Chapter 5 Two Dimensional Kinematics

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Where was the chap I saw in the picture somewhere? Ah yes, in the dead sea floating on his back, reading a book with a parasol open. Couldn't sink if you tried: so thick with salt. Because the weight of the water, no, the weight of the body in the water is equal to the weight of the what? Or is it the volume equal to the weight? It's a law something like that. Vance in High school cracking his fingerjoints, teaching. The college curriculum. Cracking curriculum. What is weight really when you say weight? Thirtytwo feet per second per second. Law of falling bodies: per second per second. They all fall to the ground. The earth. It's the force of gravity of the earth is the weight. [1]

James Joyce

5.1 Introduction to the Vector Description of Motion in Two Dimensions

So far we have introduced the concepts of kinematics to describe motion in one dimension; however we live in a multidimensional universe. In order to explore and describe motion in this universe, we begin by looking at examples of two-dimensional motion, of which there are many; planets orbiting a star in elliptical orbits or a projectile moving under the action of uniform gravitation are two common examples.

We will now extend our definitions of position, velocity, and acceleration for an object that moves in two dimensions (in a plane) by treating each direction independently, which we can do with vector quantities by resolving each of these quantities into components. For example, our definition of velocity as the derivative of position holds for each component separately. In Cartesian coordinates, in which the directions of the unit vectors do not change from place to place, the position vector $\vec{\mathbf{r}}(t)$ with respect to some choice of origin for the object at time t is given by

$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}. \tag{5.1.1}$$

The velocity vector $\vec{\mathbf{v}}(t)$ at time t is the derivative of the position vector,

$$\vec{\mathbf{v}}(t) = \frac{dx(t)}{dt}\,\hat{\mathbf{i}} + \frac{dy(t)}{dt}\,\hat{\mathbf{j}} \equiv v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}}, \tag{5.1.2}$$

where $v_x(t) \equiv dx(t)/dt$ and $v_y(t) \equiv dy(t)/dt$ denote the x- and y-components of the velocity respectively.

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[1] James Joyce, *Ulysses*, The Corrected Text edited by Hans Walter Gabler with Wolfhard Steppe and Claus Melchior, Random House, New York.

The acceleration vector $\vec{\mathbf{a}}(t)$ is defined in a similar fashion as the derivative of the velocity vector,

$$\vec{\mathbf{a}}(t) = \frac{dv_x(t)}{dt}\,\hat{\mathbf{i}} + \frac{dv_y(t)}{dt}\,\hat{\mathbf{j}} \equiv a_x(t)\,\hat{\mathbf{i}} + a_y(t)\,\hat{\mathbf{j}}, \tag{5.1.3}$$

where $a_x(t) \equiv dv_x(t)/dt$ and $a_y(t) \equiv dv_y(t)/dt$ denote the x- and y-components of the acceleration.

5.2 Projectile Motion

Consider the motion of a body that is released at time $t = 0$ with an initial velocity $\vec{v}_0$ at a height h above the ground. Two paths are shown in Figure 5.1.

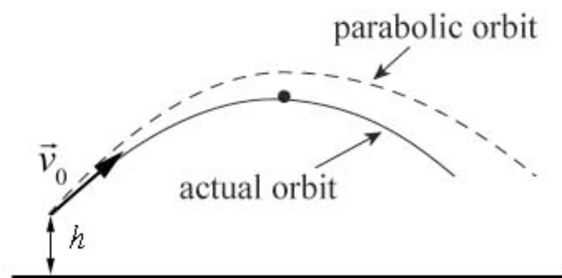

Figure 5.1 Actual orbit and parabolic orbit of a projectile

The dotted path represents a *parabolic* trajectory and the solid path represents the actual orbit. The difference between the paths is due to air resistance. There are other factors that can influence the path of motion; a rotating body or a special shape can alter the flow of air around the body, which may induce a curved motion or lift like the flight of a baseball or golf ball. We shall begin our analysis by neglecting all influences on the body except for the influence of gravity.

