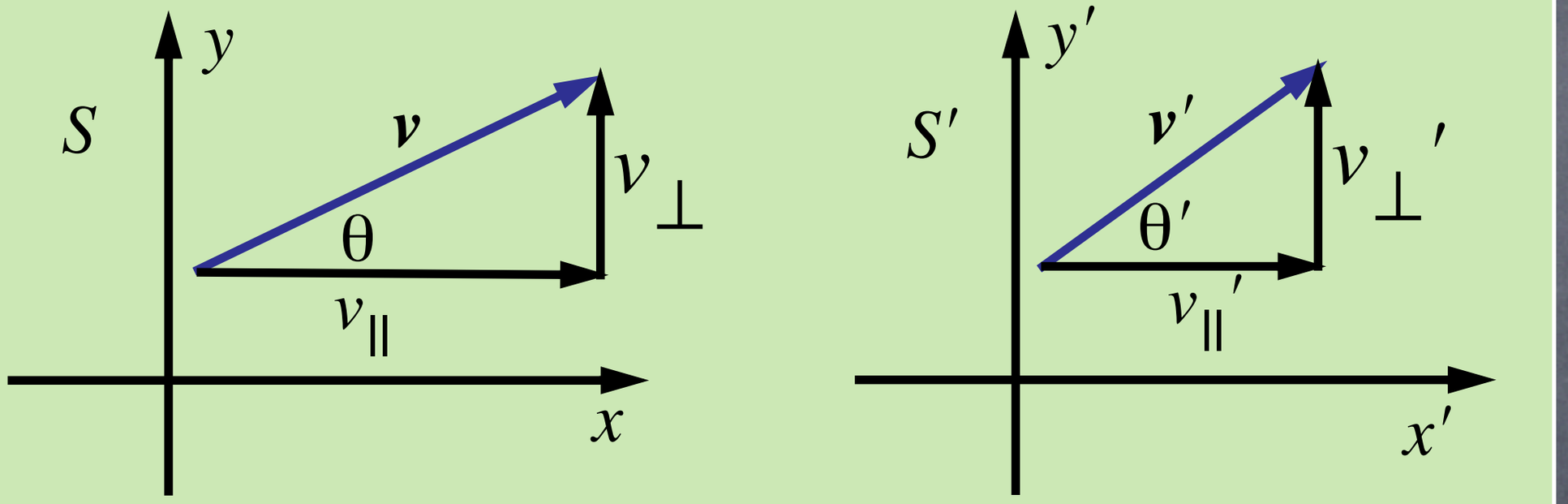
and the apparent projected length is

$$L_{\rm app} = \Delta x \sin\theta = \frac{L\sin\theta}{1-\beta\cos\theta} = \frac{L_0}{\Gamma(1-\beta\cos\theta)} = \delta L_0$$

This is another example of the appearance of the ubiquitous Doppler factor

And these can also be recovered by considering the differential form of the reverse Lorentz transformations

Aberration



Because of the law of transformation of velocities, a velocity vector make different angles with the direction of motion

From the above laws for transformation of velocities

$$\tan\theta = \frac{\mathrm{v}_\perp}{\mathrm{v}_\parallel} = \frac{\mathrm{v}'_\perp}{\Gamma(\mathrm{v}'_\parallel + \mathrm{V})} = \frac{\mathrm{v}' \sin\theta}{\Gamma(\mathrm{v}' \cos\theta + \mathrm{V})}$$

The difference from the non-relativistic case is the factor of Γ

The most important case of this is when $v = v' = c$

We put

$$v_\parallel = c\cos\theta \qquad v'_\parallel = c\cos\theta' \qquad v_\perp = c\sin\theta \qquad v'_\perp = c\sin\theta'$$

and $$\beta = \frac{V}{c}$$

and the angles made by the light rays in the two frames satisfy

$$c\cos\theta = \frac{c\cos\theta' + V}{1 + \frac{V}{c}\cos\theta'} \Rightarrow \cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

$$c\sin\theta = \frac{c\sin\theta'}{\Gamma\left(1 + \frac{V}{c}\cos\theta'\right)} \Rightarrow \sin\theta = \frac{\sin\theta'}{\Gamma(1 + \beta\cos\theta')}$$