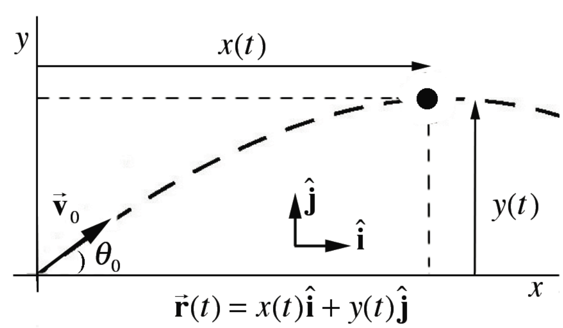

Figure 5.2 A coordinate sketch for parabolic motion.

Choose coordinates with the y-axis in the vertical direction with $\hat{\mathbf{j}}$ pointing upwards and the x-axis in the horizontal direction with $\hat{\mathbf{i}}$ pointing in the direction that the object is moving horizontally. Choose the origin to be at the point where the object is released.

Figure 5.2 shows our coordinate system with the position of the object at time t and the coordinate functions $x(t)$ and $y(t)$. The coordinate function $y(t)$ represents the distance from the body to the origin along the y-axis at time t, and the coordinate function $x(t)$ represents the distance from the body to the origin along the x-axis at time t.

Initial Conditions:

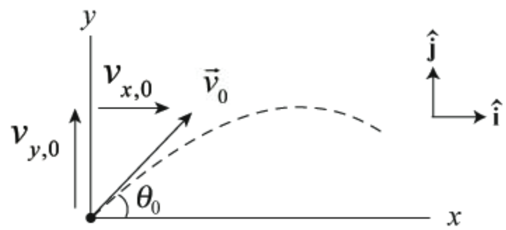

Figure 5.3 A vector decomposition of the initial velocity

We begin by making a vector decomposition of the initial velocity vector

$$\vec{\mathbf{v}}_0(t) = v_{x,0}\,\hat{\mathbf{i}} + v_{y,0}\,\hat{\mathbf{j}}. \qquad (5.1.4)$$

Often the description of the flight of a projectile includes the statement, "a body is projected with an initial speed v_0 at an angle θ_0 with respect to the horizontal." The vector decomposition diagram for the initial velocity is shown in Figure 5.3. The components of the initial velocity are given by

$$v_{x,0} = v_0 \cos\theta_0, \qquad (5.1.5)$$

$$v_{y,0} = v_0 \sin\theta_0. \qquad (5.1.6)$$

Since the initial speed is the magnitude of the initial velocity, we have

$$v_0 = (v_{x,0}^2 + v_{y,0}^2)^{1/2}. \qquad (5.1.7)$$

The angle θ_0 is related to the components of the initial velocity by

$$\theta_0 = \tan^{-1}(v_{y,0} / v_{x,0}). \qquad (5.1.8)$$

The initial position vector appears with components

$$\vec{\mathbf{r}}_0 = x_0\,\hat{\mathbf{i}} + y_0\,\hat{\mathbf{j}}. \qquad (5.1.9)$$

Note that the trajectory in Figure 5.3 has $x_0 = y_0 = 0$, but this will not always be the case.

Force Diagram:

The only force acting on the object is the gravitational interaction between the object and the earth. This force acts downward with magnitude mg, where m is the mass of the object and $g = 9.8\ \text{m} \cdot \text{s}^{-2}$. Figure 5.4 shows the force diagram on the object. The vector decomposition of the force is

$$\vec{\mathbf{F}}_{\text{grav}} = -mg\,\hat{\mathbf{j}}. \tag{5.1.10}$$

Should you include the force that gave the object its initial velocity on the force diagram? No! We are only interested in the forces acting on the object once the object has been released. It is a separate problem (and a very hard one at that) to determine exactly what forces acted on the object before the body was released. (This issue obscured the understanding of projectile motion for centuries.)

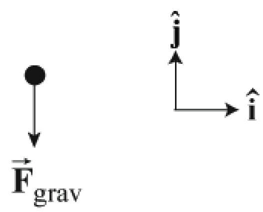

Figure 5.4 Free-body force diagram on the object with the action of gravity

Equations of Motions:

The force diagram reminds us that the only force is acting in the y-direction. Newton's Second Law states that the total vector force $\vec{\mathbf{F}}^{\text{total}}$ acting on the object is equal to the product of the mass m and the acceleration vector $\vec{\mathbf{a}}$,

$$\vec{\mathbf{F}}^{\text{total}} = m\vec{\mathbf{a}}. \tag{5.1.11}$$

This is a vector equation; the components are equated separately:

$$F_y^{\text{total}} = m a_y, \tag{5.1.12}$$

$$F_x^{\text{total}} = m a_x. \tag{5.1.13}$$

Thus the vector equation in the y-direction becomes

$$-mg = m a_y. \tag{5.1.14}$$

Therefore the y-component of the acceleration is

$$a_y = -g. \tag{5.1.15}$$

We see that the acceleration is a constant and is independent of the mass of the object. Notice that $a_y < 0$. This is because we chose our positive y-direction to point upwards. The sign of the y-component of acceleration is determined by how we chose our coordinate system. Because there are no horizontal forces acting on the object,

$$F_x^{\text{total}} = 0, \tag{5.1.16}$$

and we conclude that the acceleration in the horizontal direction is also zero,

$$a_x = 0. \tag{5.1.17}$$

This tells us that the x-component of the velocity remains unchanged throughout the flight of the object.

Newton's Second Law provides an analysis that determines that the acceleration in the vertical direction is constant for all bodies independent of the mass of the object, thus confirming Galileo's Law of Free Falling Bodies. Notice that the equation of motion (Equation (5.1.15)) generalizes the experimental observation that objects fall with constant acceleration. Our prediction is predicated on this force law and if subsequent observations show the acceleration is not constant then we either must include additional forces (for example, air resistance) or modify the force law (for objects that are no longer near the surface of the earth).

We can now integrate the equation of motion separately for the x- and y-directions to find expressions for the x- and y-components of position and velocity:

$$v_x(t) - v_{x,0} = \int_{t'=0}^{t'=t} a_x(t')\,dt' = 0 \Rightarrow v_x(t) = v_{x,0}$$

$$x(t) - x_0 = \int_{t'=0}^{t'=t} v_x(t')\,dt' = \int_{t'=0}^{t'=t} v_{x,0}\,dt' = v_{x,0}t \Rightarrow x(t) = x_0 + v_{x,0}t$$

$$v_y(t) - v_{y,0} = \int_{t'=0}^{t'=t} a_y(t')\,dt' = -\int_{t'=0}^{t'=t} g\,dt' = -gt \Rightarrow v_y(t) = v_{y,0} - gt$$

$$y(t) - y_0 = \int_{t'=0}^{t'=t} v_y(t')\,dt' = \int_{t'=0}^{t'=t} (v_{y,0} - gt)\,dt' = v_{y,0}t - (1/2)gt^2 \Rightarrow y(t) = y_0 + v_{y,0}t - (1/2)gt^2.$$

Then the complete set of vector equations for position and velocity for each independent direction of motion are given by

$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} = (x_0 + v_{x,0}t)\,\hat{\mathbf{i}} + (y_0 + v_{y,0}\,t + (1/2)a_y\,t^2)\,\hat{\mathbf{j}} \qquad (5.1.18)$$

$(x(t)\,, y(t))$

$$\vec{\mathbf{v}}(t) = v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}} = v_{x,0}\,\hat{\mathbf{i}} + (v_{y,0} + a_y\,t)\,\hat{\mathbf{j}}, \qquad (5.1.19)$$

$$\vec{\mathbf{a}}(t) = a_x(t)\,\hat{\mathbf{i}} + a_y(t)\,\hat{\mathbf{j}} = a_y\,\hat{\mathbf{j}}. \qquad (5.1.20)$$

Example 5.1 Time of Flight and Maximum Height of a Projectile

A person throws a stone at an initial angle $\theta_0 = 45^\circ$ from the horizontal with an initial speed of $v_0 = 20\ \mathrm{m \cdot s^{-1}}$. The point of release of the stone is at a height $d = 2\ \mathrm{m}$ above the ground. You may neglect air resistance. a) How long does it take the stone to reach the highest point of its trajectory? b) What was the maximum vertical displacement of the stone? Ignore air resistance.

Solution: Choose the origin on the ground directly underneath the point where the stone is released. We choose upwards for the positive y-direction and along the projection of the path of the stone along the ground for the positive x-direction. Set $t = 0$ the instant the stone is released. At $t = 0$ the initial conditions are then $x_0 = 0$ and $y_0 = d$. The initial x- and y-components of the velocity are given by Eqs. (5.1.5) and (5.1.6).

At time t the stone has coordinates $(x(t), y(t))$. These coordinate functions are shown in Figure 5.5.