Half-angle formula

There is a useful expression for aberration involving half-angles

Using the identity

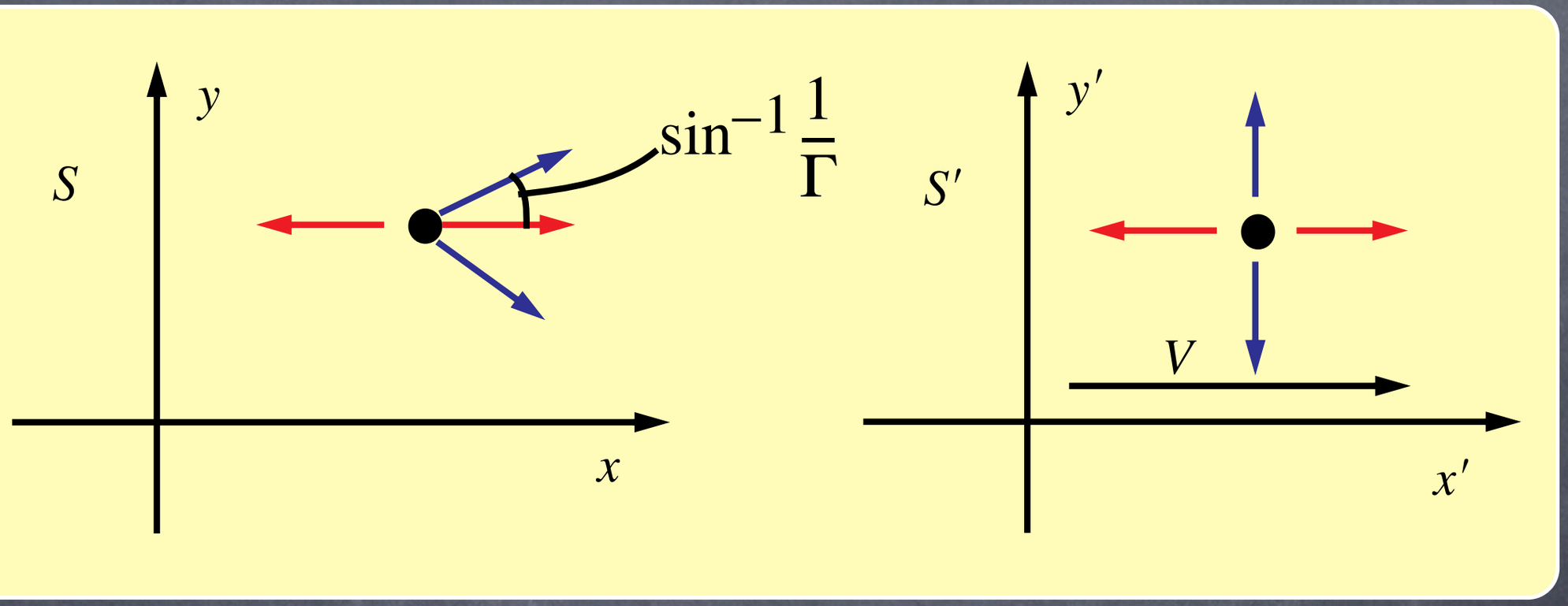
$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$$

The aberration formulae can be written as

$$\tan\left(\frac{\theta}{2}\right) = \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}} \tan\left(\frac{\theta'}{2}\right)$$

Isotropic radiation source

Consider a source of radiation which emits isotropically in its rest frame and which is moving with velocity V with respect to an observer (in frame S) ➤ The source is at rest in S′



Consider rays emitted at right angles to the direction of motion

This has $\theta = \pm\frac{\pi}{2}$ The angle of these rays in S is $\sin\theta = \pm\frac{1}{\Gamma}$

These rays enclose half the light emitted by the source, so that in the reference frame of the observer the light is emitted in a forward cone (when Γ is large) of half-angle $\theta \approx \Gamma^{-1}$