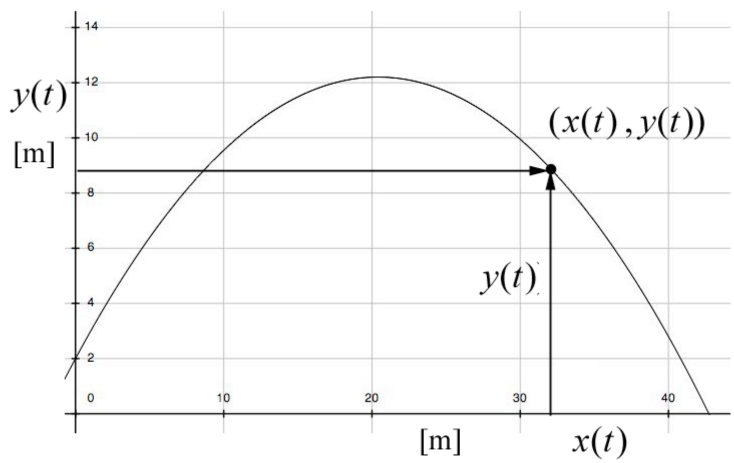

Figure 5.5: Coordinate functions for stone

The slope of this graph at any time t yields the instantaneous y-component of the velocity $v_y(t)$ at that time t.

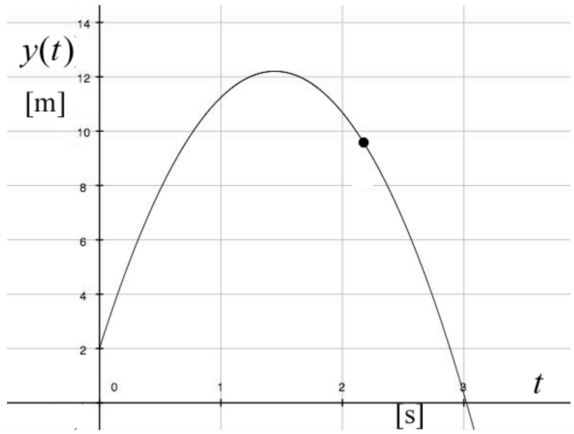

Figure 5.6 Plot of the y-component of the position as a function of time

There are several important things to notice about Figures 5.5 and 5.6. The first point is that the abscissa axes are different in both figures, Figure 5.5 is a plot of y vs. x and Figure 5.6 is a plot of y vs. t. The second thing to notice is that at $t = 0$, the slope of the graph in Figure 5.5 is equal to $(dy/dx)(t=0) = v_{y,0} / v_{x,0} = \tan\theta_0$, while at $t = 0$ the slope of the graph in Figure 5.6 is equal to $v_{y,0}$. Let $t = t_{top}$ correspond to the instant the stone is at its maximal vertical position, the highest point in the flight. The final thing to notice about Figure 5.6 is that at $t = t_{top}$ the slope is zero or $v_y(t = t_{top}) = 0$. Therefore

$$v_y(t_{top}) = v_0 \sin\theta_0 - g t_{top} = 0 .$$

We can solve this equation for t_{top},

$$t_{top} = \frac{v_0 \sin\theta_0}{g} = \frac{(20\ \mathrm{m \cdot s^{-1}})\sin(45^\circ)}{9.8\ \ \mathrm{m \cdot s^{-2}}} = 1.44\ \mathrm{s} . \qquad (5.1.21)$$

The y-component of the velocity as a function of time is graphed in Figure 5.7.

Notice that at $t = 0$ the intercept is positive indicting the initial y-component of the velocity is positive which means that the stone was thrown upwards. The y-component of the velocity changes sign at $t = t_{top}$ indicating that the stone is reversing its direction and starting to move downwards.

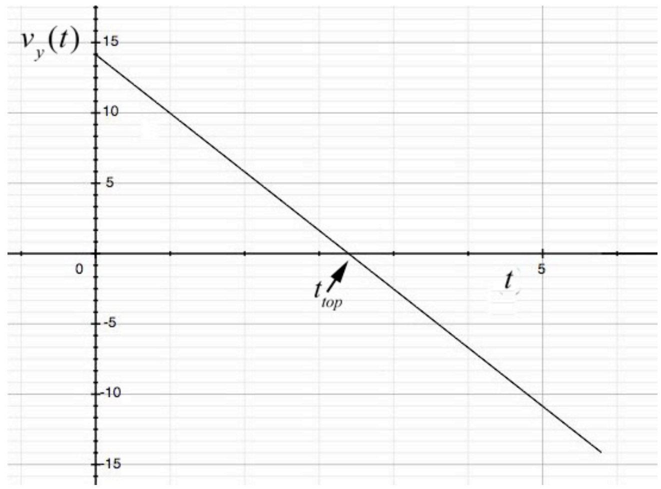

Figure 5.7 y-component of the velocity as a function of time

We can now substitute the expression for $t = t_{top}$ (Eq. (5.1.21)) into the y-component of the position in Eq. (5.1.18) to find the maximal height of the stone above the ground

$$\begin{aligned} y(t = t_{top}) &= d + v_0 \sin\theta_0 \frac{v_0 \sin\theta_0}{g} - \frac{1}{2} g \left(\frac{v_0 \sin\theta_0}{g} \right)^2 \\ &= d + \frac{v_0^{\,2} \sin^2\theta_0}{2g} = 2 \text{ m} + \frac{(20 \text{ m}\cdot\text{s}^{-1})^2 \sin^2(45^\circ)}{2(9.8 \text{ m}\cdot\text{s}^{-2})} = 12.2 \text{ m} \end{aligned}, \qquad (5.1.22)$$

5.2.1 Orbit equation

So far our description of the motion has emphasized the independence of the spatial dimensions, treating all of the kinematic quantities as functions of time. We shall now eliminate time from our equation and find the *orbit equation* of the body. We begin with the x-component of the position in Eq. (5.1.18),

$$x(t) = x_0 + v_{x,0} t \qquad (5.1.23)$$

and solve Equation (5.1.23) for time t as a function of $x(t)$,

$$t = \frac{x(t) - x_0}{v_{x,0}}. \qquad (5.1.24)$$

The y-component of the position in Eq. (5.1.18) is given by

$$y(t) = y_0 + v_{y,0} t - \frac{1}{2} g t^2. \qquad (5.1.25)$$

We then substitute the above expression, Equation (5.1.24) for time t into our equation for the y-component of the position yielding

$$y(t) = y_0 + v_{y,0}\left(\frac{x(t)-x_0}{v_{x,0}}\right) - \frac{1}{2}g\left(\frac{x(t)-x_0}{v_{x,0}}\right)^2. \tag{5.1.26}$$

This expression can be simplified to give

$$y(t) = y_0 + \frac{v_{y,0}}{v_{x,0}}(x(t)-x_0) - \frac{1}{2}\frac{g}{v_{x,0}^2}(x(t)^2 - 2x(t)x_0 + x_0^2). \tag{5.1.27}$$

This is seen to be an equation for a parabola by rearranging terms to find

$$y(t) = -\frac{1}{2}\frac{g}{v_{x,0}^2}x(t)^2 + \left(\frac{g\,x_0}{v_{x,0}^2} + \frac{v_{y,0}}{v_{x,0}}\right)x(t) - \frac{v_{y,0}}{v_{x,0}}x_0 - \frac{1}{2}\frac{g}{v_{x,0}^2}x_0^2 + y_0. \tag{5.1.28}$$

The graph of $y(t)$ as a function of $x(t)$ is shown in Figure 5.8.

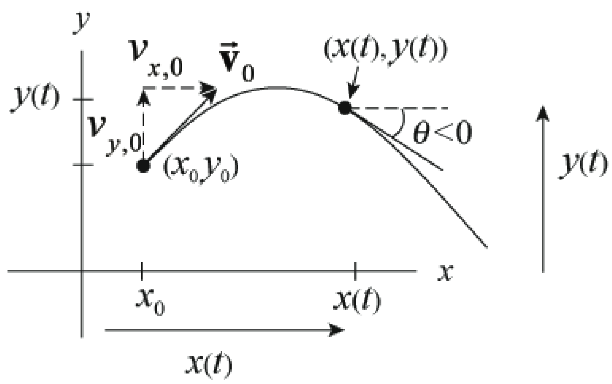

Figure 5.8 The parabolic orbit

Note that at any point $(x(t), y(t))$ along the parabolic trajectory, the direction of the tangent line at that point makes an angle θ with the positive x-axis as shown in Figure 5.8. This angle is given by

$$\theta = \tan^{-1}\left(\frac{dy}{dx}\right), \tag{5.1.29}$$

where dy/dx is the derivative of the function $y(x) = y(x(t))$ at the point $(x(t), y(t))$.

The velocity vector is given by

$$\vec{\mathbf{v}}(t) = \frac{dx(t)}{dt}\,\hat{\mathbf{i}} + \frac{dy(t)}{dt}\,\hat{\mathbf{j}} \equiv v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}} \tag{5.1.30}$$

The direction of the velocity vector at a point $(x(t), y(t))$ can be determined from the components. Let ϕ be the angle that the velocity vector forms with respect to the positive x-axis. Then

$$\phi = \tan^{-1}\left(\frac{v_y(t)}{v_x(t)}\right) = \tan^{-1}\left(\frac{dy/dt}{dx/dt}\right) = \tan^{-1}\left(\frac{dy}{dx}\right). \tag{5.1.31}$$

Comparing our two expressions we see that $\phi = \theta$; the slope of the graph of $y(t)$ *vs.* $x(t)$ at any point determines the direction of the velocity at that point. We cannot tell from our graph of $y(x)$ how fast the body moves along the curve; the magnitude of the velocity cannot be determined from information about the tangent line.

If we choose our origin at the initial position of the body at $t = 0$, then $x_0 = 0$ and $y_0 = 0$. Our orbit equation, Equation (5.1.28) can now be simplified to

$$y(t) = -\frac{1}{2}\frac{g}{v_{x,0}^2}x(t)^2 + \frac{v_{y,0}}{v_{x,0}}x(t). \tag{5.1.32}$$

Example 5.2 Hitting the Bucket

A person is standing on a ladder holding a pail. The person releases the pail from rest at a height h_1 above the ground. A second person standing a horizontal distance s_2 from the pail aims and throws a ball the instant the pail is released in order to hit the pail. The person releases the ball at a height h_2 above the ground, with an initial speed v_0, and at an angle θ_0 with respect to the horizontal. You may ignore air resistance.

a) Find an expression for the angle θ_0 that the person aims the ball in order to hit the pail.

b) Find an expression for the height above the ground where the collision occurred as a function of the initial speed of the ball v_0, and the quantities h_1, h_2, and s_2.

Solution: 1. Understand – get a conceptual grasp of the problem

There are two objects involved in this problem. Each object is undergoing free fall, so there is only one stage each. The pail is undergoing one-dimensional motion. The ball is undergoing two-dimensional motion. The parameters h_1, h_2, and s_2 are unspecified, so our answers will be functions of those symbolic expressions for the quantities. Figure 5.9 shows a sketch of the motion of all the bodies in this problem.

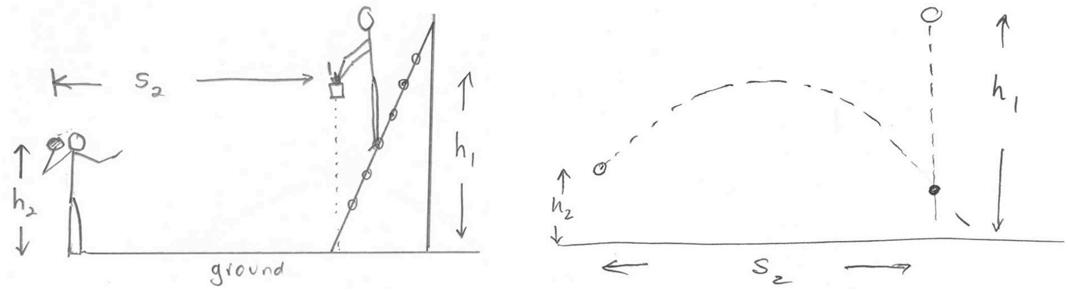

Figure 5.9: Sketch of motion of ball and bucket.

Since the acceleration is unidirectional and constant, we will choose Cartesian coordinates, with one axis along the direction of acceleration. Choose the origin on the ground directly underneath the point where the ball is released. We choose upwards for the positive y-direction and towards the pail for the positive x-direction.

We choose position coordinates for the pail as follows. The horizontal coordinate is constant and given by $x_1 = s_2$. The vertical coordinate represents the height above the ground and is denoted by $y_1(t)$. The ball has coordinates $(x_2(t), y_2(t))$. We show these coordinates in the Figure 5.10.

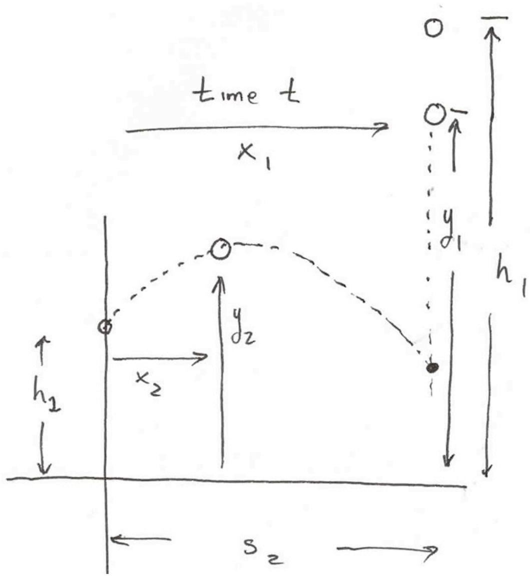

Figure 5.10: Coordinate System

2. Devise a Plan - set up a procedure to obtain the desired solution

(a) Find an expression for the angle θ_0 that the person throws the ball as a function of h_1, h_2, and s_2.

(b) Find an expression for the time of collision as a function of the initial speed of the ball v_0, and the quantities h_1, h_2, and s_2.

(c) Find an expression for the height above the ground where the collision occurred as a function of the initial speed of the ball v_0, and the quantities h_1, h_2, and s_2.

Model: The pail undergoes constant acceleration $(a_y)_1 = -g$ in the vertical direction downwards and the ball undergoes uniform motion in the horizontal direction and constant acceleration downwards in the vertical direction, with $(a_x)_2 = 0$ and $(a_y)_2 = -g$.

Equations of Motion for Pail: The initial conditions for the pail are $(v_{y,0})_1 = 0$, $x_1 = s_2$, $(y_0)_1 = h_1$. Since the pail moves vertically, the pail always satisfies the constraint condition $x_1 = s_2$ and $v_{x,1} = 0$. The equations for position and velocity of the pail simplify to

$$y_1(t) = h_1 - \frac{1}{2}gt^2 \tag{5.1.33}$$

$$v_{y,1}(t) = -gt\,. \tag{5.1.34}$$

Equations of Motion for Ball: The initial position is given by $(x_0)_2 = 0$, $(y_0)_2 = h_2$. The components of the initial velocity are given by $(v_{y,0})_2 = v_0\sin(\theta_0)$ and $(v_{x,0})_2 = v_0\cos(\theta_0)$, where v_0 is the magnitude of the initial velocity and θ_0 is the initial angle with respect to the horizontal. So the equations for position and velocity of the ball simplify to

$$x_2(t) = v_0\cos(\theta_0)t \tag{5.1.35}$$

$$v_{x,2}(t) = v_0\cos(\theta_0) \tag{5.1.36}$$

$$y_2(t) = h_2 + v_0\sin(\theta_0)t - \frac{1}{2}gt^2 \tag{5.1.37}$$

$$v_{y,2}(t) = v_0\sin(\theta_0) - gt\,. \tag{5.1.38}$$

Note that the quantities h_1, h_2, and s_2 should be treated as known quantities although no numerical values were given, only symbolic expressions. There are six independent equations with 9 as yet unspecified quantities $y_1(t)$, t, $y_2(t)$, $x_2(t)$, $v_{y,1}(t)$, $v_{y,2}(t)$, $v_{x,2}(t)$, v_0, and θ_0.

So we need two more conditions, in order to find expressions for the initial angle, θ_0, the time of collision, t_a, and the spatial location of the collision point specified by $y_1(t_a)$ or $y_2(t_a)$ in terms of the one unspecified parameter v_0. At the collision time $t = t_a$, the collision occurs when the two balls are located at the same position. Therefore

$$y_1(t_a) = y_2(t_a) \tag{5.1.39}$$

$$x_2(t_a) = x_1 = s_2\,. \tag{5.1.40}$$

We shall now apply these conditions that must be satisfied in order for the ball to hit the pail.

$$h_1 - \frac{1}{2}gt_a^{\ 2} = h_2 + v_0\sin\left(\theta_0\right)t_a - \frac{1}{2}gt_a^{\ 2} \tag{5.1.41}$$

$$s_2 = v_0\cos(\theta_0)t_a\,. \tag{5.1.42}$$

From the first equation, the term $(1/2)gt_a^{\ 2}$ cancels from both sides. Therefore we have that

$$h_1 = h_2 + v_0\sin(\theta_0)t_a$$

$$s_2 = v_0\cos(\theta_0)t_a\,.$$

We can now solve these equations for $\tan(\theta_0) = \sin(\theta_0)/\cos(\theta_0)$, and thus the angle the person throws the ball in order to hit the pail.

3. Carry our your plan – solve the problem!

We rewrite these equations as

$$v_0\sin(\theta_0)t_a = h_1 - h_2 \tag{5.1.43}$$

$$v_0\cos(\theta_0)t_a = s_2\,. \tag{5.1.44}$$

Dividing these equations yields

$$\frac{v_0\sin(\theta_0)t_a}{v_0\cos(\theta_0)t_a} = \tan(\theta_0) = \frac{h_1 - h_2}{s_2}\,. \tag{5.1.45}$$

So the initial angle is independent of v_0, and is given by

$$\theta_0 = \tan^{-1}((h_1 - h_2)/s_2)\,. \tag{5.1.46}$$

From the Figure 5.11 we can see that $\tan(\theta_0) = (h_1 - h_2)/s_2$ implies that the second person aims the ball at the initial position of the pail.

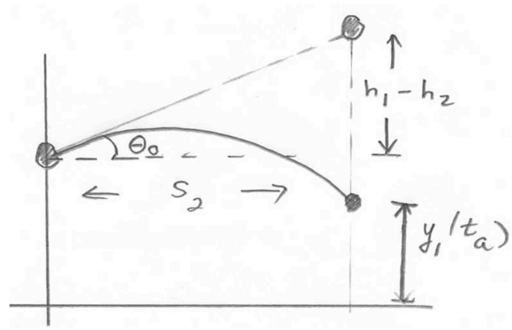

Figure 5.11: Geometry of collision

In order to find the time that the ball collides with the pail, we begin by squaring both Eqs. (5.1.43) and (5.1.44), then utilize the trigonometric identity $\sin^2(\theta_0) + \cos^2(\theta_0) = 1$. Our squared equations become

$$v_0^{\,2}\sin^2(\theta_0)t_a^{\,2} = (h_1 - h_2)^2 \tag{5.1.47}$$

$$v_0^{\,2}\cos^2(\theta_0)t_a^{\,2} = s_2^{\,2}. \tag{5.1.48}$$

Adding these equations together yields

$$v_0^{\,2}(\sin^2(\theta_0) + \cos^2(\theta_0))t_a^{\,2} = s_2^{\,2} + (h_1 - h_2)^2, \tag{5.1.49}$$

$$v_0^{\,2}t_a^{\,2} = s_2^{\,2} + (h_1 - h_2)^2. \tag{5.1.50}$$

We can solve Eq. (5.1.50) for the time of collision

$$t_a = \left(\frac{s_2^{\,2} + (h_1 - h_2)^2}{v_0^{\,2}}\right)^{1/2}. \tag{5.1.51}$$

We can now use the y-coordinate function of either the ball or the pail at $t = t_a$ to find the height that the ball collides with the pail. Since it had no initial y-component of the velocity, it's easier to use the pail,

$$y_1(t_a) = h_1 - \frac{g(s_2^{\,2} + (h_1 - h_2)^2)}{2v_0^{\,2}}. \tag{5.1.52}$$

4. Look Back – check your solution and method of solution

The person aims at the pail at the point where the pail was released. Both undergo free fall so the key result was that the vertical position obeys

$$h_1 - \frac{1}{2} g t_a{}^2 = h_2 + v_0 \sin(\theta_0) t_a - \frac{1}{2} g t_a{}^2 .$$

The distance traveled due to gravitational acceleration are the same for both so all that matters is the contribution from the initial positions and the vertical component of velocity $h_1 = h_2 + v_0 \sin(\theta_0) t_a$. Because the time is related to the horizontal distance by $s_2 = v_0 \cos(\theta_0) t_a$, it's as if both objects were moving at constant velocity.